Rewriting History in Integrable Stochastic Particle Systems

Leonid Petrov (University of Virginia)

November 2, 2022

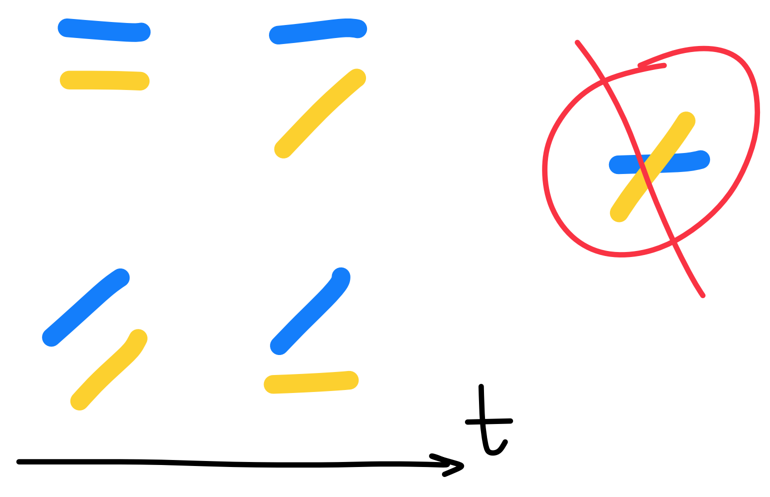
joint with Axel Saenz (Oregon State University), preprint upcoming

A tale of two cars on a one-lane road

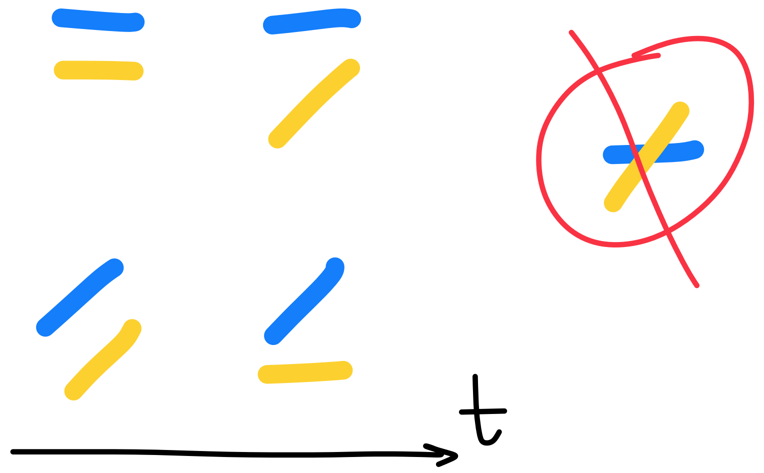
Two cars (discrete time TASEP with Bernoulli jumps)

Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1+a_i)$

One-lane highway: no passing. Consider two systems: FS and SF

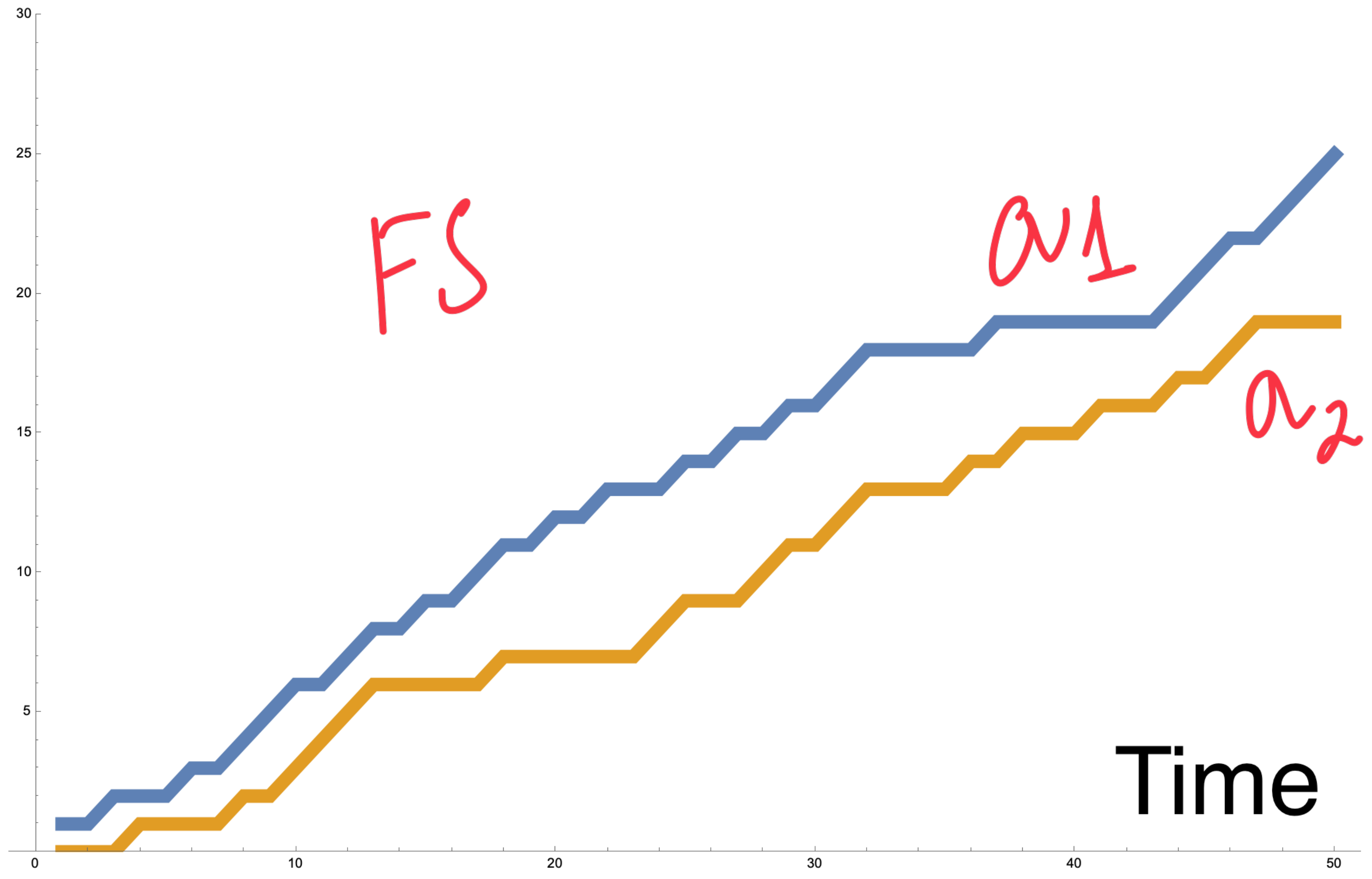


Two cars (discrete time TASEP with Bernoulli jumps)

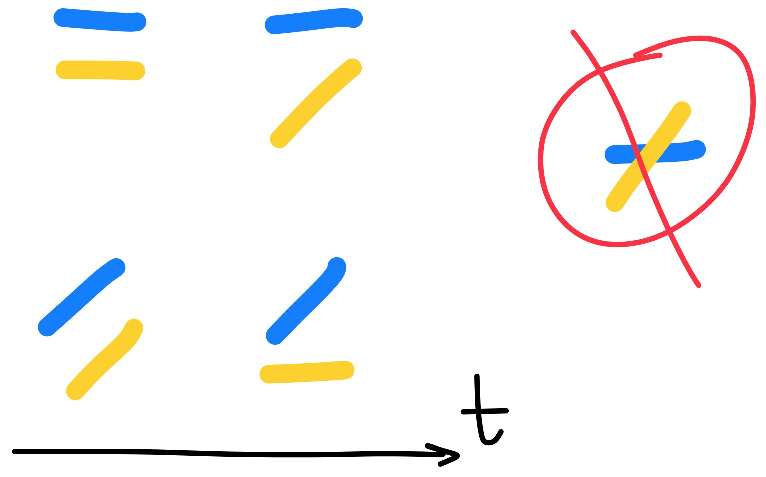


Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1+a_i)$

One-lane highway: no passing. Consider two systems: FS and SF

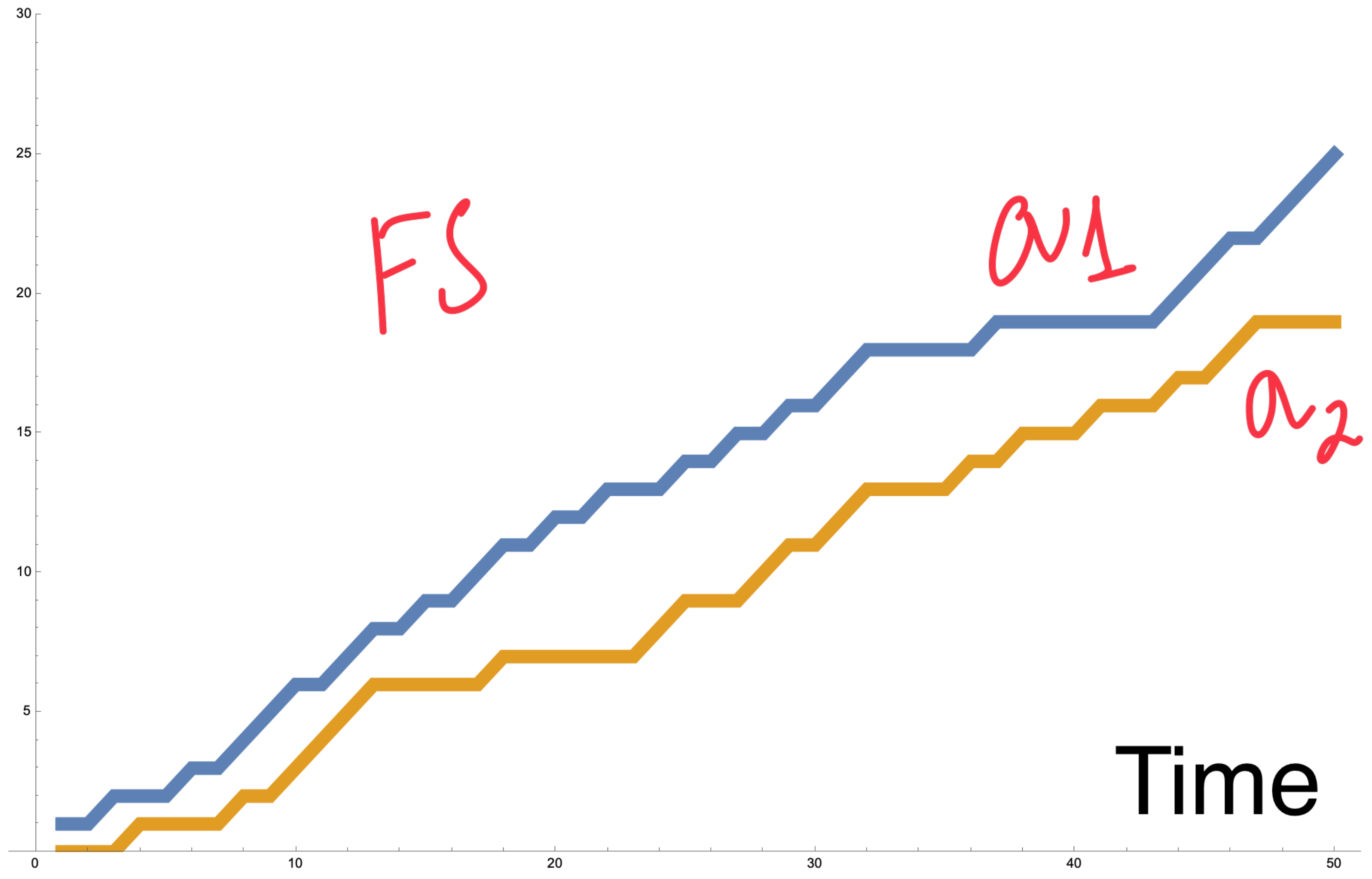


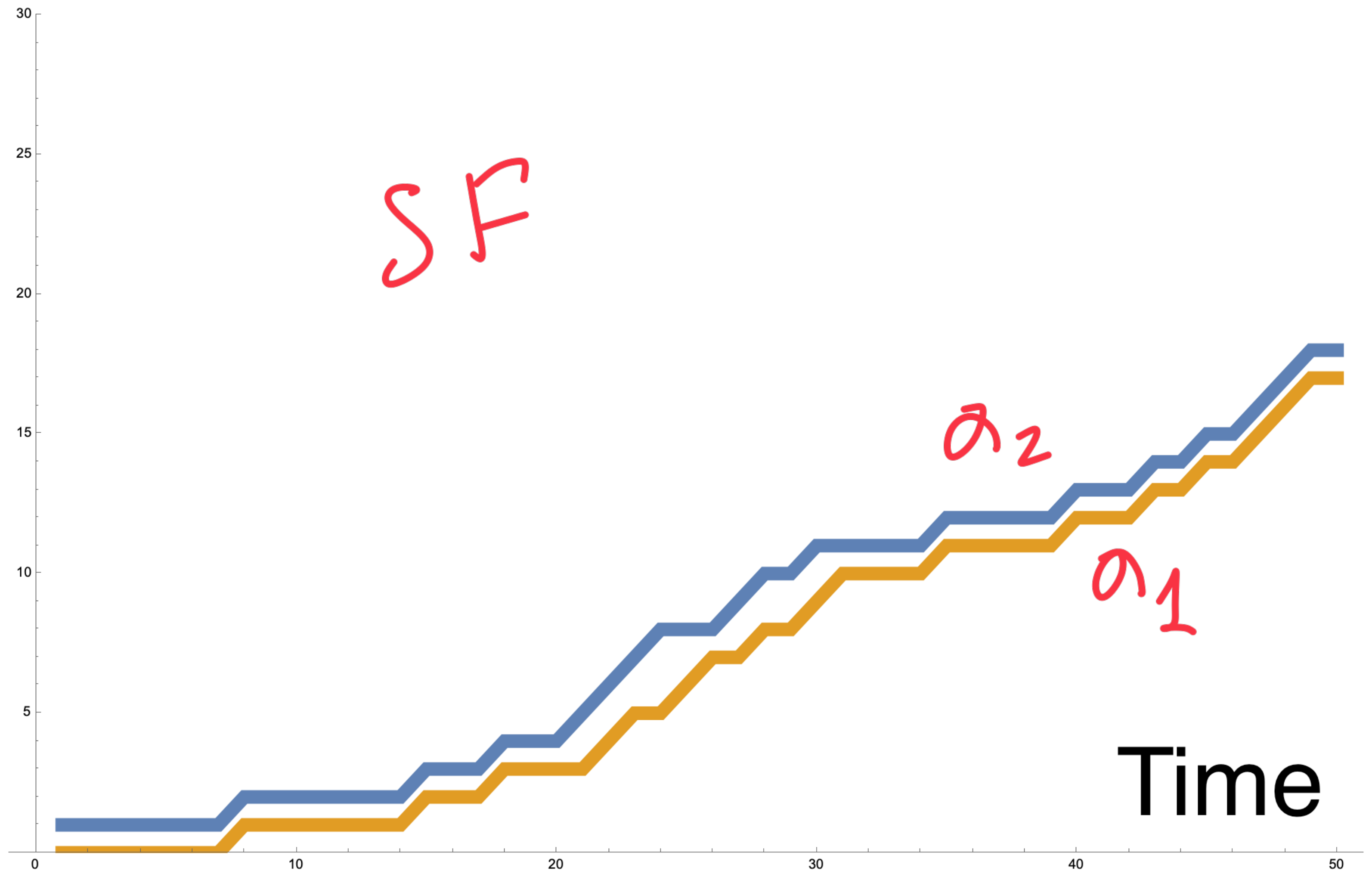
Two cars (discrete time TASEP with Bernoulli jumps)



Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1+a_i)$

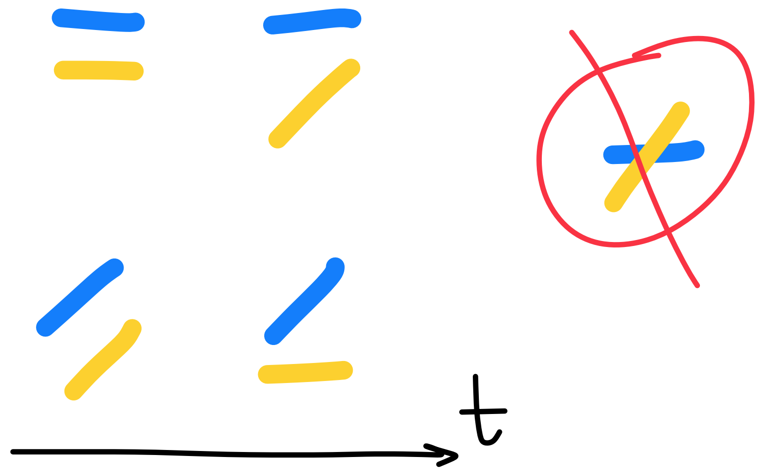
One-lane highway: no passing. Consider two systems: FS and SF



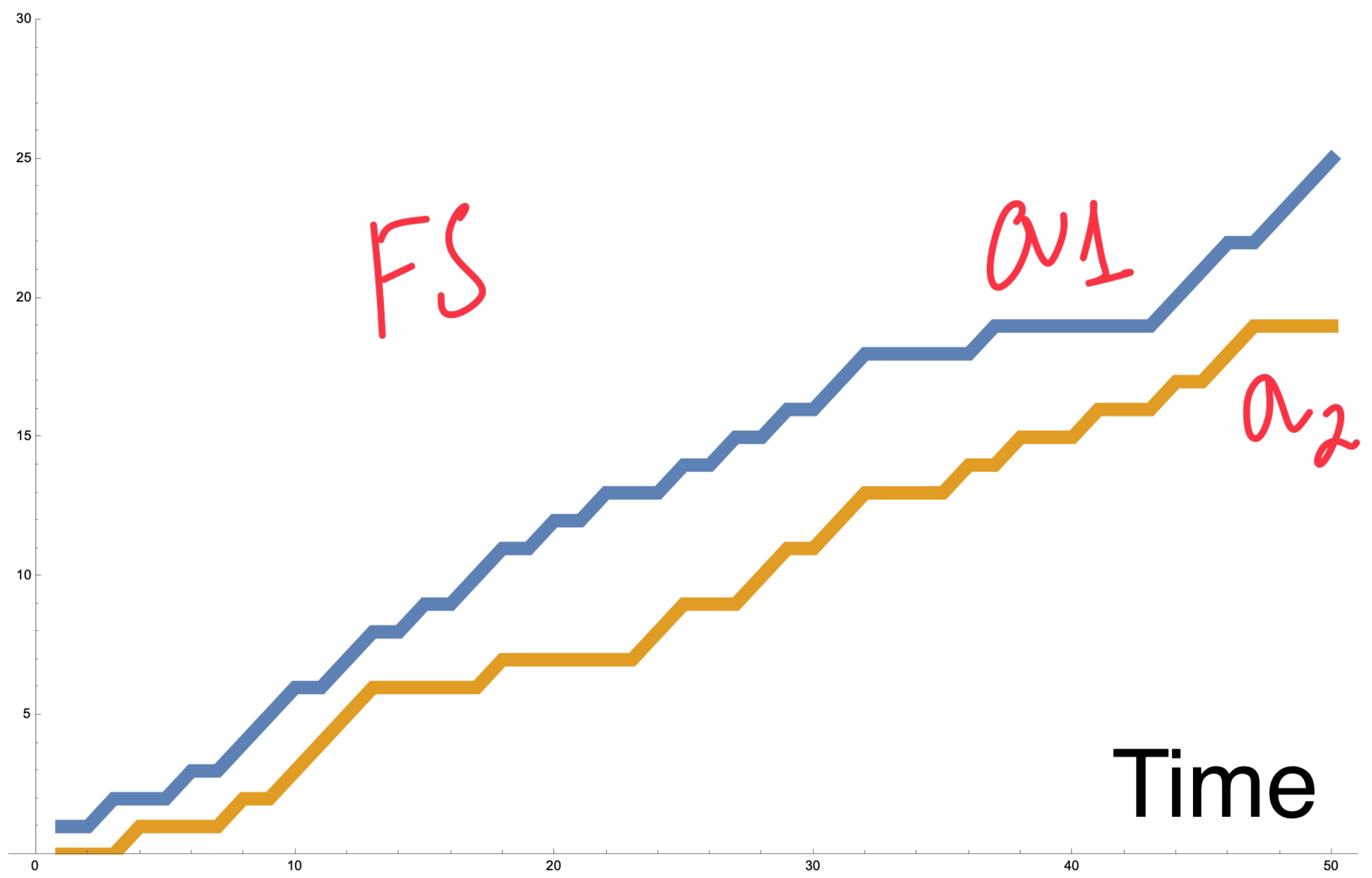


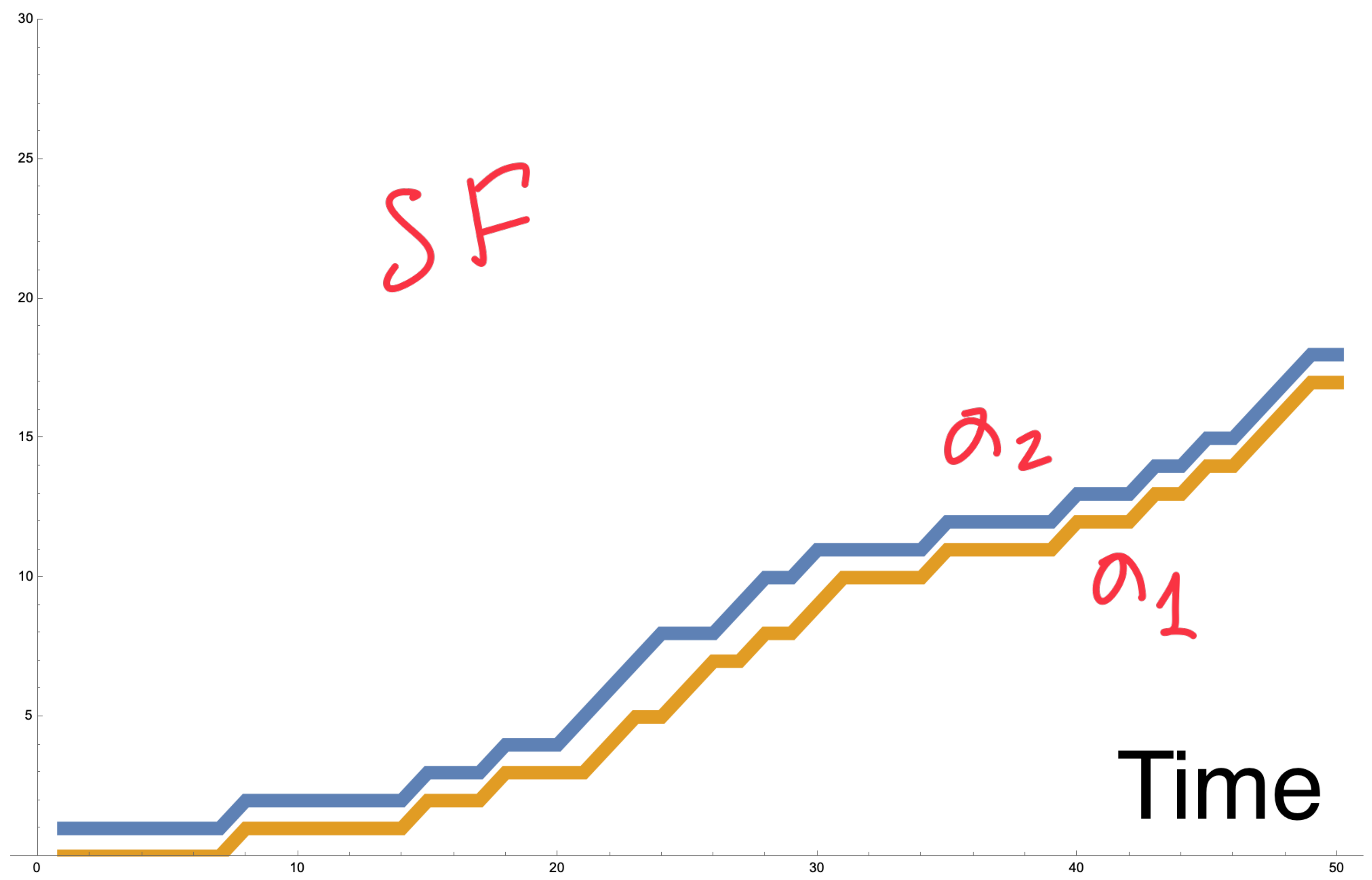
Two cars (discrete time TASEP with Bernoulli jumps)

Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1+a_i)$



One-lane highway: no passing. Consider two systems: FS and SF



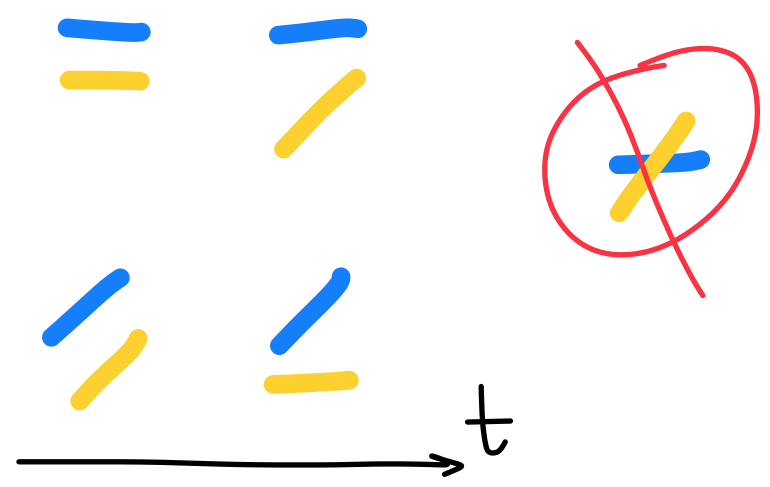


The long-time speed of the car ahead (blue) depends on which car is first;

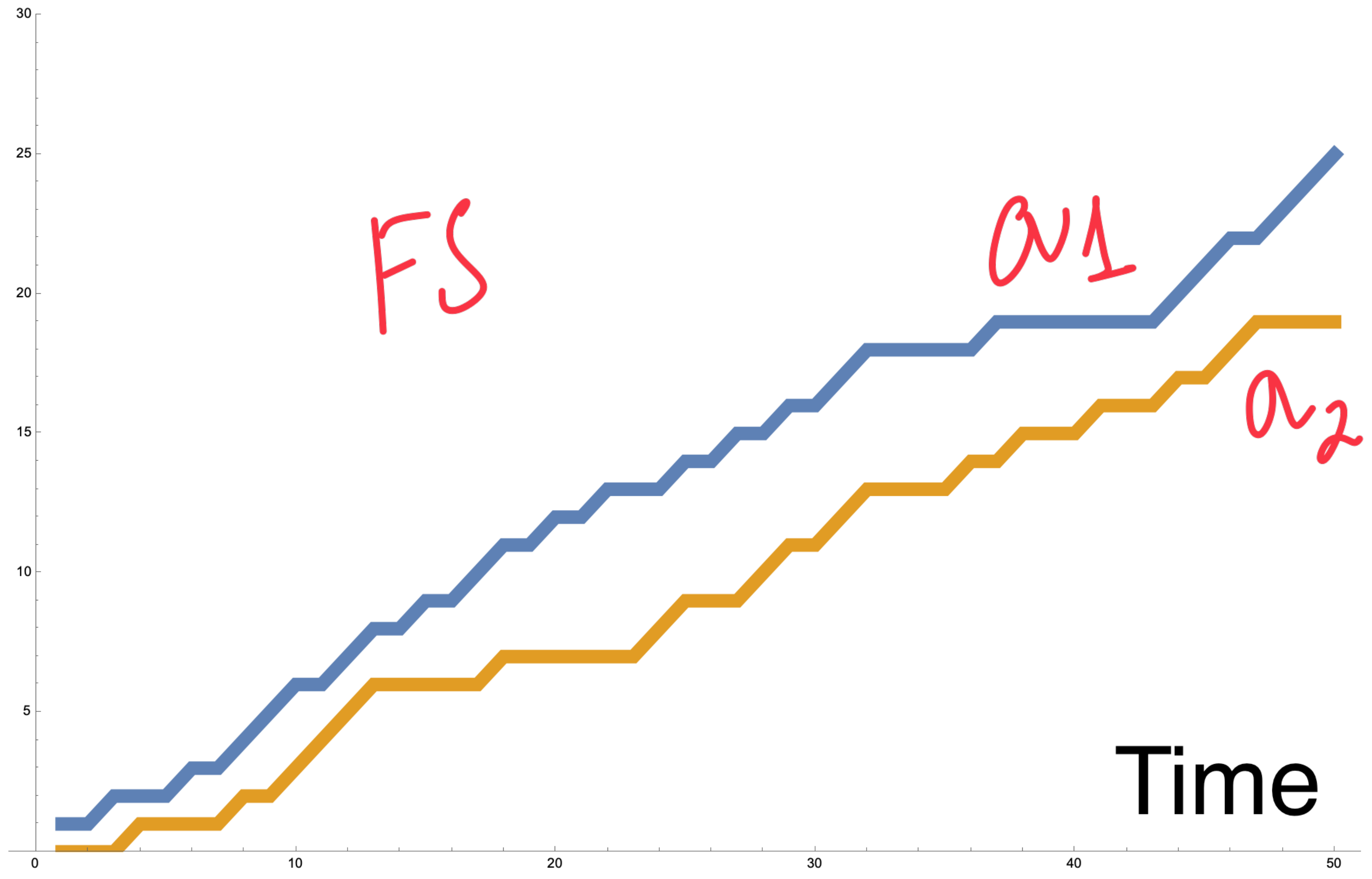
for the car behind (yellow) it **does not** depend on the order

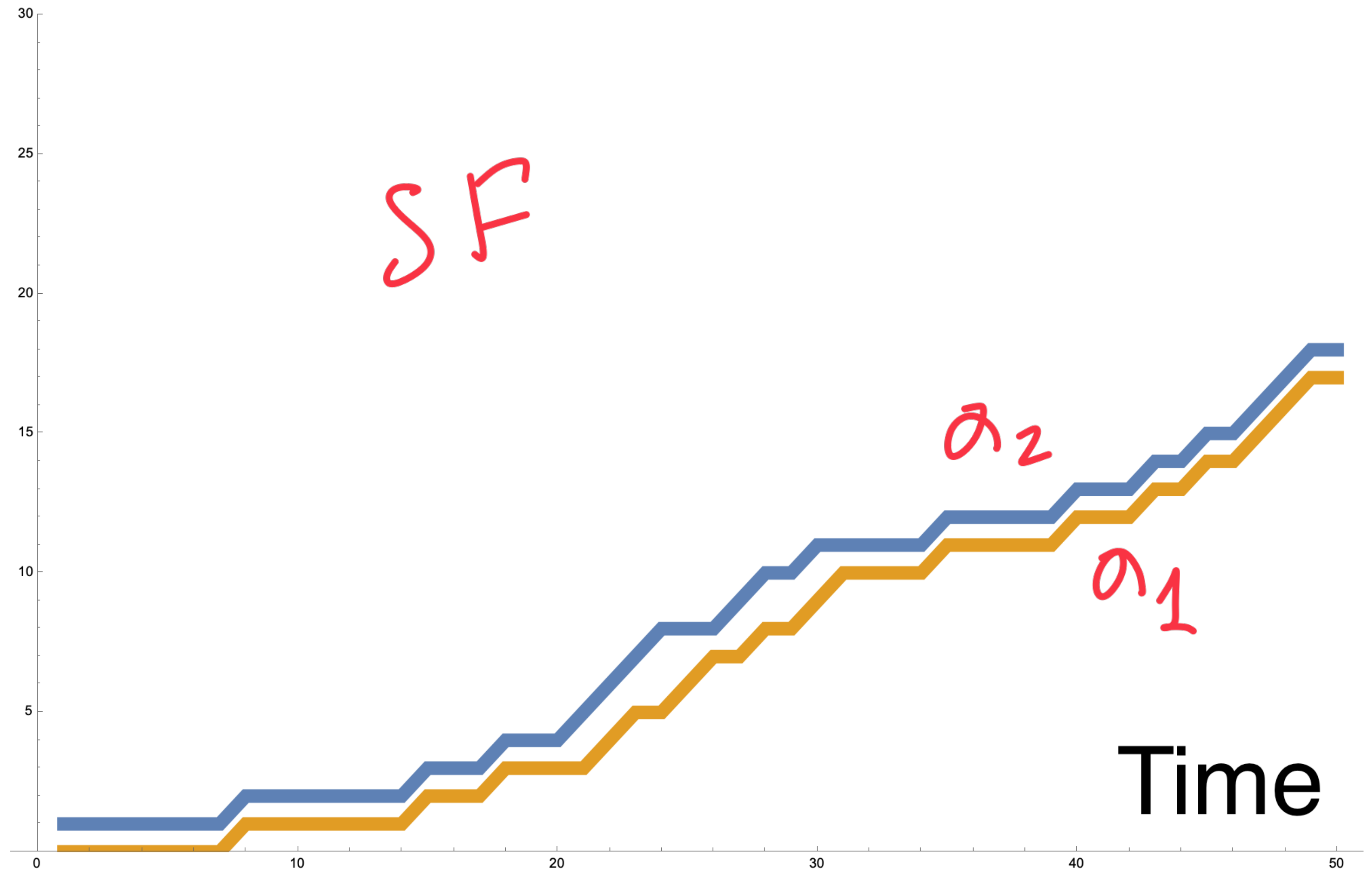
Two cars (discrete time TASEP with Bernoulli jumps)

Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1+a_i)$



One-lane highway: no passing. Consider two systems: FS and SF





The long-time speed of the car ahead (blue) depends on which car is first;

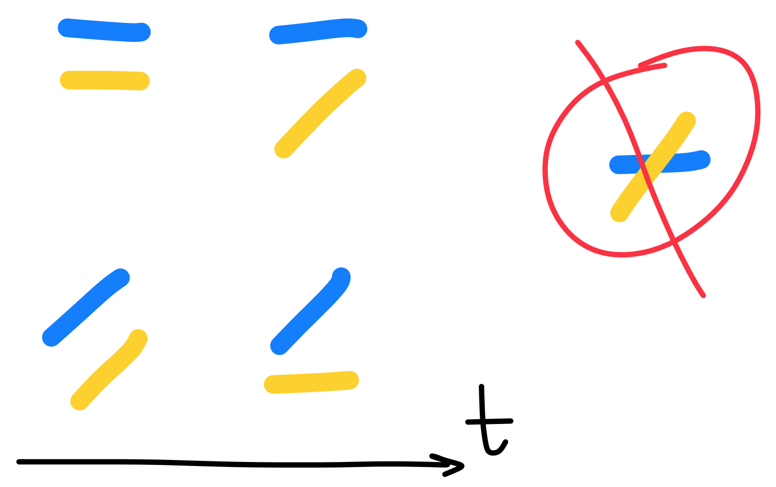
for the car behind (yellow) it **does not** depend on the order

Theorem. (Vershik-Kerov ~1981; O'Connell 2003)

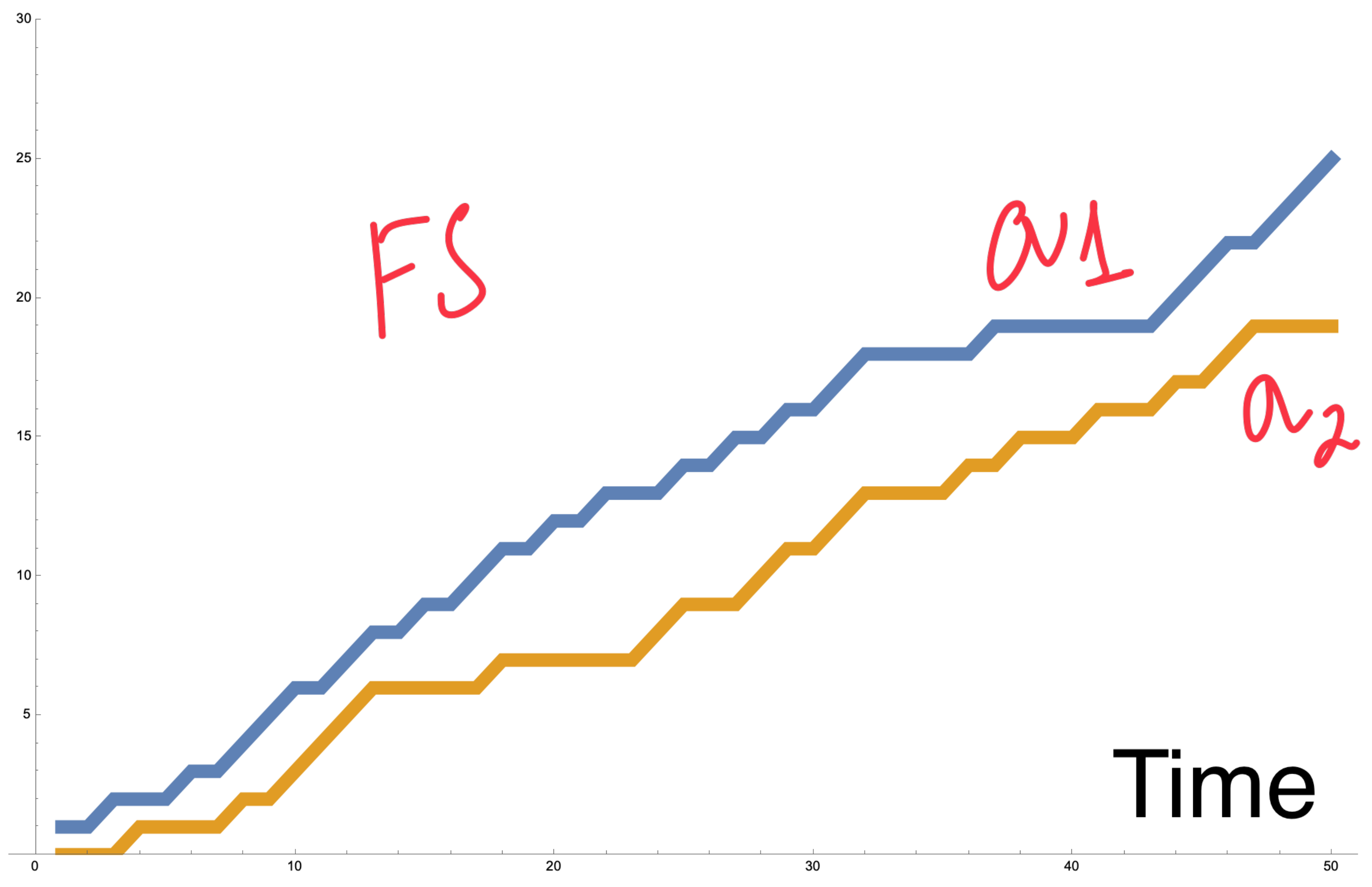
If the cars started at locations 0,1 (immediate neighbors; called *step initial configuration*), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

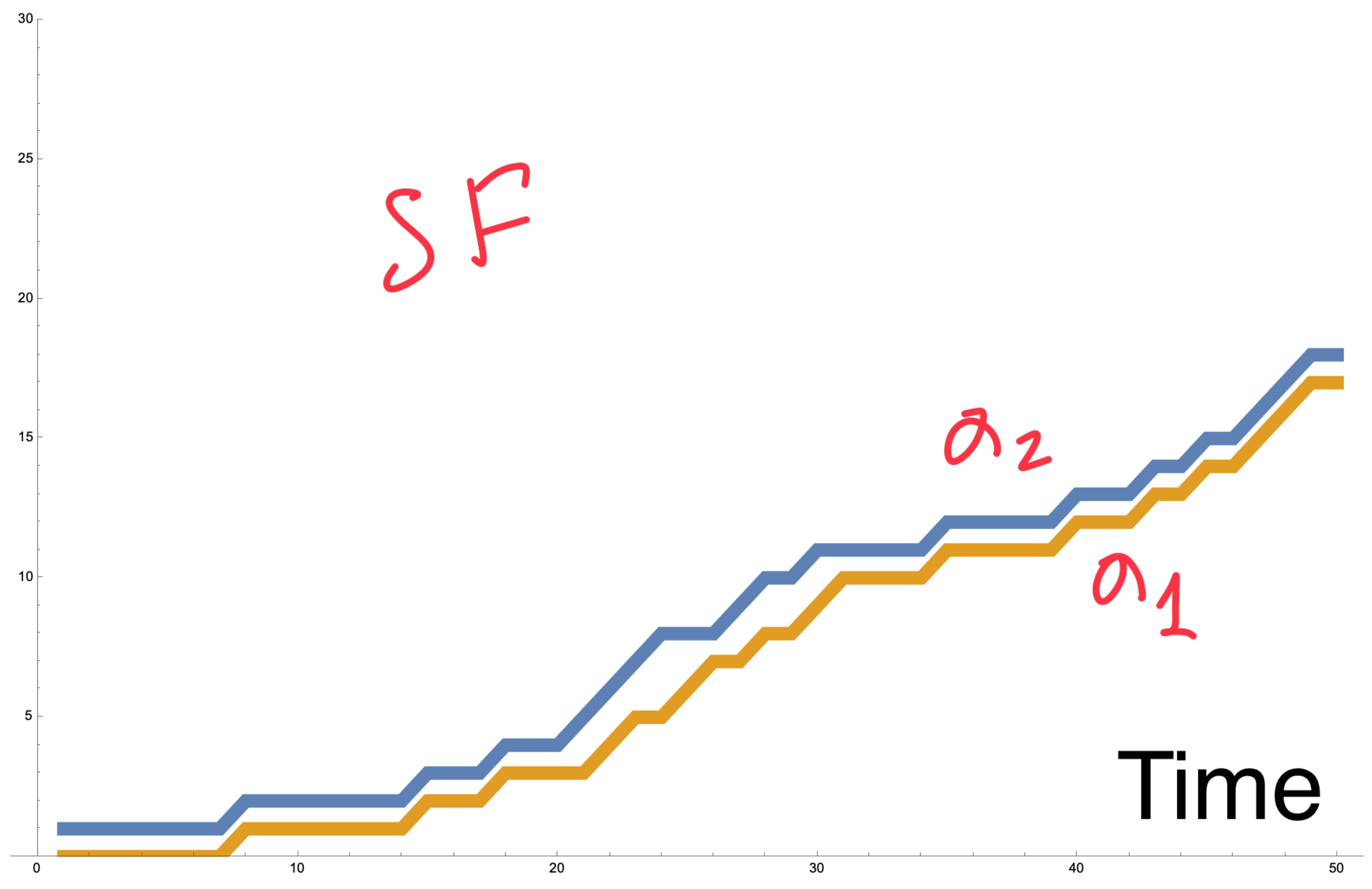
Two cars (discrete time TASEP with Bernoulli jumps)

Time $t \in \mathbb{Z}_{\geq 0}$, 2 cars with speeds $a_1 > a_2 > 0$, probabilities of jumps $a_i/(1 + a_i)$



One-lane highway: no passing. Consider two systems: FS and SF





The long-time speed of the car ahead (blue) depends on which car is first;

for the car behind (yellow) it **does not** depend on the order

Theorem. (Vershik-Kerov ~1981; O'Connell 2003)

If the cars started at locations 0,1 (immediate neighbors; called *step initial configuration*), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

follows from Robinson-Schensted-Knuth correspondence which encodes unused jumps

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem. If the cars started at locations 0,1 (immediate neighbors), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

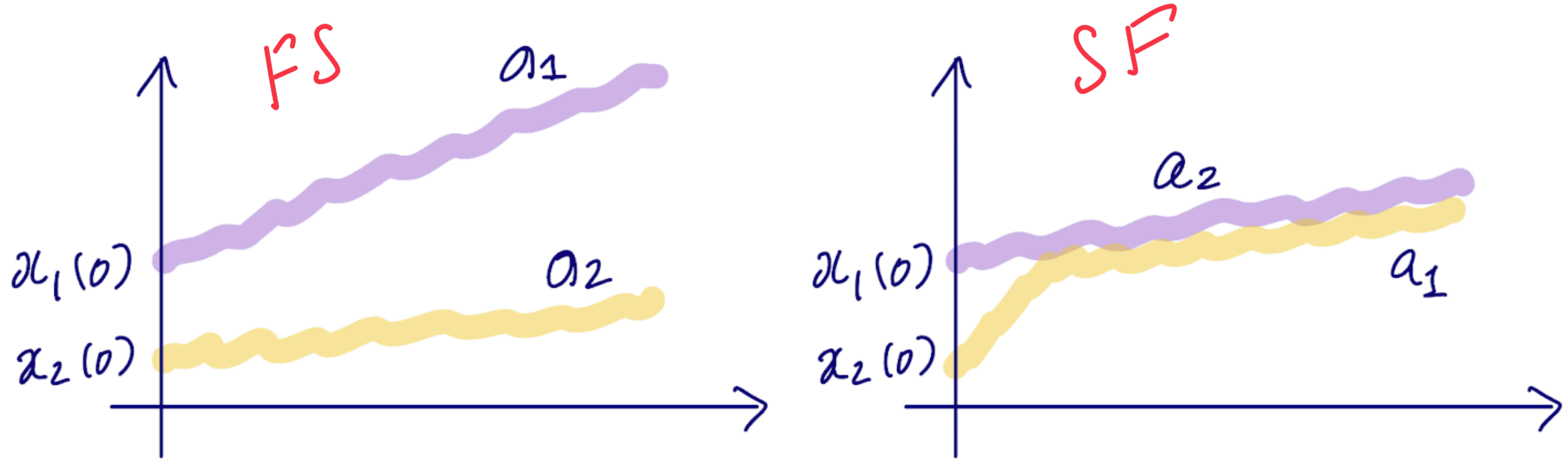
Theorem. If the cars started at locations 0,1 (immediate neighbors), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem. If the cars started at locations 0,1 (immediate neighbors), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

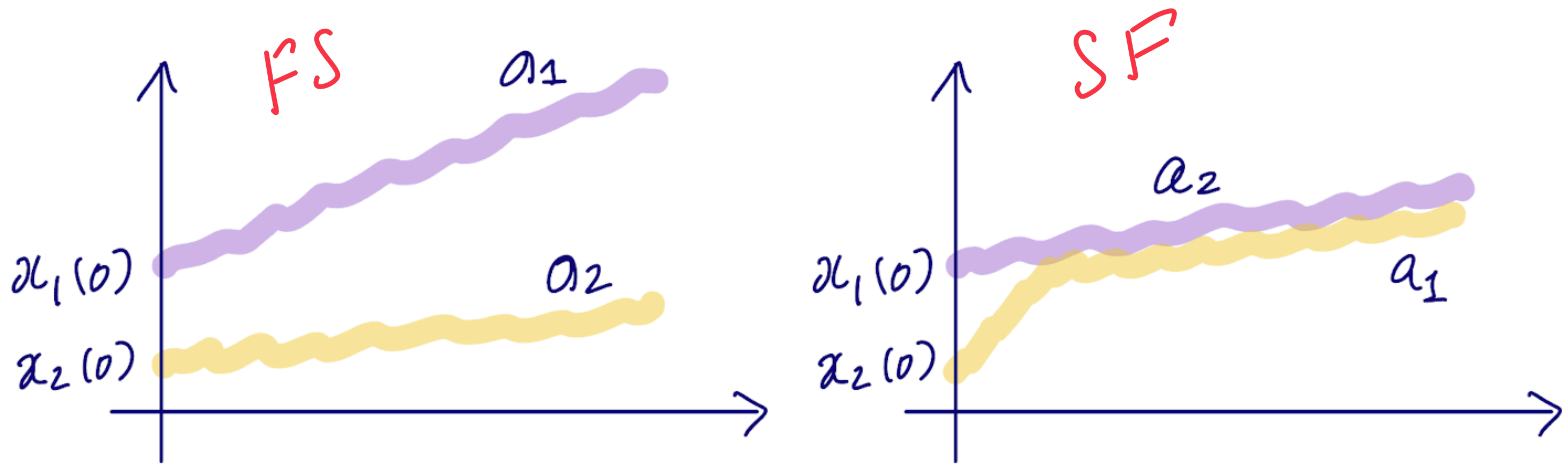
Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$



Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem. If the cars started at locations 0,1 (immediate neighbors), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$



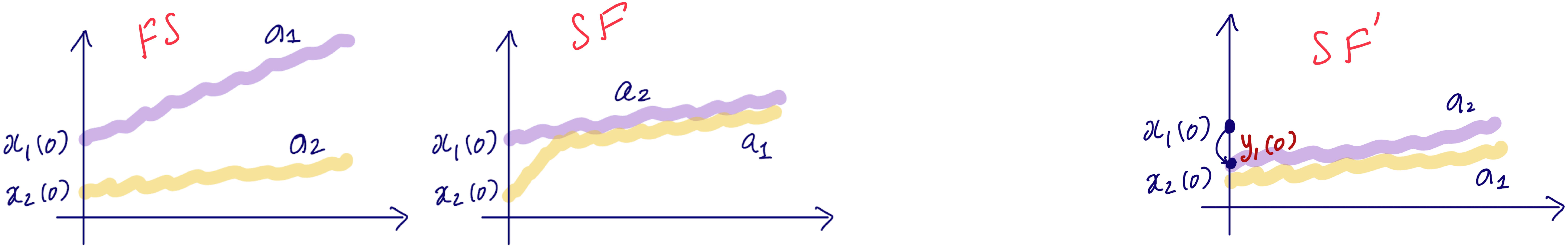
Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

Two cars (discrete time TASEP with Bernoulli jumps). Randomized initial conditions

Theorem. If the cars started at locations 0,1 (immediate neighbors), then the distribution of the trajectory of the car behind is **independent** of the order of the speeds

Theorem **fails** when cars are not initially neighbors, $x_1(0) - x_2(0) - 1 > 0$



Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

Two cars (discrete time TASEP with Bernoulli jumps). Intertwining

Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$.
Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

Two cars (discrete time TASEP with Bernoulli jumps). Intertwining

Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$.
Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

- B_{a_2/a_1} - Markov jump operator on configurations of two particles
- T_{a_1,a_2} - FS transition operator
- T_{a_2,a_1} - SF transition operator

Two cars (discrete time TASEP with Bernoulli jumps). Intertwining

Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

- B_{a_2/a_1} - Markov jump operator on configurations of two particles
- T_{a_1,a_2} - FS transition operator
- T_{a_2,a_1} - SF transition operator

Theorem (P.-Saenz 2022). We have the intertwining $T_{a_1,a_2}B_{a_2/a_1} = B_{a_2/a_1}T_{a_2,a_1}$ (T's can be iterated)

Two cars (discrete time TASEP with Bernoulli jumps). Intertwining

Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$.
Let SF' be the system started from $(y_1(0), x_2(0))$.

Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

- B_{a_2/a_1} - Markov jump operator on configurations of two particles
- T_{a_1,a_2} - FS transition operator
- T_{a_2,a_1} - SF transition operator

Theorem (P.-Saenz 2022). We have the intertwining $T_{a_1,a_2}B_{a_2/a_1} = B_{a_2/a_1}T_{a_2,a_1}$ (T's can be iterated)

When applied to $x_1(0) = 1,\ x_2(0) = 0$, the operator B_{a_2/a_1} acts as identity, so

$$\delta_{step}T_{a_1,a_2}B_{a_2/a_1} = \delta_{step}B_{a_2/a_1}T_{a_2,a_1} = \delta_{step}T_{a_2,a_1}$$

Two cars (discrete time TASEP with Bernoulli jumps). Intertwining

Theorem (P.-Saenz 2022). Define $y_1(0) = x_2(0) + 1 + \min(G, x_1(0) - x_2(0) - 1)$, where $G \in \mathbb{Z}_{\geq 0}$ is an independent geometric random variable with $P(G = k) = (a_2/a_1)^k(1 - a_2/a_1)$. Let SF' be the system started from $(y_1(0), x_2(0))$.

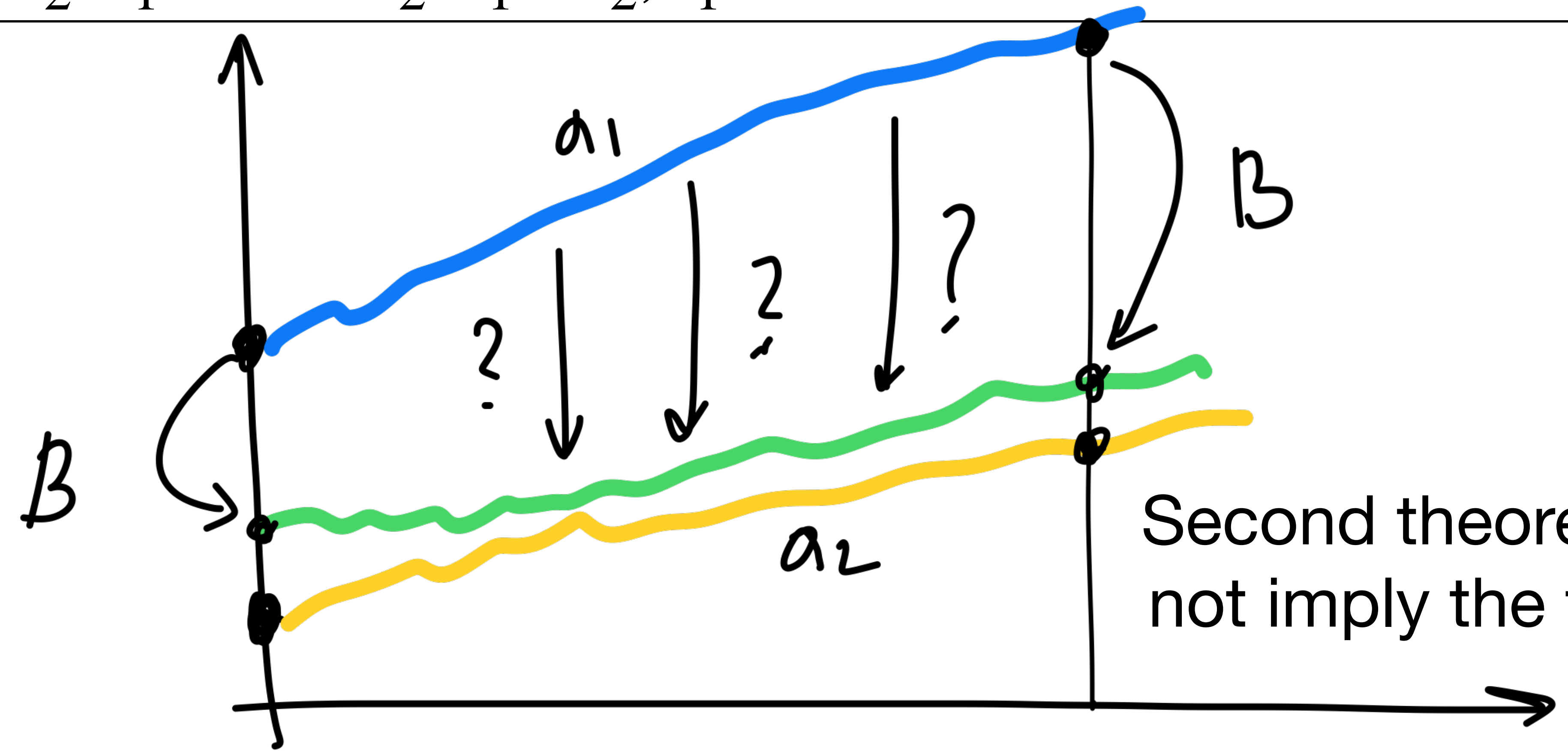
Then the distributions of the trajectories of the second particle in FS and SF' are **the same**.

- B_{a_2/a_1} - Markov jump operator on configurations of two particles
- T_{a_1,a_2} - FS transition operator
- T_{a_2,a_1} - SF transition operator

Theorem (P.-Saenz 2022). We have the intertwining $T_{a_1,a_2}B_{a_2/a_1} = B_{a_2/a_1}T_{a_2,a_1}$ (T's can be iterated)

When applied to $x_1(0) = 1$, $x_2(0) = 0$, the operator B_{a_2/a_1} acts as identity, so

$$\delta_{step}T_{a_1,a_2}B_{a_2/a_1} = \delta_{step}B_{a_2/a_1}T_{a_2,a_1} = \delta_{step}T_{a_2,a_1}$$



Second theorem does not imply the first one

For questions and remarks

Proof via Yang-Baxter equation

Jumps and Yang-Baxter equation

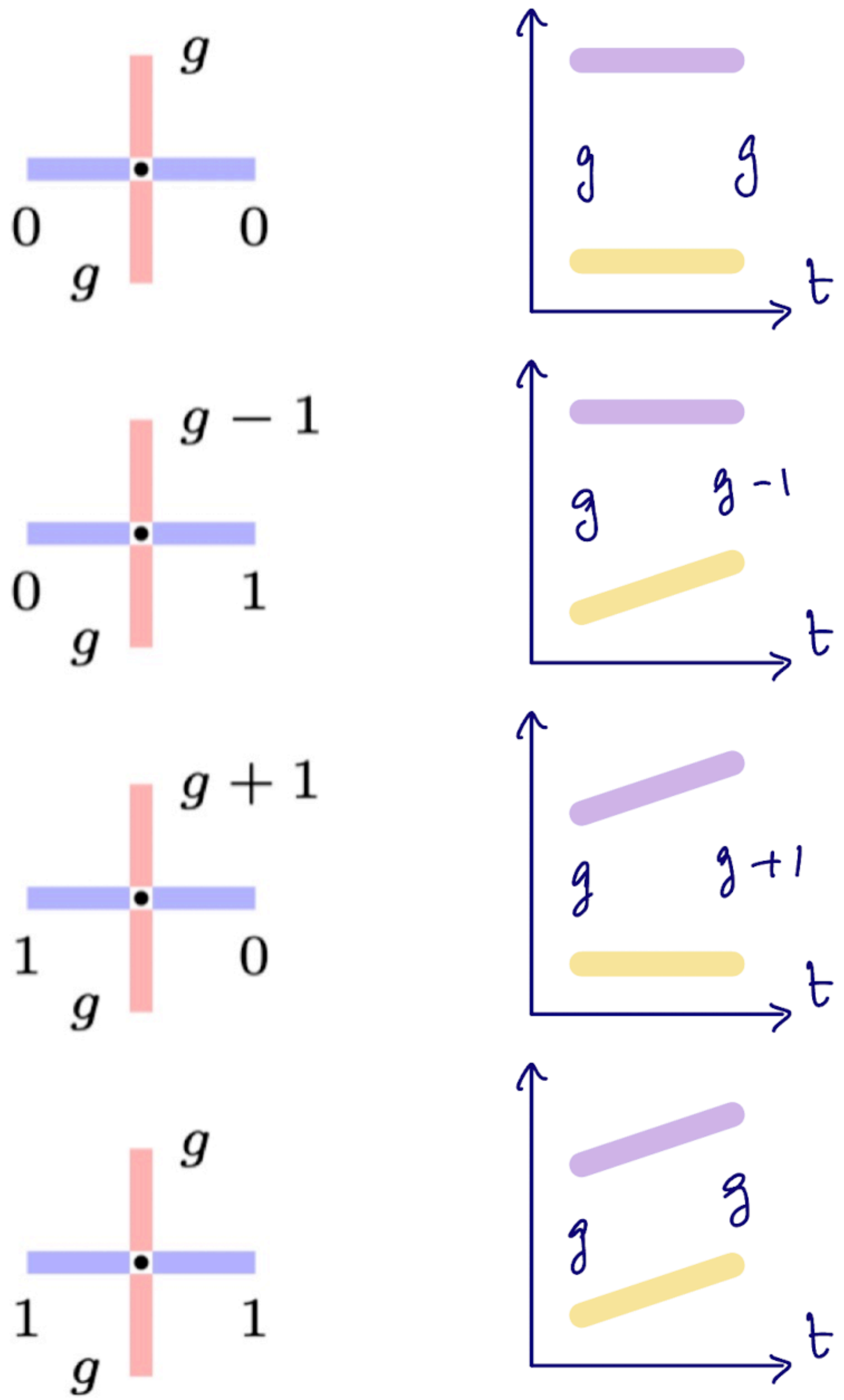
Translate into the “vertex model” language. In vertex models, time runs **up**.

$g = x_1 - x_2 - 1$ is the **gap**

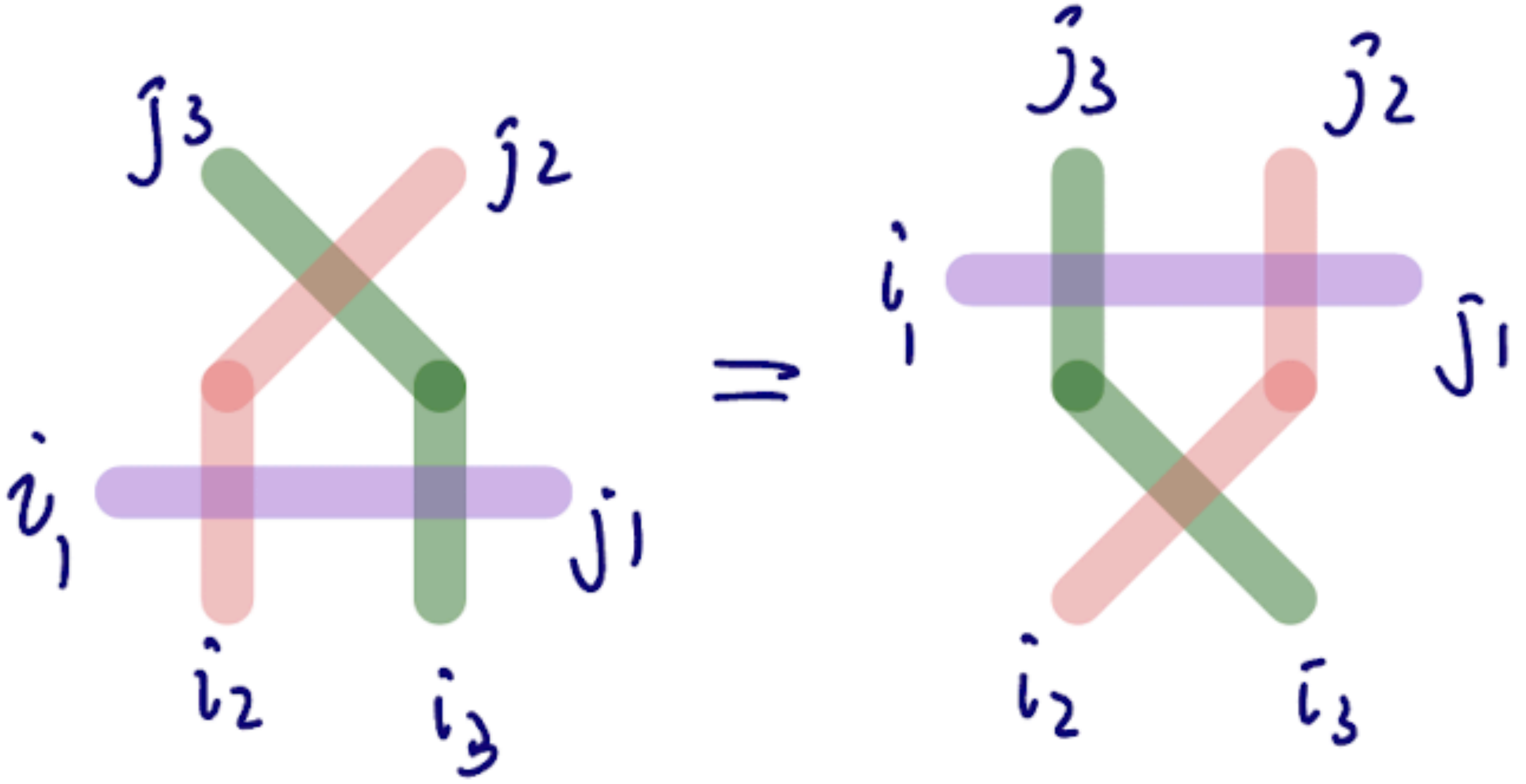
Jumps and Yang-Baxter equation

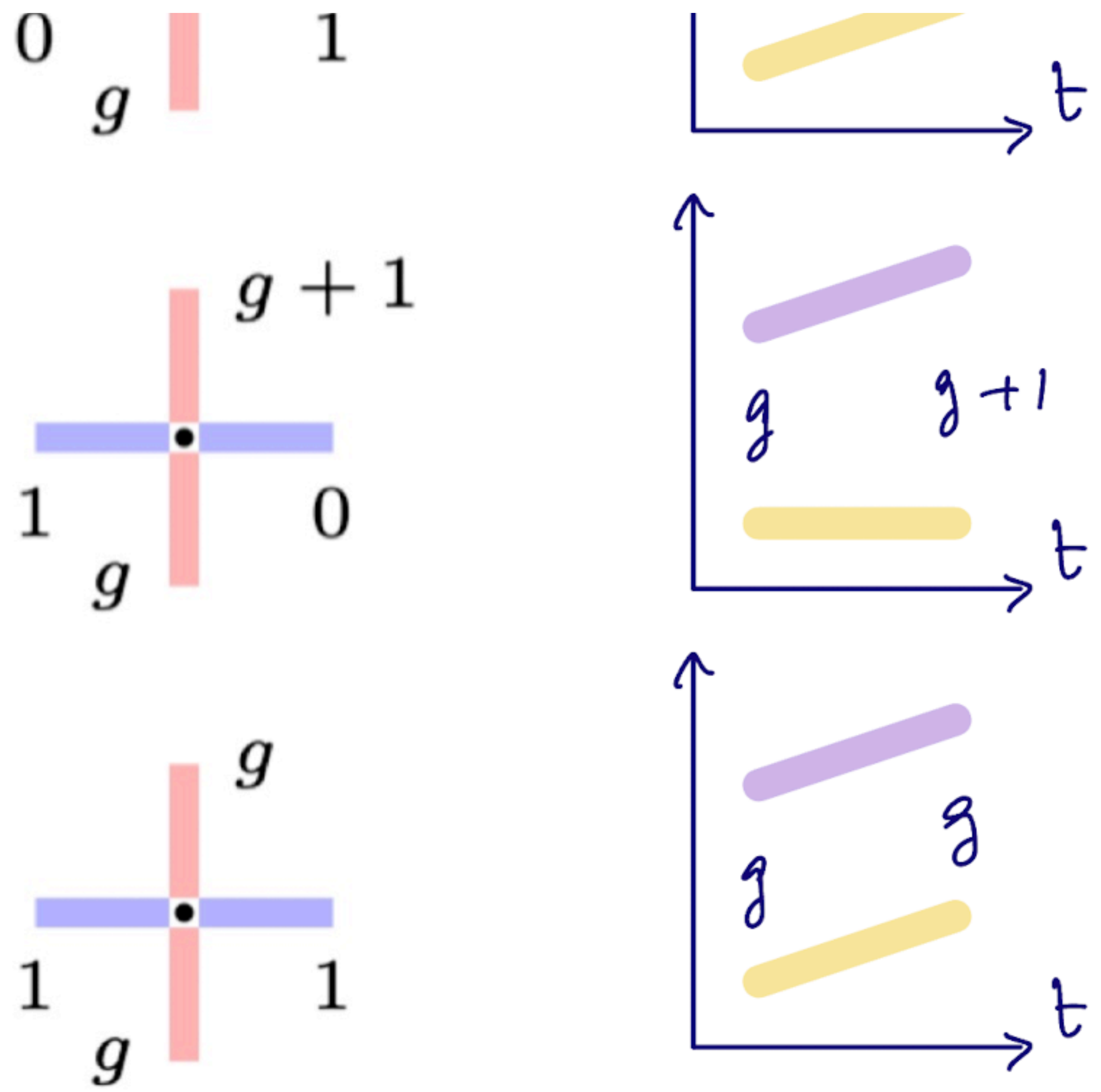
Translate into the "vertex model" language. In vertex models, time runs **up**.

$g = x_1 - x_2 - 1$ is the **gap**

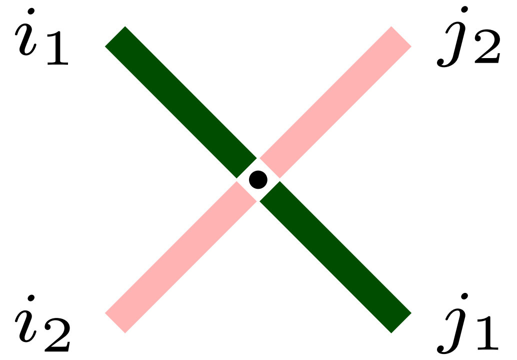


Jumps and Yang-Baxter equation

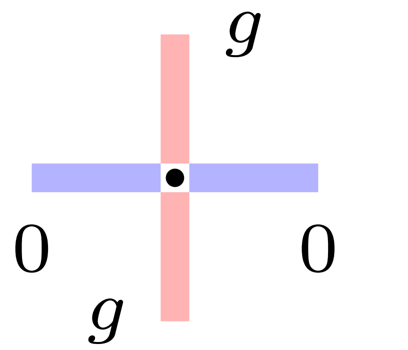




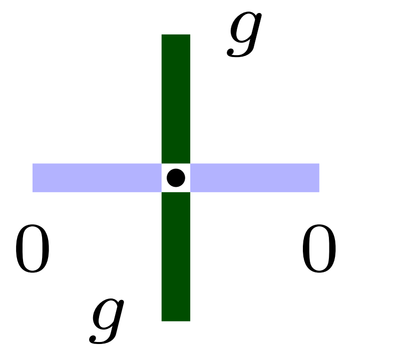
Theorem. Vertex weights satisfy the **Yang-Baxter equation**



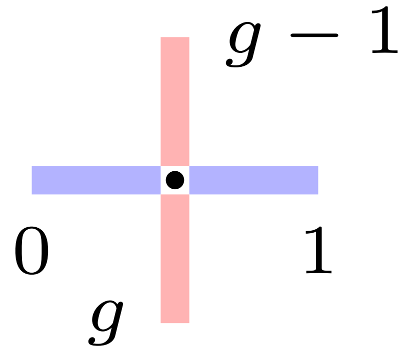
$$(a_2/a_1)^{j_2}(1-\mathbf{1}_{j_2<j_1}\,a_2/a_1)\,\mathbf{1}_{j_2\le j_1}$$



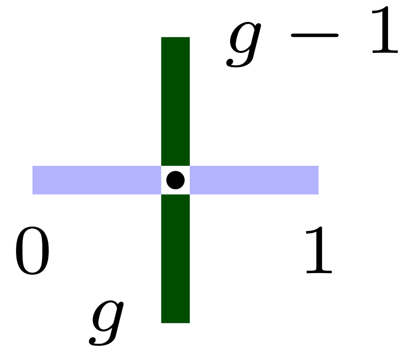
$$\frac{1}{1+a_1}$$



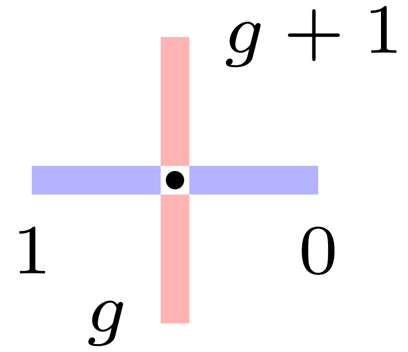
$$\frac{1}{1+a_2}$$



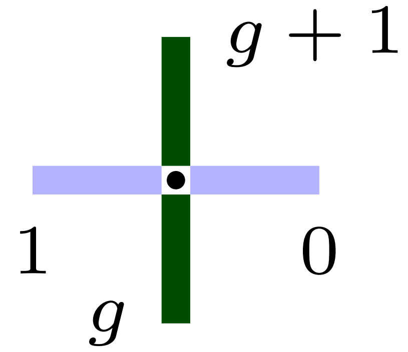
$$\frac{a_1}{1+a_1}$$



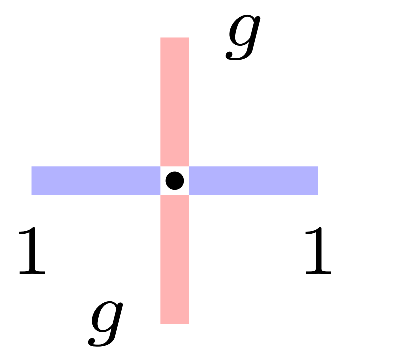
$$\frac{a_2}{1+a_2}$$



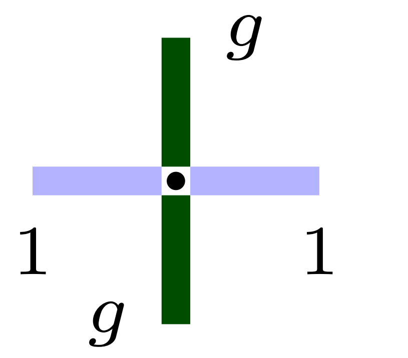
$$\frac{1}{1+a_1}$$



$$\frac{1}{1+a_2}$$

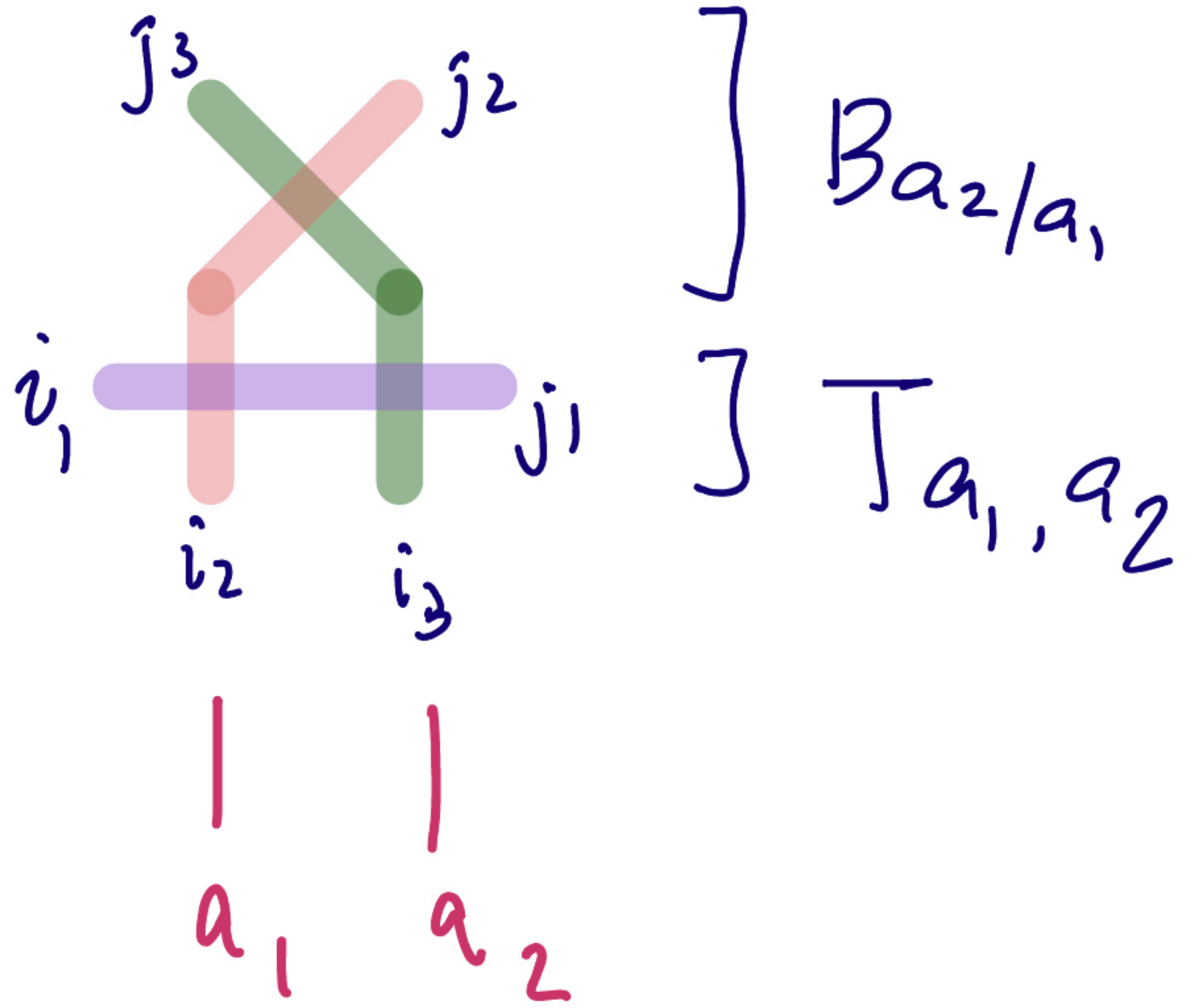


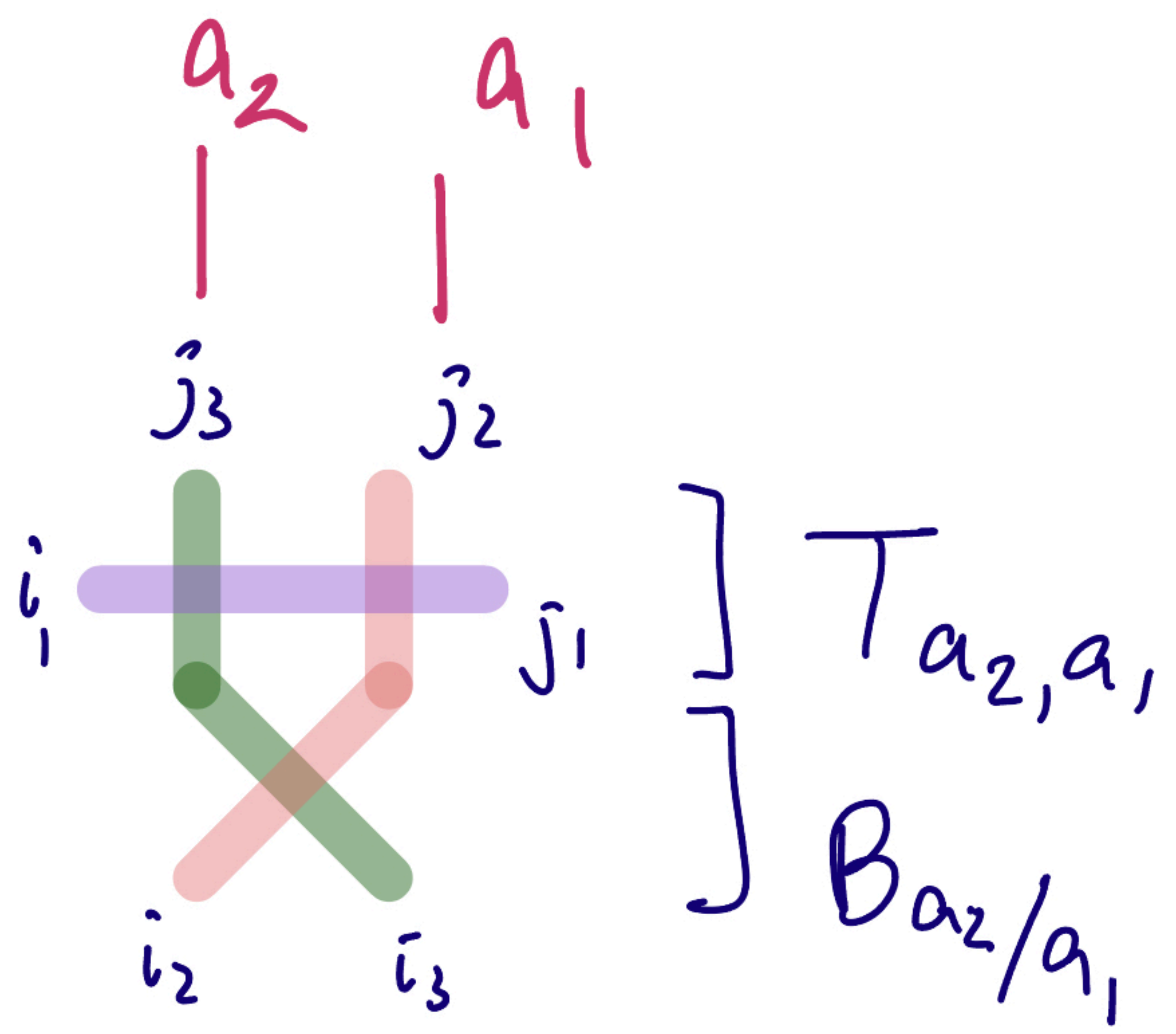
$$\frac{a_1}{1+a_1}$$



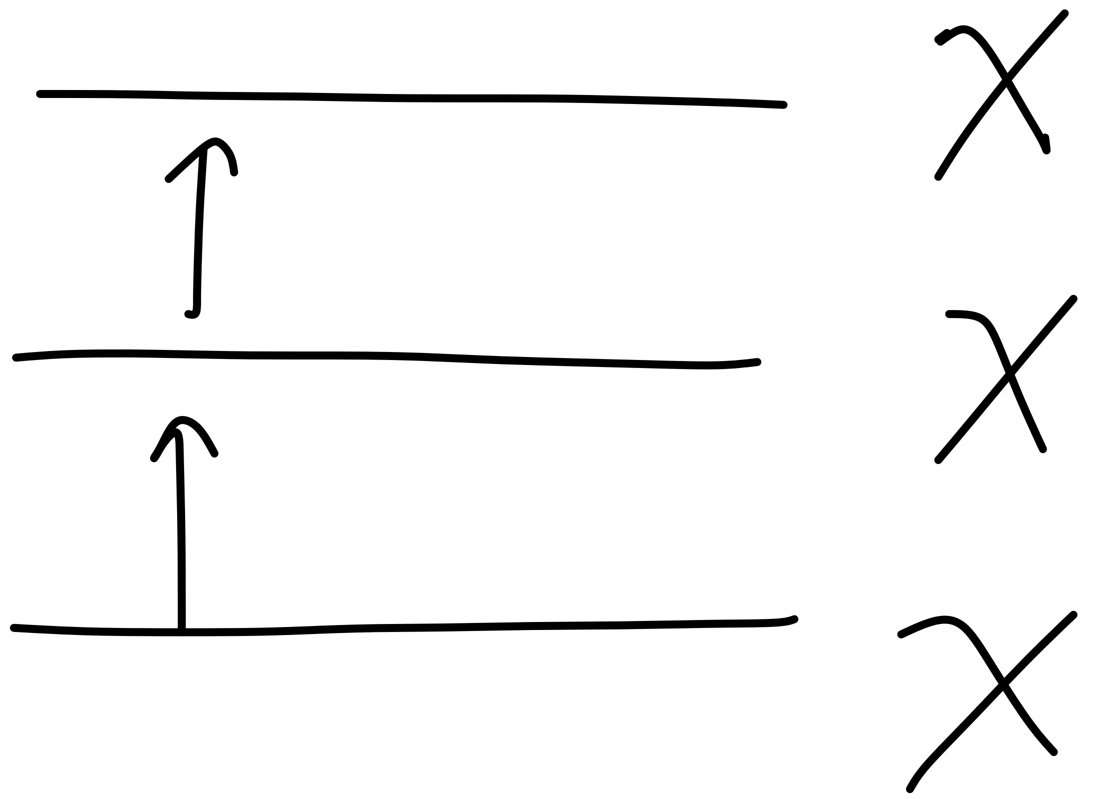
$$\frac{a_2}{1+a_2}$$

Intertwining relation





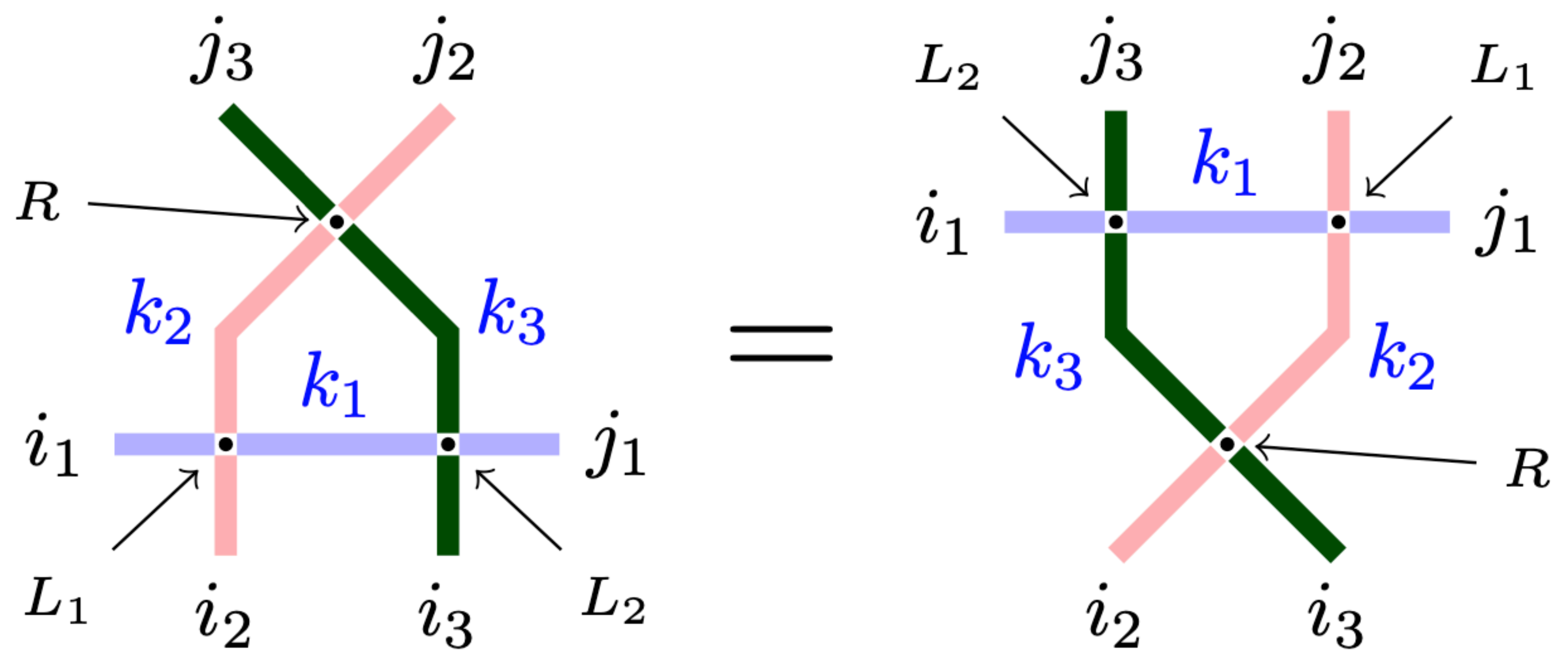
X - space of particle configurations



- B_{a_2/a_1} - Markov jump operator on configurations of two particles
- T_{a_1,a_2} - FS transition operator
- T_{a_2,a_1} - SF transition operator

Theorem (P.-Saenz 2022). We have the intertwining $T_{a_1,a_2}B_{a_2/a_1} = B_{a_2/a_1}T_{a_2,a_1}$

Intertwining relation for general stochastic R matrices



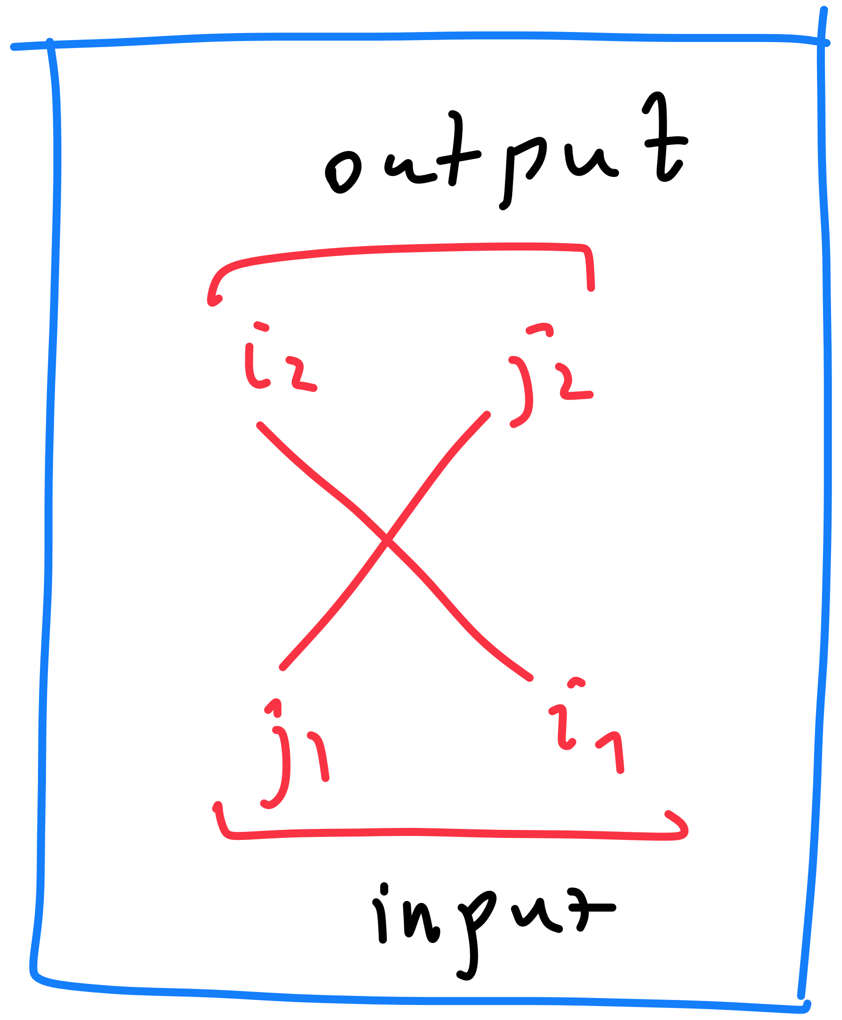
$$L^{(1)}_{u,s}(g,0;g,0) = \frac{1-q^g su}{1-su}, \qquad L^{(1)}_{u,s}(g,0;g-1,1) = \frac{-su(1-q^g)}{1-su};$$

$$L^{(1)}_{u,s}(g,1;g,1) = \frac{-su+q^g s^2}{1-su}, \qquad L^{(1)}_{u,s}(g,1;g+1,0) = \frac{1-q^g s^2}{1-su}.$$

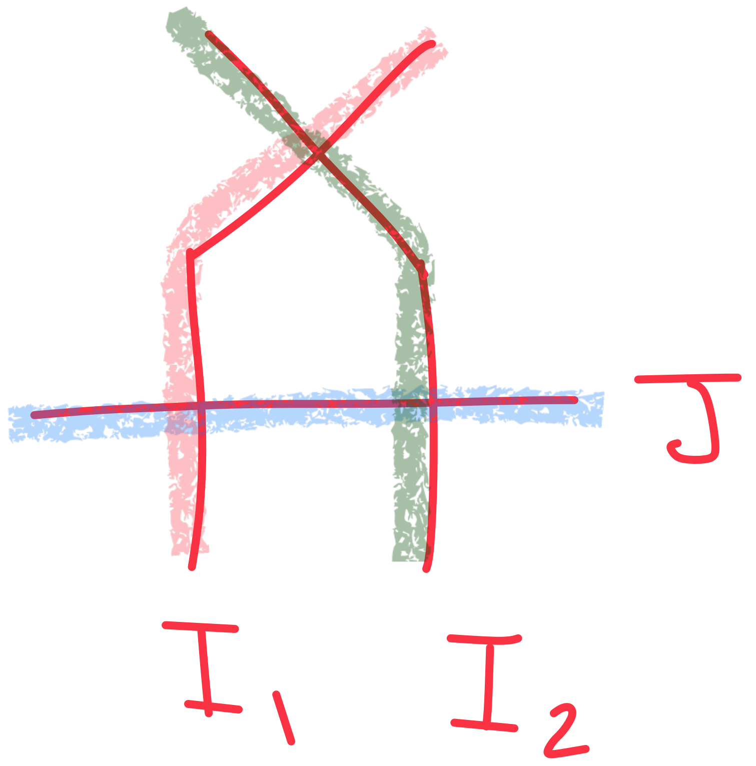
$L_i = L^{(J)}_{u_i, s_i}$ (fused)

$R = R_{u_2/u_1, s_1, s_2} = L^{(I_1)}_{\frac{s_1 u_2}{u_1}, s_2}$

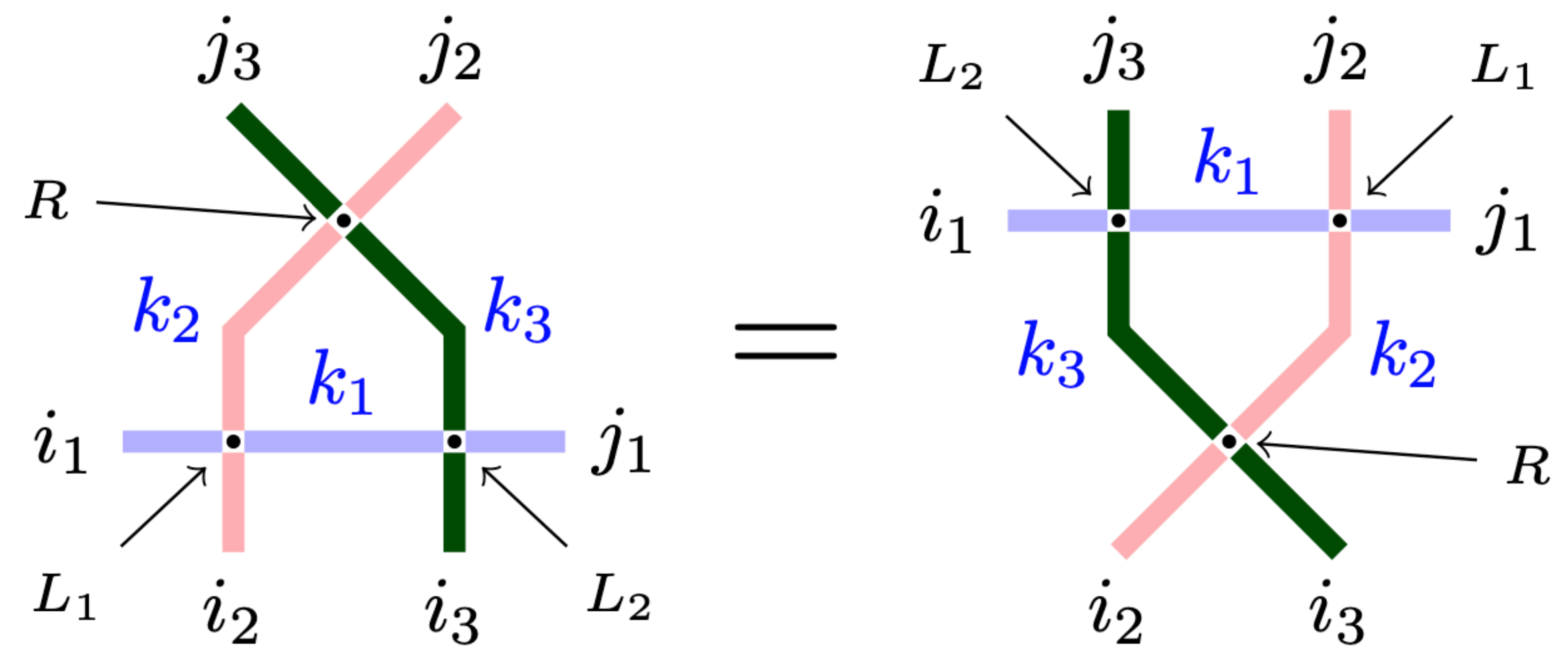
Stochastic:



$s_1 = q^{-I_1/2}$ (fusion)

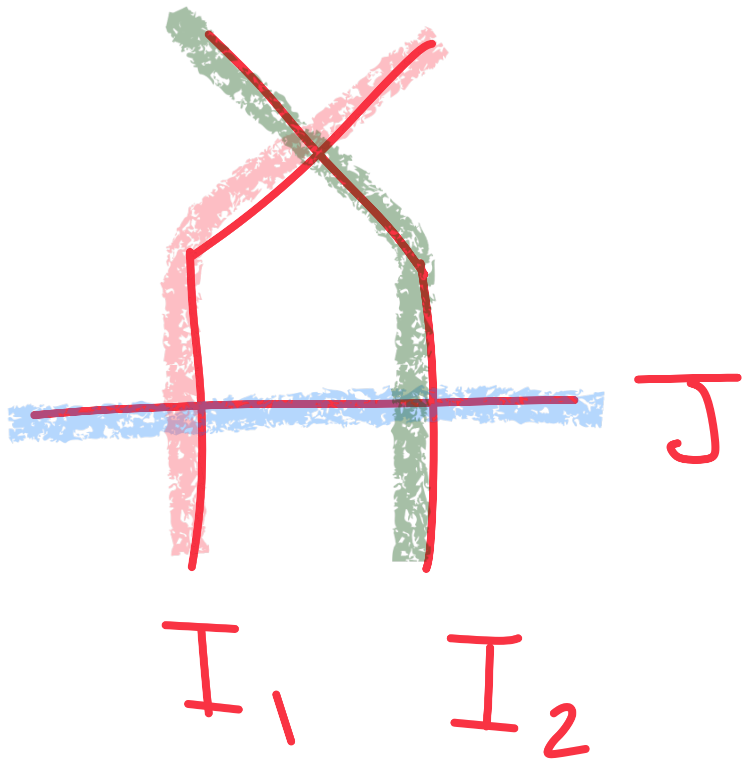


Intertwining relation for general stochastic R matrices



$$L^{(1)}_{u,s}(g,0;g,0) = \frac{1-q^g su}{1-su}, \qquad L^{(1)}_{u,s}(g,0;g-1,1) = \frac{-su(1-q^g)}{1-su};$$

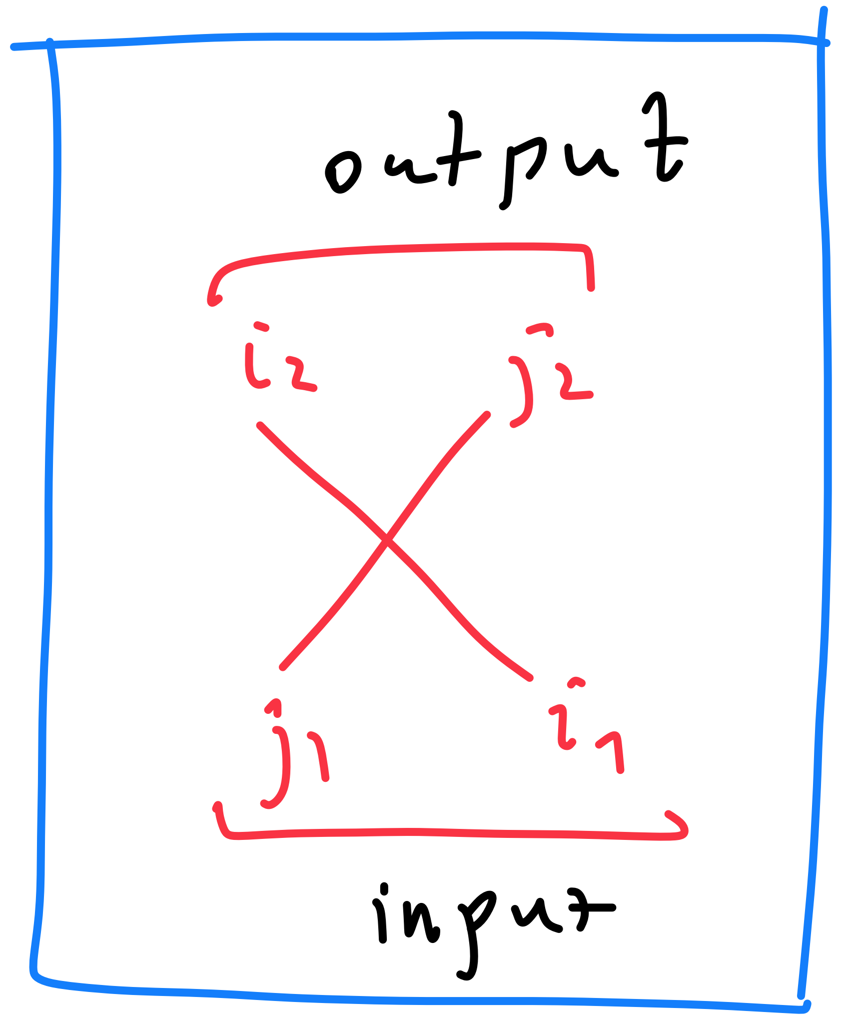
$$L^{(1)}_{u,s}(g,1;g,1) = \frac{-su+q^g s^2}{1-su}, \qquad L^{(1)}_{u,s}(g,1;g+1,0) = \frac{1-q^g s^2}{1-su}.$$



$L_i = L^{(J)}_{u_i, s_i}$ (fused)

$R = R_{u_2/u_1, s_1, s_2} = L^{(I_1)}_{\frac{s_1 u_2}{u_1}, s_2}$

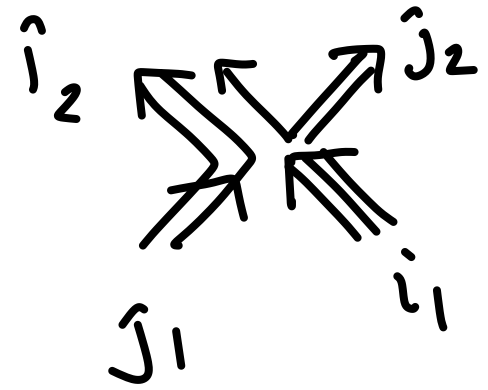
Stochastic :



$s_1 = q^{-I_1/2}$ (fusion)

q-Hahn (monotone)

$u_2/u_1 = s_2/s_1$ $\Rightarrow$ specialization

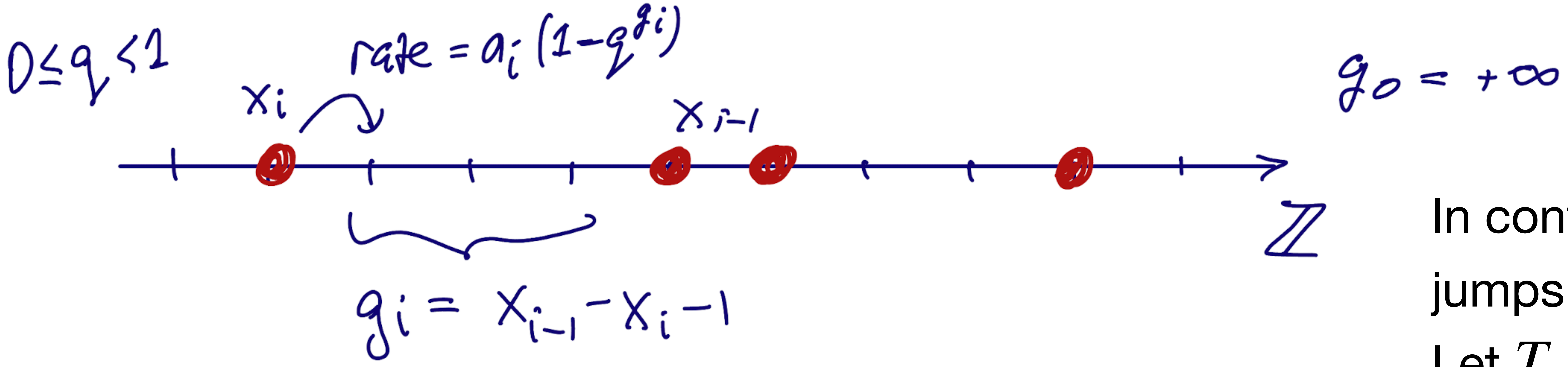


$j_2 \le i_1$

For questions and remarks

General intertwining and Lax equations

Intertwining for the q-TASEP in continuous time

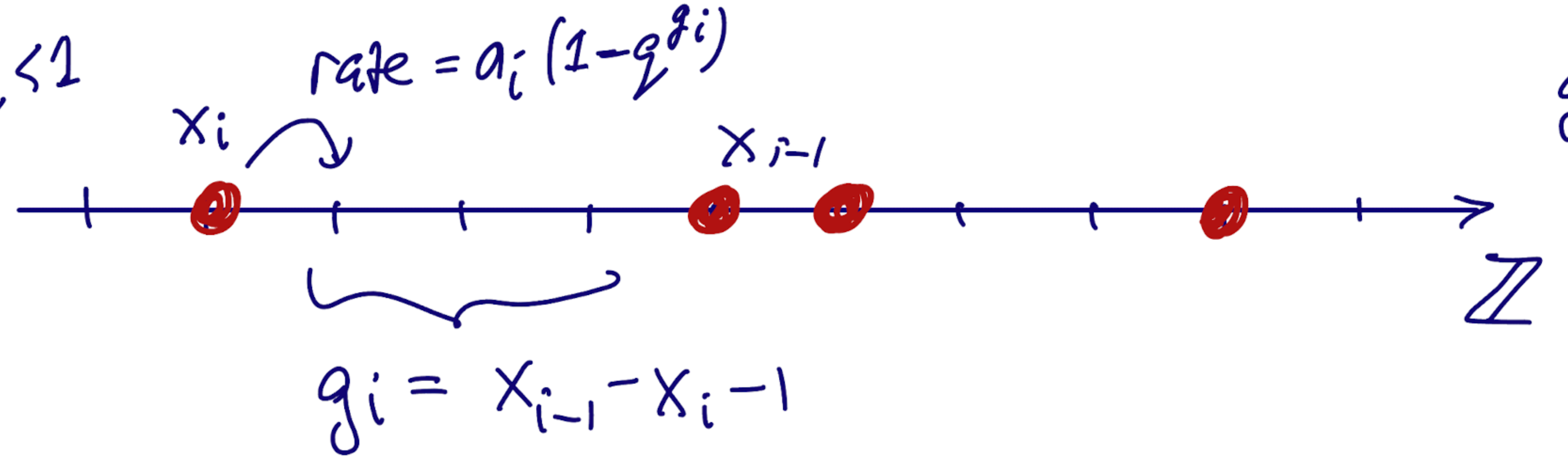


In continuous time, each particle x_i jumps forward at rate $a_i(1 - q^{gap_i})$.

Let $T_{\mathbf{a}}(t)$ be the q-TASEP semigroup

Intertwining for the q-TASEP in continuous time

$0 \le q < 1$

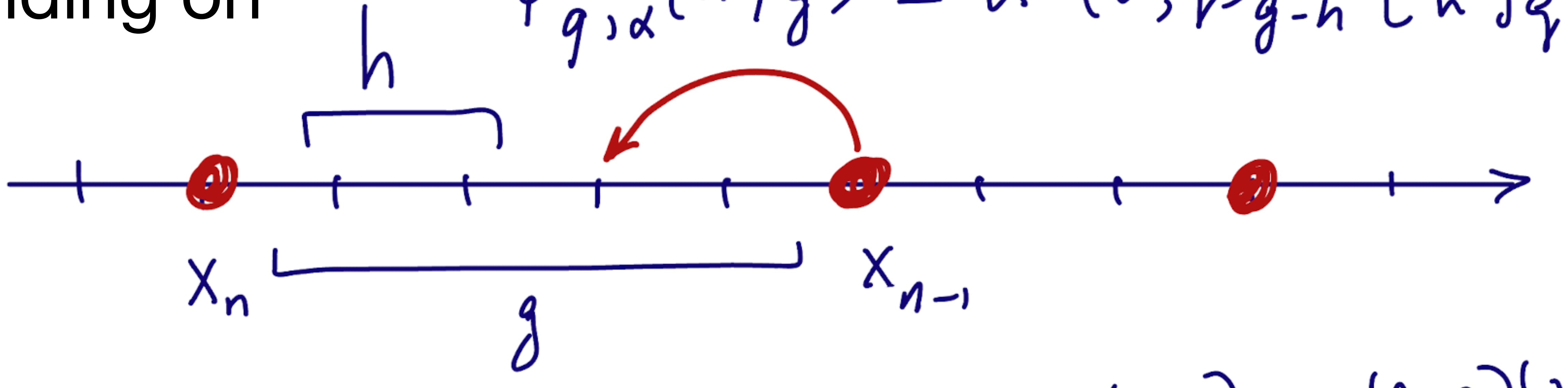


$g_0 = +\infty$

In continuous time, each particle x_i jumps forward at rate $a_i(1 - q^{gap_i})$. Let $T_{\mathbf{a}}(t)$ be the q-TASEP semigroup

Let B_{n-1} be the operator of randomly moving particle x_{n-1} back closer to x_n, depending on $\alpha = a_n/a_{n-1} < 1$.

$$\varphi_{q,\alpha}(h|g) = \alpha^{j}(\alpha;q)_{g-h} \begin{bmatrix} g \\ h \end{bmatrix}_q$$

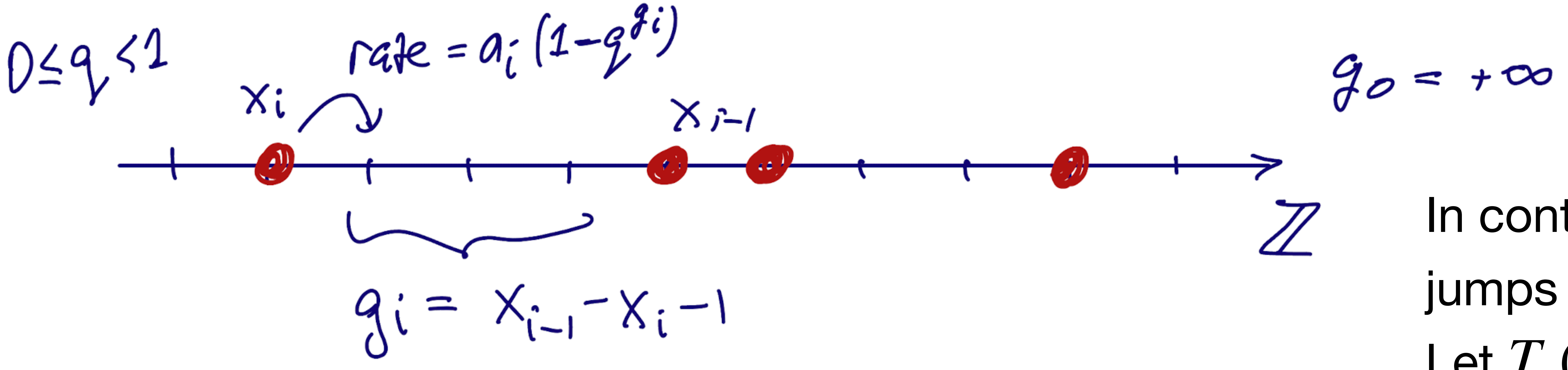


$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

$$\alpha = a_n/a_{n-1} < 1$$

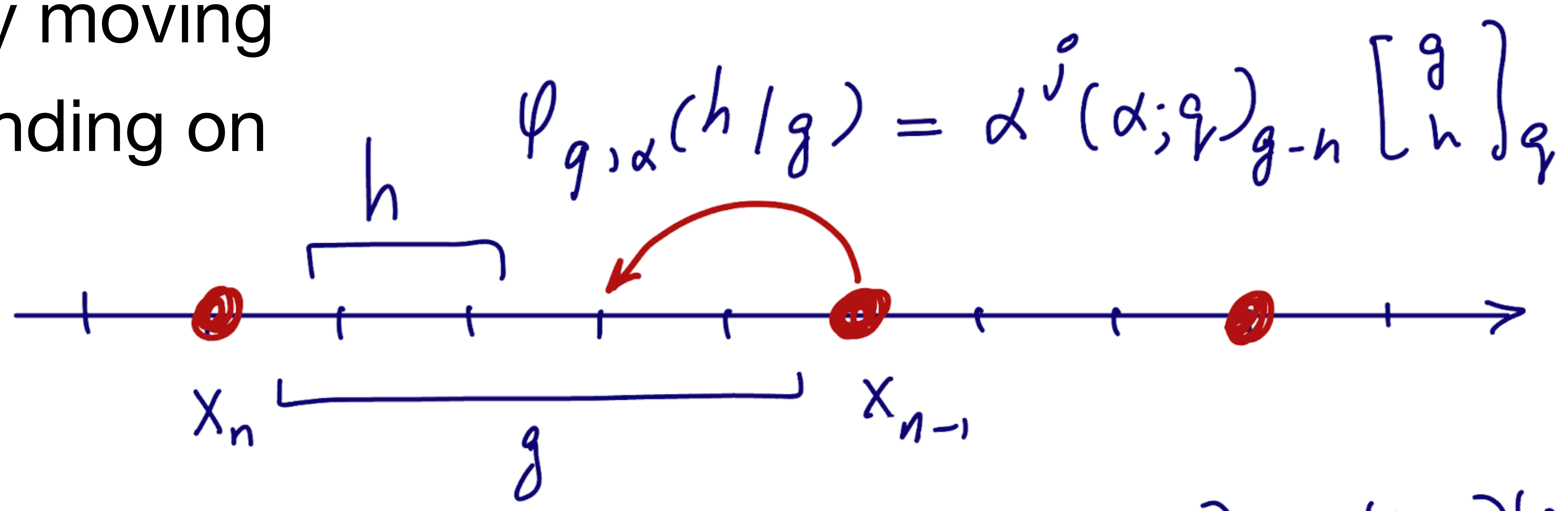
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$$

Intertwining for the q-TASEP in continuous time



In continuous time, each particle x_i jumps forward at rate $a_i(1 - q^{gap_i})$. Let $T_{\mathbf{a}}(t)$ be the q-TASEP semigroup

Let B_{n-1} be the operator of randomly moving particle x_{n-1} back closer to x_n, depending on $\alpha = a_n / a_{n-1} < 1$.



$(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$

$\alpha = a_n / a_{n-1} < 1$

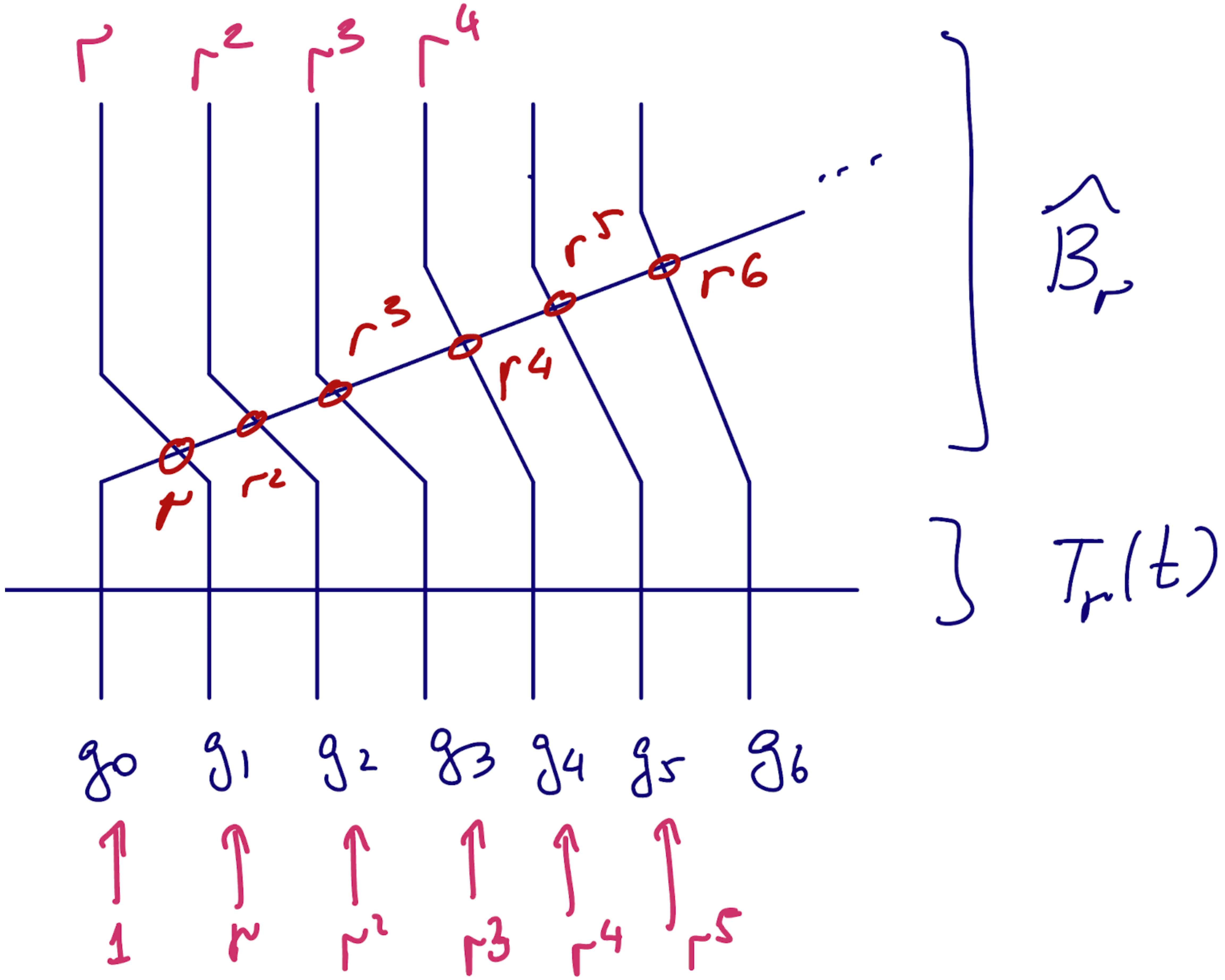
$\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$

Theorem (P.-Saenz 2022). We have the intertwining $T_{\mathbf{a}}(t) B_{n-1} = B_{n-1} T_{\sigma_{n-1}\mathbf{a}}(t)$, where σ swaps $a_{n-1} \leftrightarrow a_n$.

Application to densely packed configurations: [P.-Saenz 2019], [P. 2019]

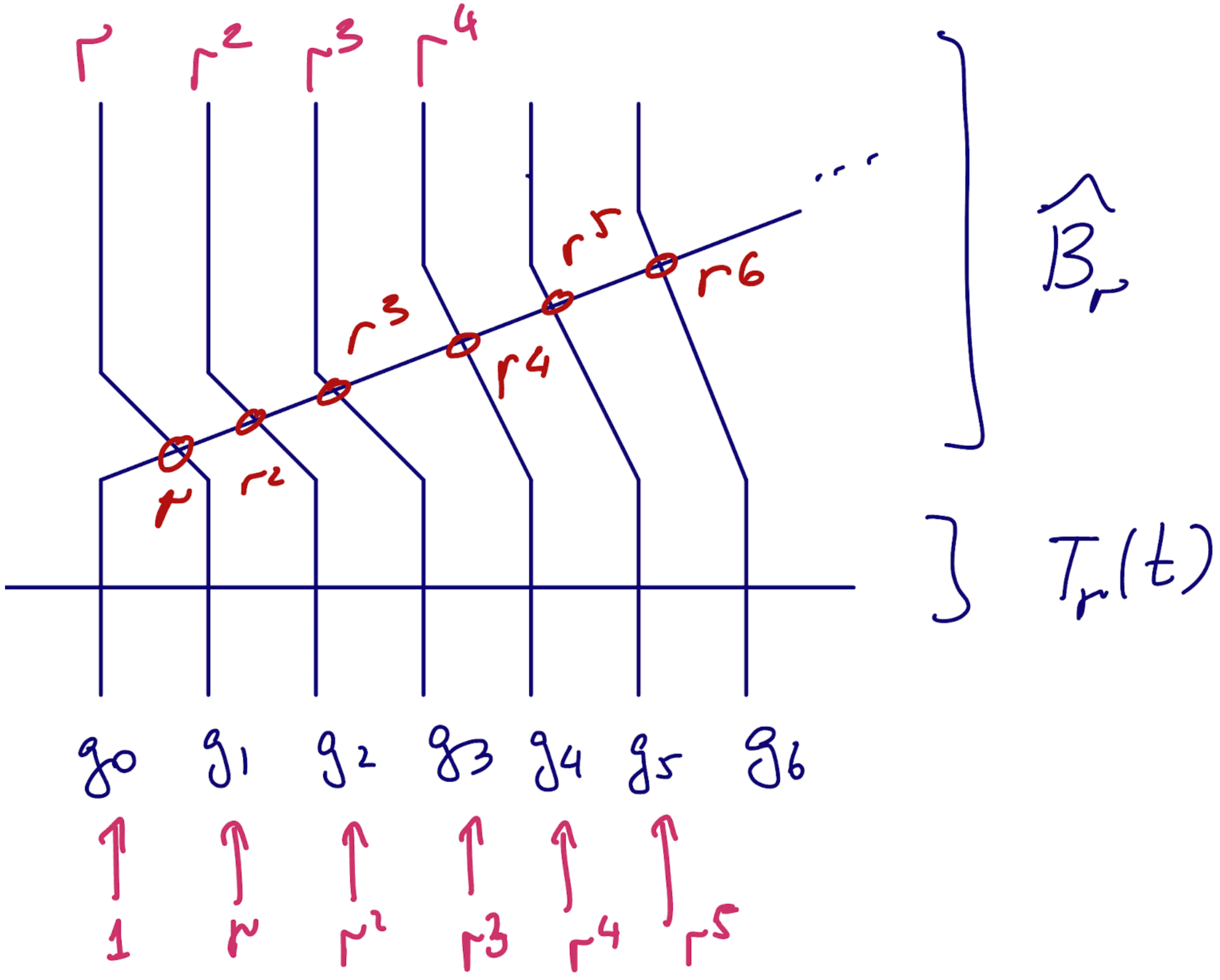
From intertwining to Lax equations for q-TASEP

Let $a_n = r^n$. Let us organize the tower of applications of $B_1, B_2, B_3, \ldots$



From intertwining to Lax equations for q-TASEP

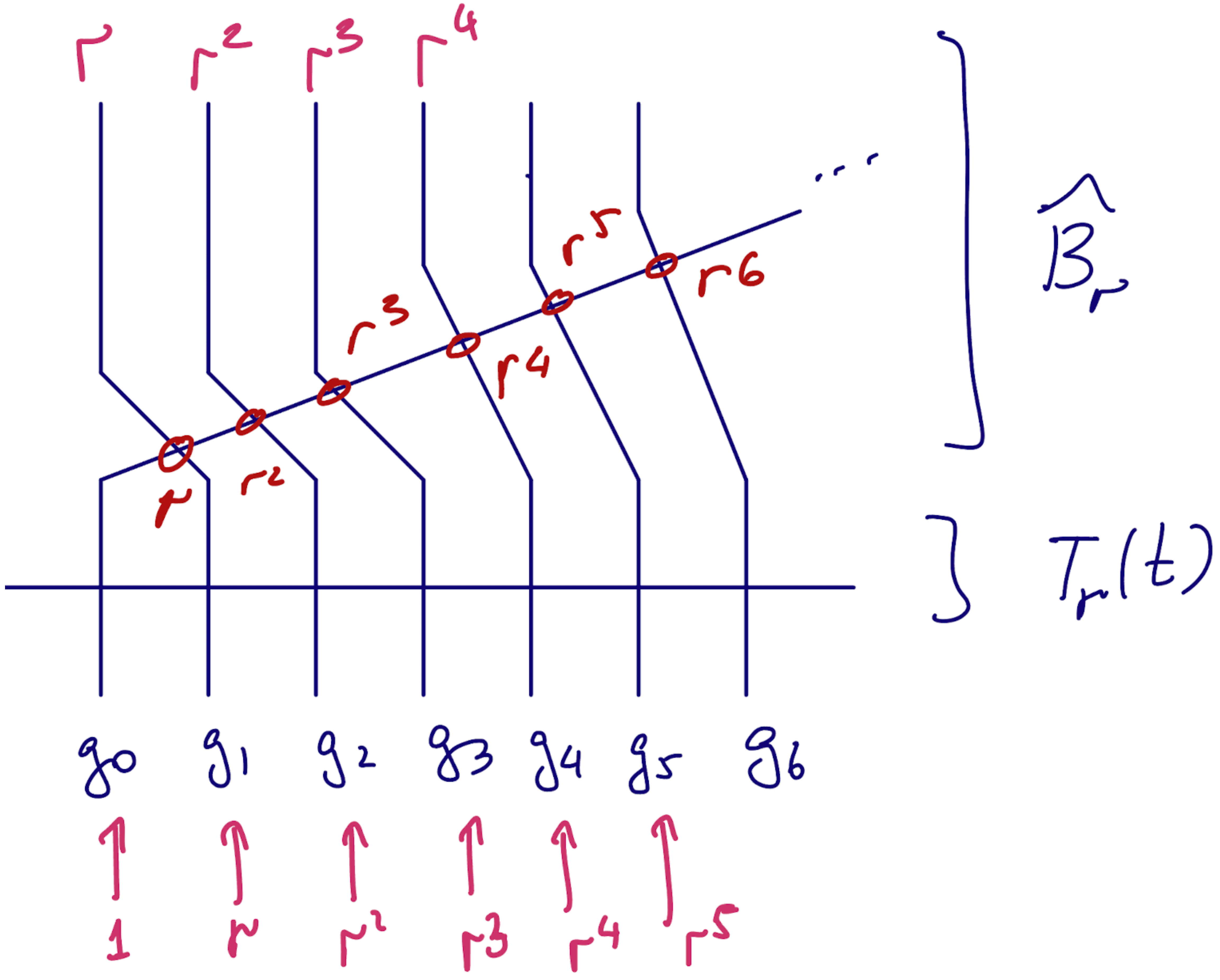
Let $a_n = r^n$. Let us organize the tower of applications of $B_1, B_2, B_3, \dots$



Iterated intertwining leads to $T_r(t)\hat{B}_r = \hat{B}_r T_r(rt)$, which is a **time change** in the continuous time q-TASEP

From intertwining to Lax equations for q-TASEP

Let $a_n = r^n$. Let us organize the tower of applications of $B_1, B_2, B_3, \ldots$



Iterated intertwining leads to $T_r(t)\hat{B}_r = \hat{B}_r T_r(rt)$, which is a **time change** in the continuous time q-TASEP

Now send $r \to 1$, so that the q-TASEP becomes homogeneous, all rates $a_i = 1$.

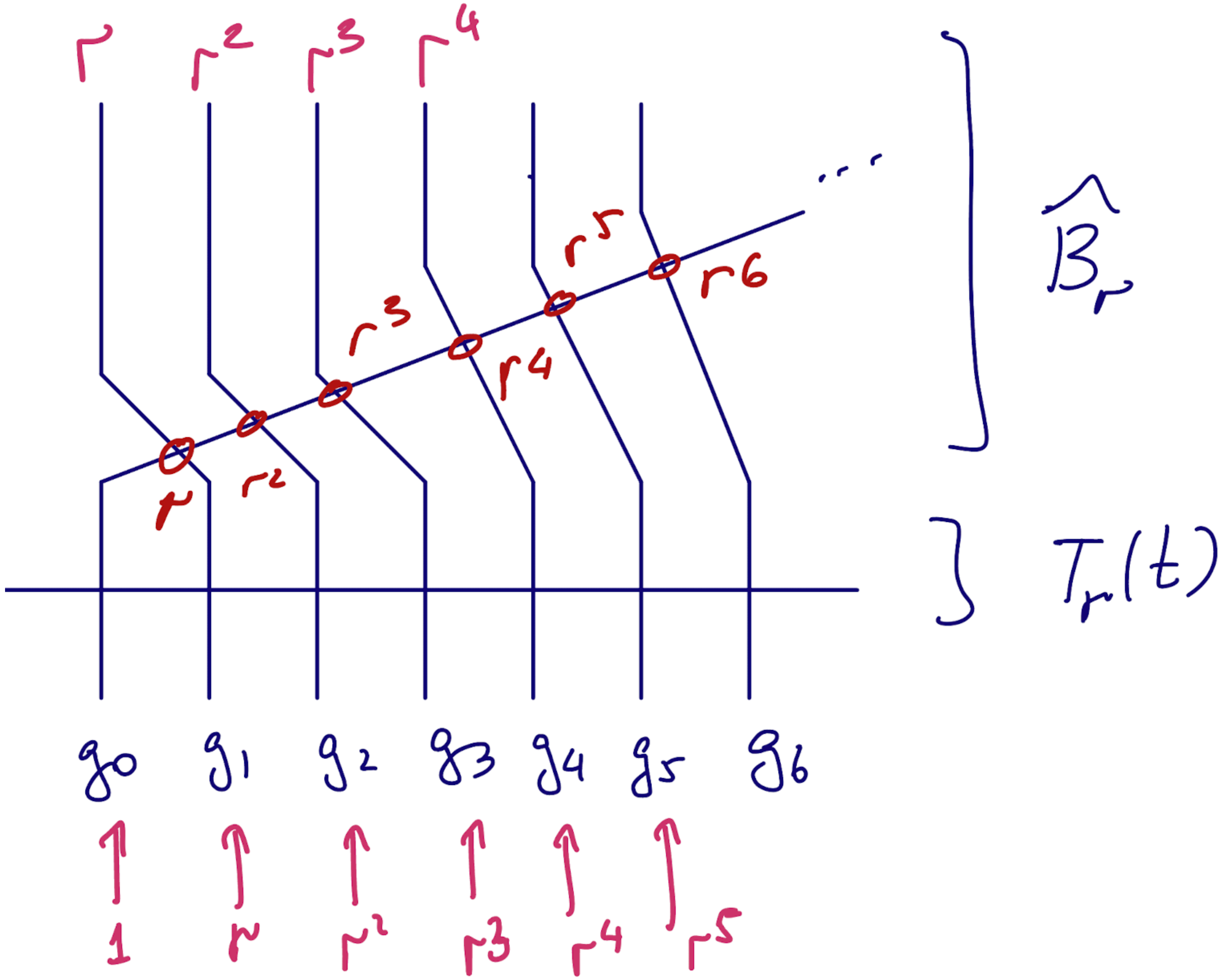
Take a continuous time Poisson limit of $(\hat{B}_r)^{\tau/(1-r)}$. We get a Markov semigroup $B(\tau)$, and intertwining

$$T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$$

(details on $B(\tau)$ later)

From intertwining to Lax equations for q-TASEP

Let $a_n = r^n$. Let us organize the tower of applications of $B_1, B_2, B_3, \ldots$



Iterated intertwining leads to $T_r(t)\hat{B}_r = \hat{B}_r T_r(rt)$, which is a **time change** in the continuous time q-TASEP

Now send $r \to 1$, so that the q-TASEP becomes homogeneous, all rates $a_i = 1$.

Take a continuous time Poisson limit of $(\hat{B}_r)^{\tau/(1-r)}$. We get a Markov semigroup $B(\tau)$, and intertwining

$$T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$$

(details on $B(\tau)$ later)

Note: Outside the q-Hahn case $u_2/u_1 = s_2/s_1$, this cross-vertex system B_r does **not** preserve the empty configuration!

Lax equation for q-TASEP and TASEP

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have
- $T(t)\mathsf{T} = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, where $[A, C] = AC - CA$ is the commutator, and B, T are Markov generators

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have
- $T(t)\mathsf{T} = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, where $[A, C] = AC - CA$ is the commutator, and B, T are Markov generators
- We have $T(t)\mathsf{T} = \frac{d}{dt}T(t)$ by Kolmogorov (Fokker-Planck) equation

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have
- $T(t)\mathsf{T} = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, where $[A, C] = AC - CA$ is the commutator, and B, T are Markov generators
- We have $T(t)\mathsf{T} = \frac{d}{dt}T(t)$ by Kolmogorov (Fokker-Planck) equation
- Therefore, $\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, which is a **Lax equation** for the TASEP semigroup.

Lax equation for q-TASEP and TASEP

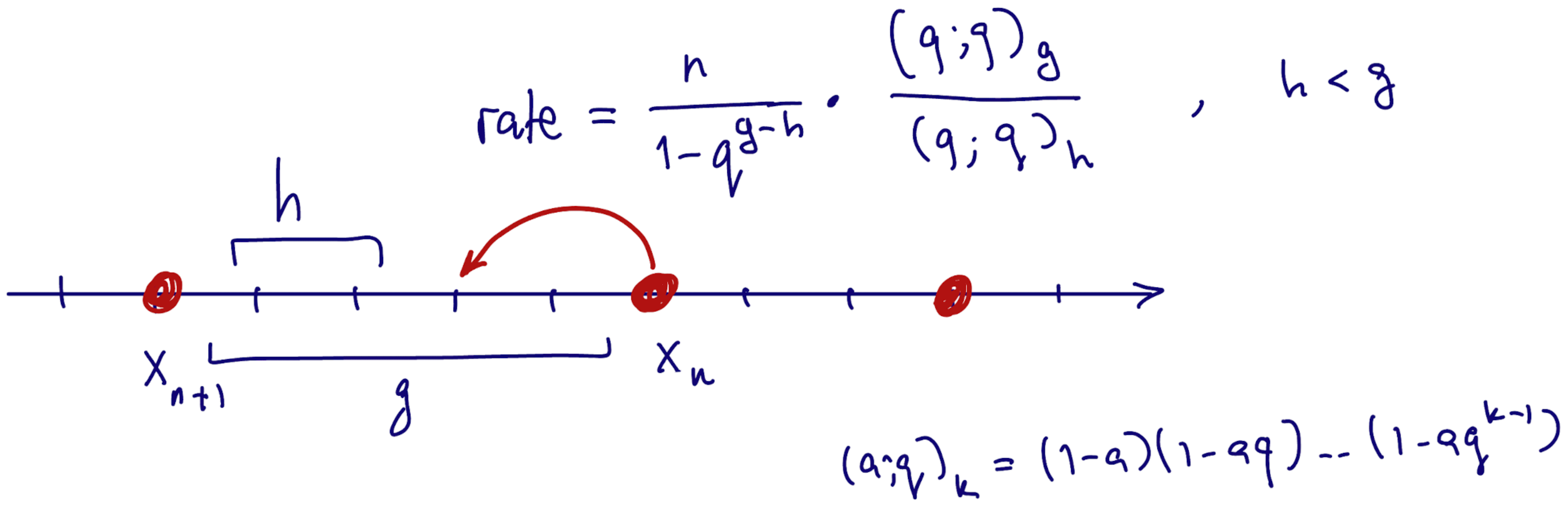
- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have
- $T(t)\mathsf{T} = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, where $[A, C] = AC - CA$ is the commutator, and B, T are Markov generators
- We have $T(t)\mathsf{T} = \frac{d}{dt}T(t)$ by Kolmogorov (Fokker-Planck) equation
- Therefore, $\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, which is a **Lax equation** for the TASEP semigroup.
- For expectations, this means $t\,\mathbb{E}_y\left[(\mathsf{T}F)(x(t))\right] = \mathsf{B}\,\mathbb{E}_y\left[F(x(t))\right] - \mathbb{E}_y\left[(\mathsf{B}F)(x(t))\right]$ for any function of the configuration (note, first term in LHS vanishes if $y = step$)

Lax equation for q-TASEP and TASEP

- Intertwining relation for the homogeneous q-TASEP: $T(t)B(\tau) = B(\tau)T(e^{-\tau}t)$
- Differentiate in τ at $\tau = 0$, and divide by t. We have
- $T(t)\mathsf{T} = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, where $[A, C] = AC - CA$ is the commutator, and B, T are Markov generators
- We have $T(t)\mathsf{T} = \frac{d}{dt}T(t)$ by Kolmogorov (Fokker-Planck) equation
- Therefore, $\frac{d}{dt}T(t) = \left[\frac{1}{t}\mathsf{B}, T(t)\right]$, which is a **Lax equation** for the TASEP semigroup.
- For expectations, this means $t\,\mathbb{E}_y\left[(\mathsf{T}F)(x(t))\right] = \mathsf{B}\,\mathbb{E}_y\left[F(x(t))\right] - \mathbb{E}_y\left[(\mathsf{B}F)(x(t))\right]$ for any function of the configuration (note, first term in LHS vanishes if $y = step$)
- The Lax equation should give access to all multipoint observables for q-TASEP, but this information is not that easy to extract… [Quastel-Remenik 2019] show KP equations for KPZ fixed point

The backwards dynamics B for q-TASEP and TASEP

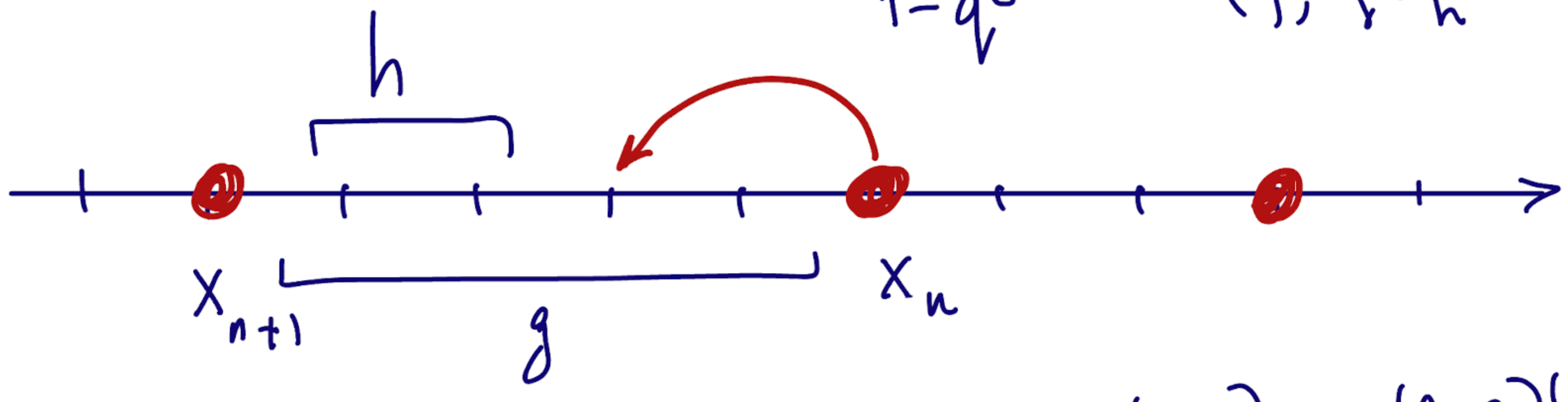
The backwards dynamics $B(\tau)$ is a continuous time Markov process on particle configurations which are densely packed to the left. Each particle x_n jumps back independently in continuous time:



The backwards dynamics B for q-TASEP and TASEP

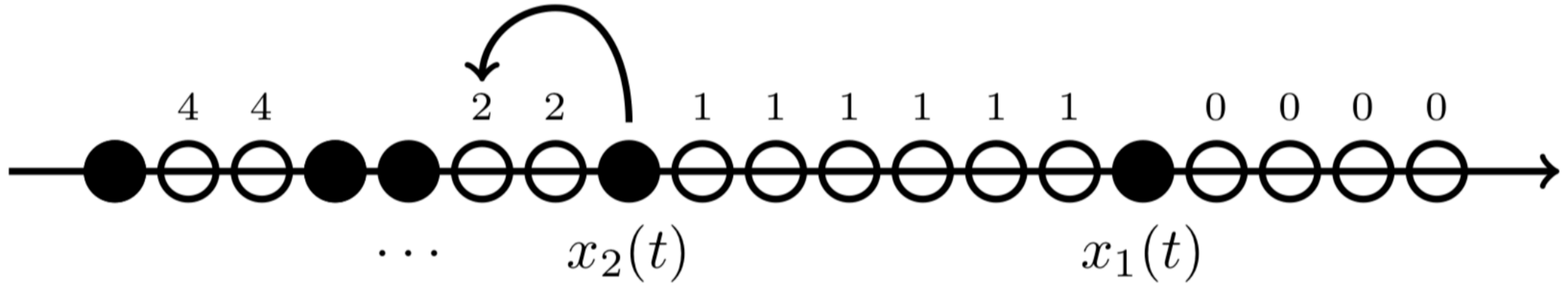
The backwards dynamics $B(\tau)$ is a continuous time Markov process on particle configurations which are densely packed to the left. Each particle x_n jumps back independently in continuous time:

$$\text{rate} = \frac{n}{1-q^{g-h}} \cdot \frac{(q;q)_g}{(q;q)_h}, \quad h < g$$



$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

Case $q = 0$

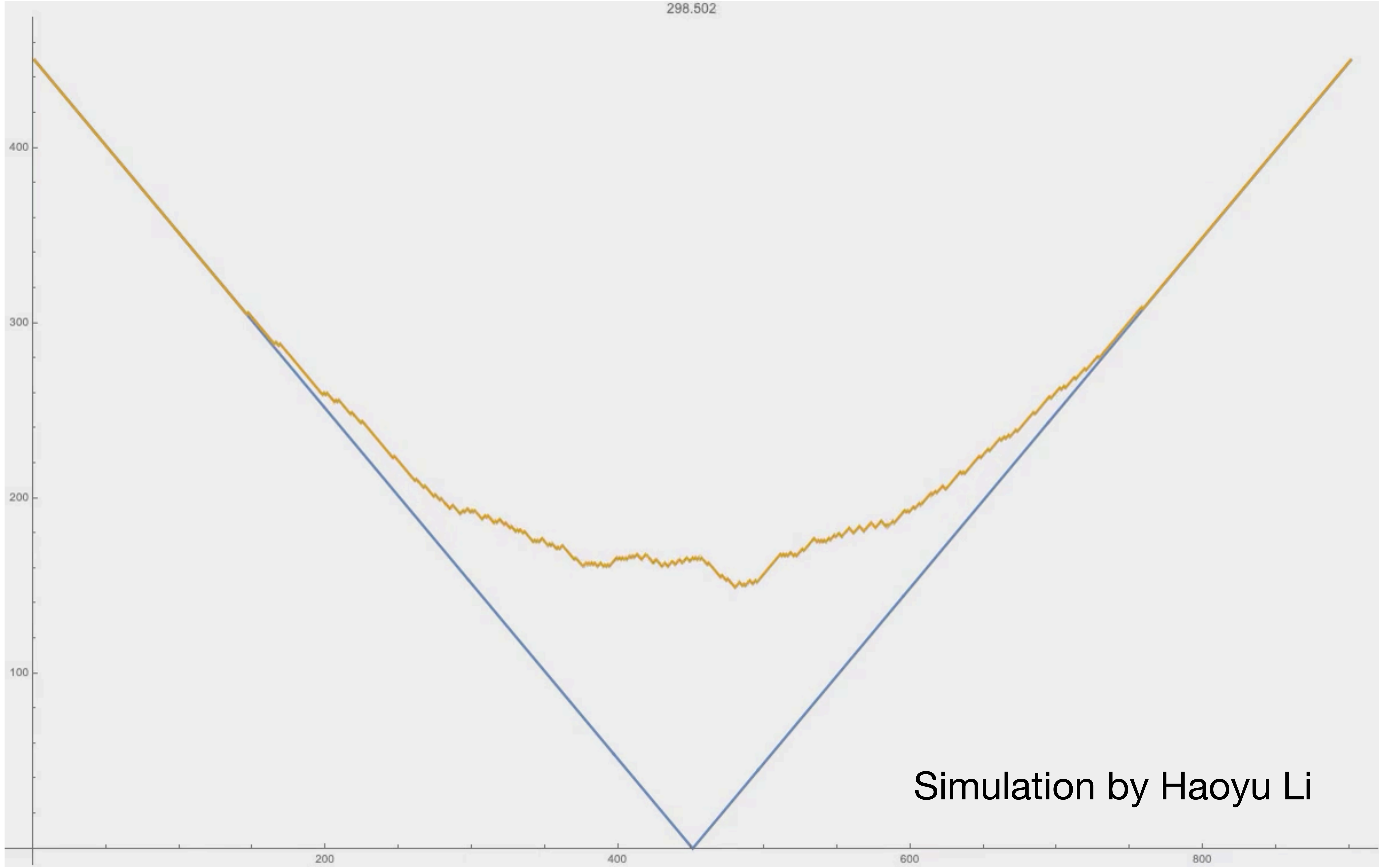


- Each hole has an independent exponential clock with rate equal to the number m of particles to its right, $\mathbb{P}(\text{wait} > s) = e^{-m \cdot s}$, $s > 0$.
- When the clock at a hole rings, the leftmost of the particles that are to the right of the hole instantaneously jumps into this hole
- Because total rate of jump is proportional to the size of the gap, this is a discrete space inhomogeneous version of the Hammersley process **[Hammersley '72], [Aldous-Diaconis '95]**

Running TASEP back in time

Theorem **[P.-Saenz '19].** $\delta_{step}T(t)B(\tau) = \delta_{step}T(e^{-\tau}t)$, which means that if we run TASEP from the step (densely packed) initial configuration, and then run the backwards process, then the result is a TASEP distribution at an earlier time

TASEP as a growth process
- the graph shows the height function

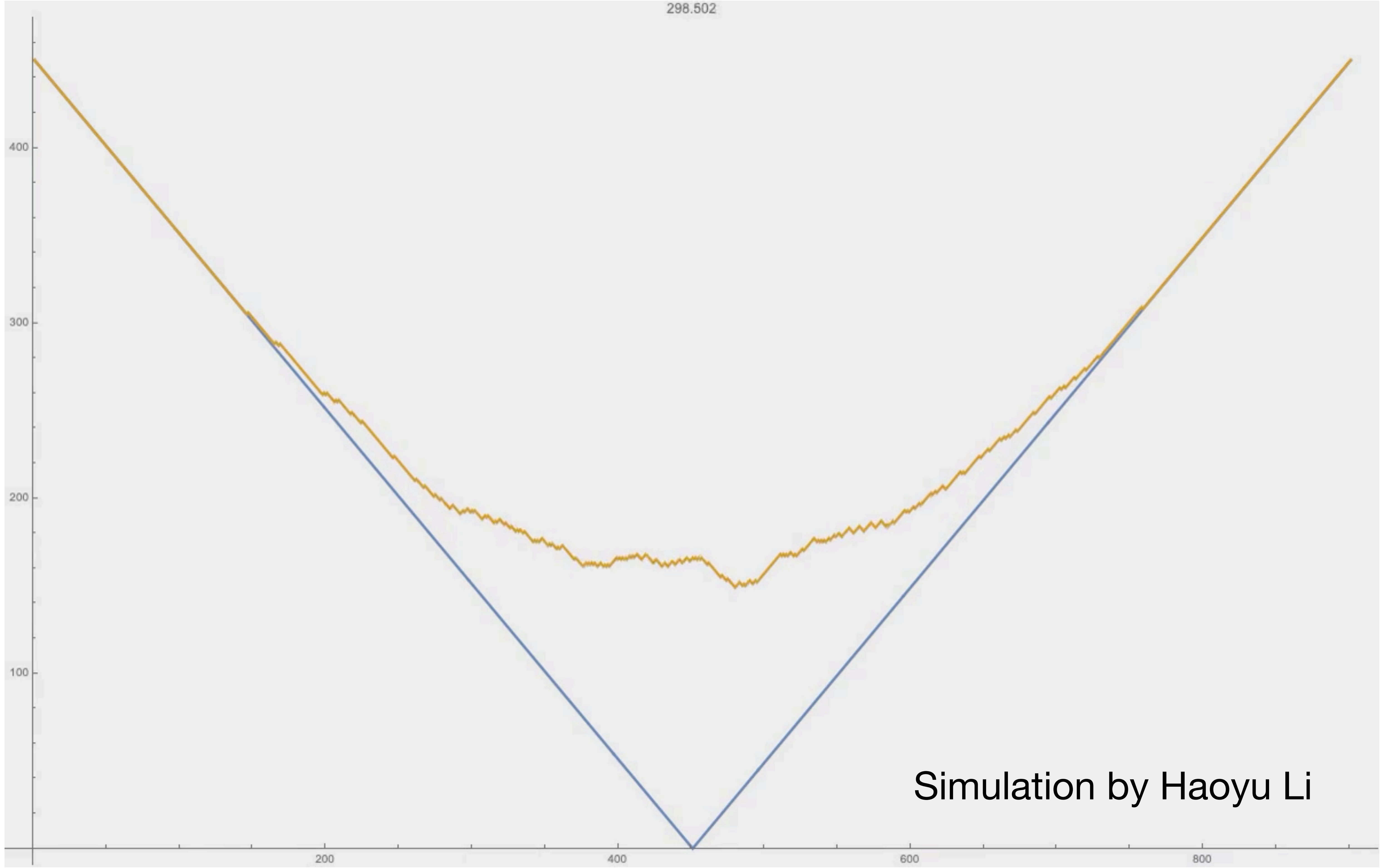


Simulation by Haoyu Li

Running TASEP back in time

Theorem **[P.-Saenz '19].** $\delta_{step}T(t)B(\tau) = \delta_{step}T(e^{-\tau}t)$, which means that if we run TASEP from the step (densely packed) initial configuration, and then run the backwards process, then the result is a TASEP distribution at an earlier time

TASEP as a growth process
- the graph shows the height function



Simulation by Haoyu Li

For questions and remarks

Coupling and rewriting history

Bijectivisation of the Yang-Baxter equation

Let A, B be finite sets and $\sum_{a \in A} w(a) = \sum_{b \in B} w(b)$ (with positive terms)

A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities
$p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A,\ b \in B$.

If all probabilities are equal to 0 or 1 and $|A| = |B|$, then this is a usual bijection.

Bijectivisation of the Yang-Baxter equation

Let A, B be finite sets and $\sum_{a \in A} w(a) = \sum_{b \in B} w(b)$ (with positive terms)

A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities
$p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A, \ b \in B$.

If all probabilities are equal to 0 or 1 and $|A| = |B|$, then this is a usual bijection.

Example: **1 + 3 = 2 + 2**

	2	2
1	1	0
3	1/3	2/3

(maximally dependent)

	2	2
1	1/2	1/2
3	1/2	1/2

(independent)

Bijectivisation of the Yang-Baxter equation

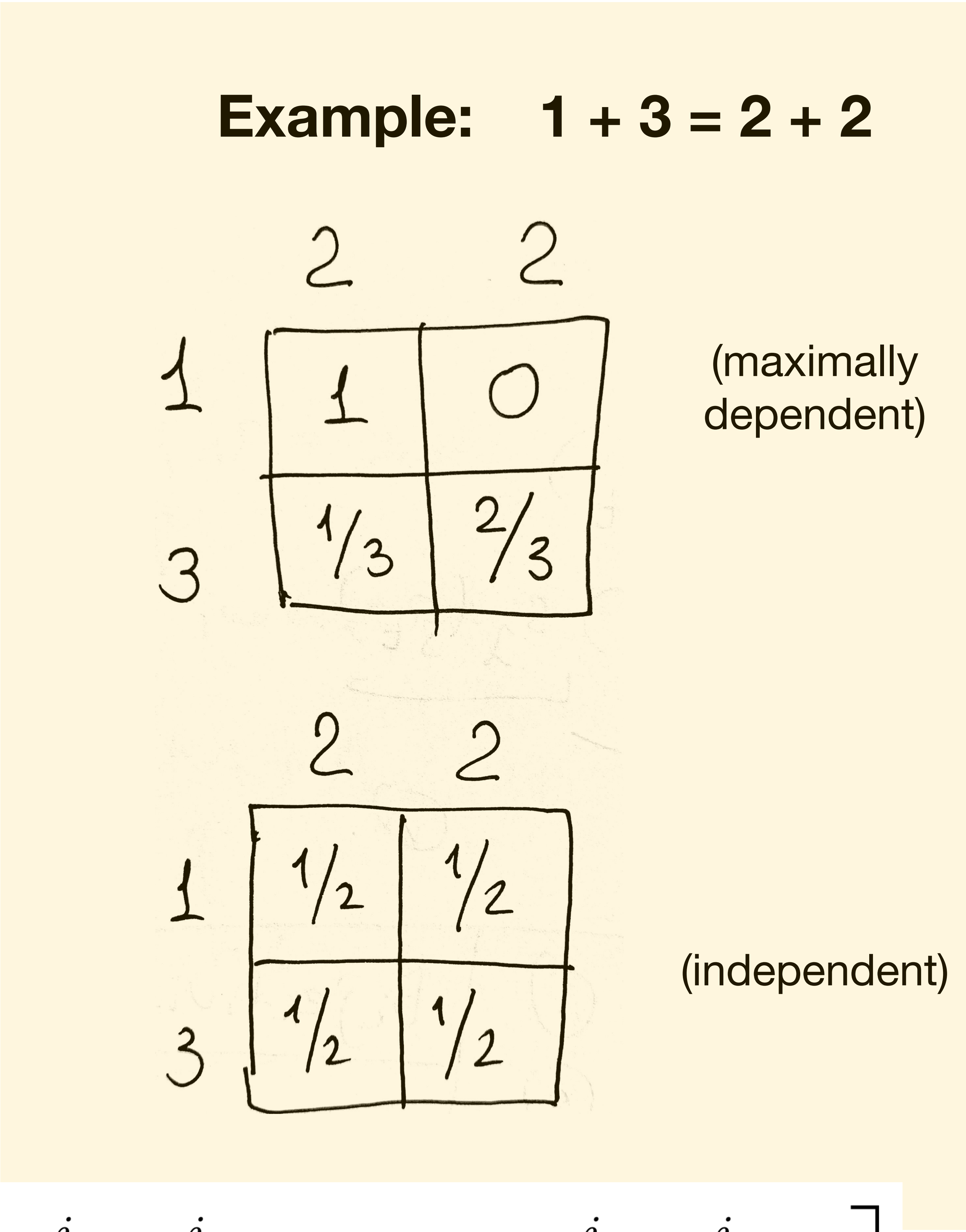
Let A, B be finite sets and $\sum_{a \in A} w(a) = \sum_{b \in B} w(b)$ (with positive terms)

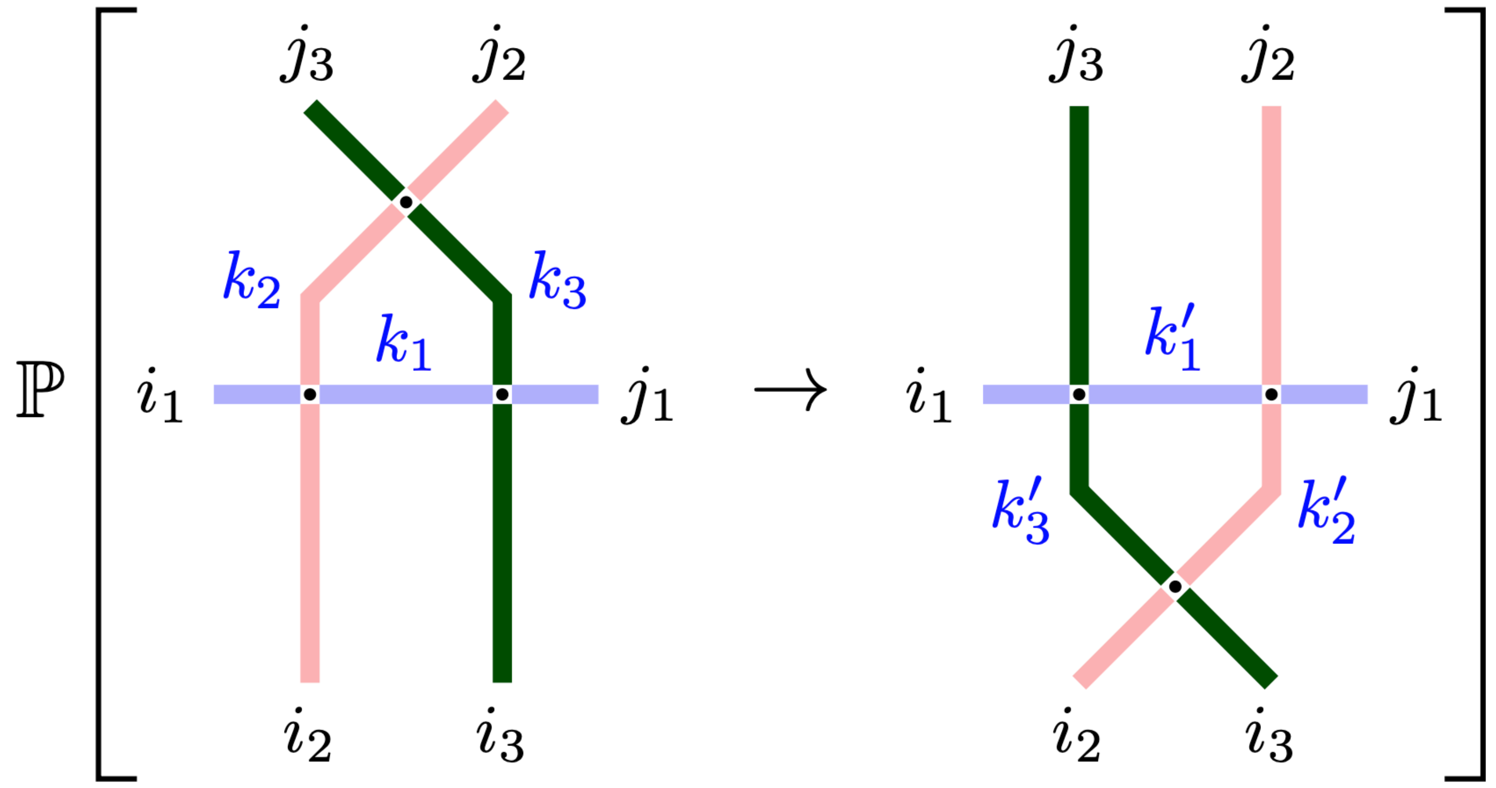
A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities
$p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A,\ b \in B$.

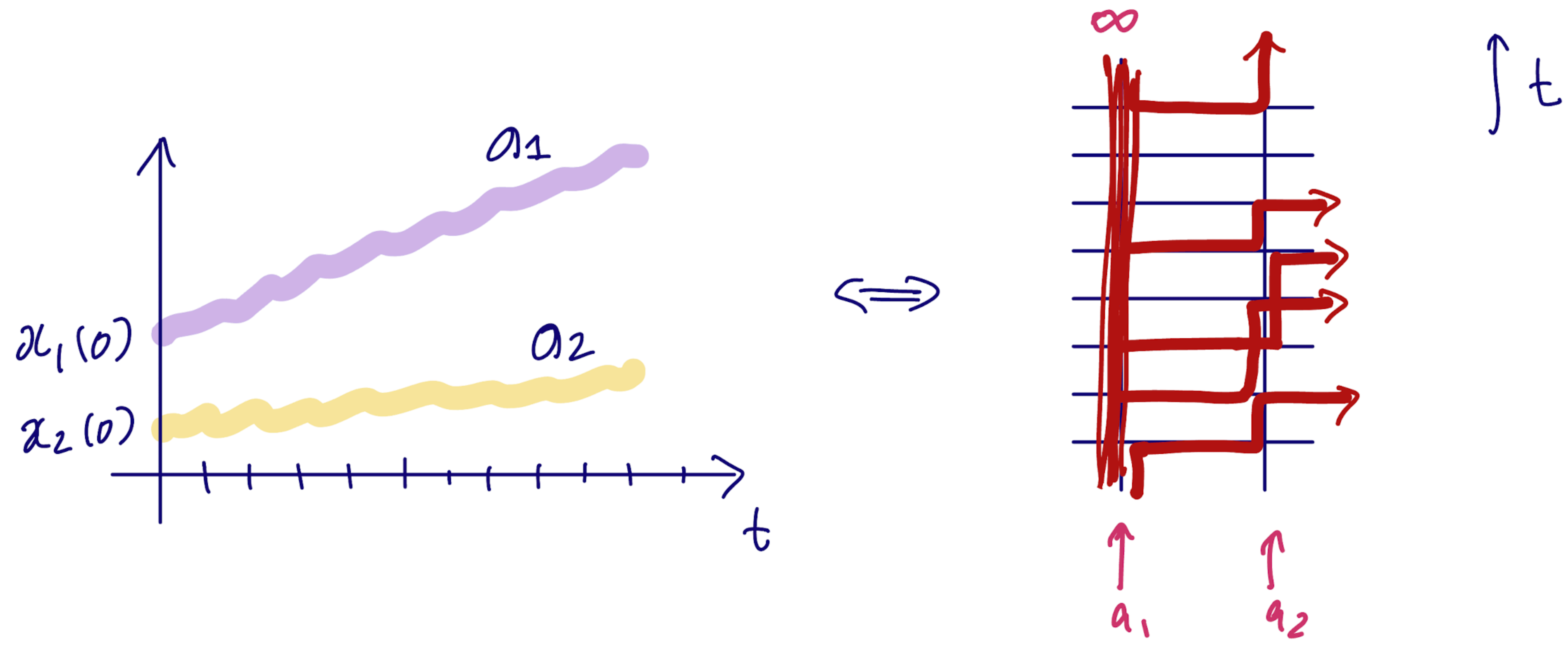
If all probabilities are equal to 0 or 1 and $|A| = |B|$, then this is a usual bijection.



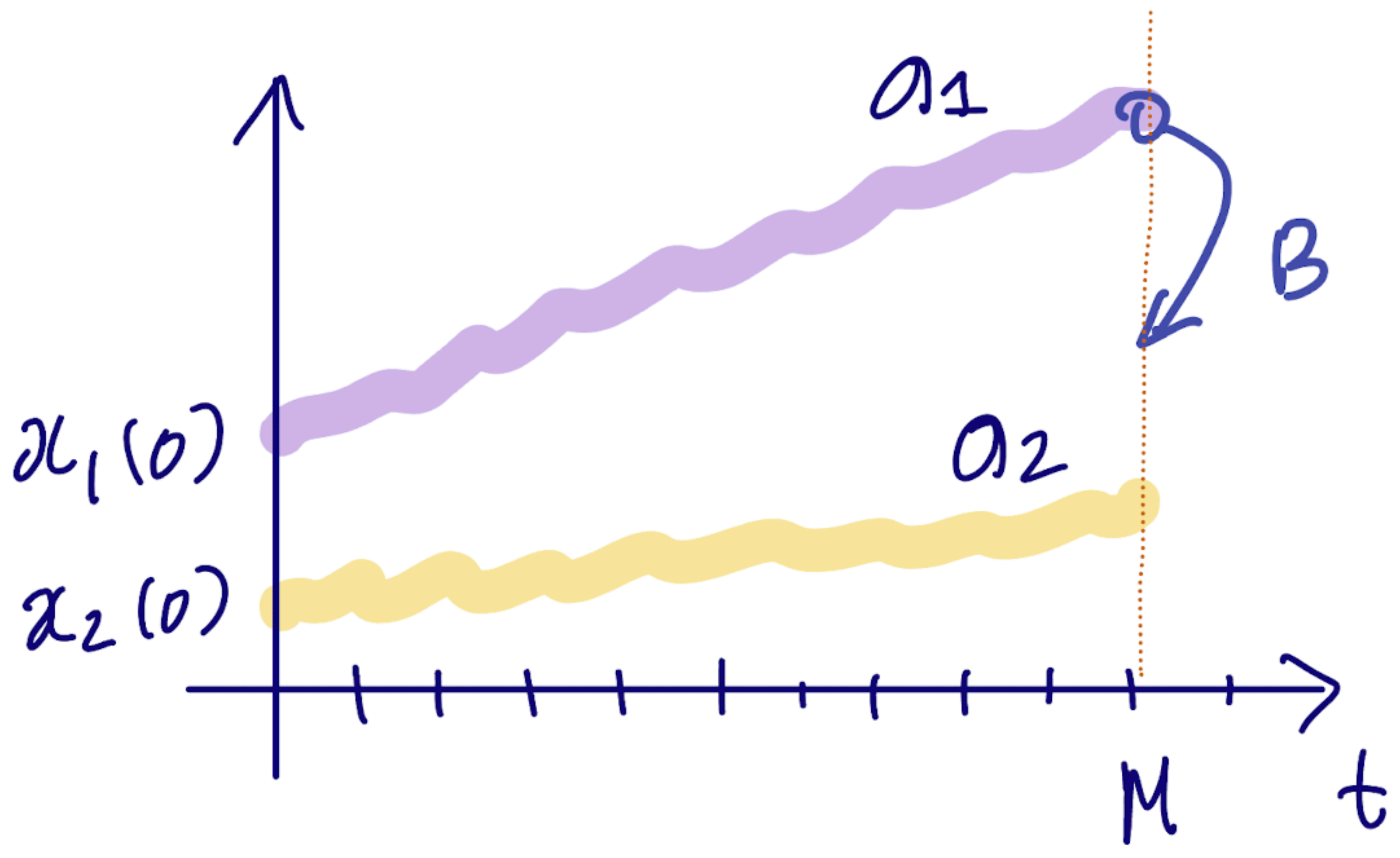


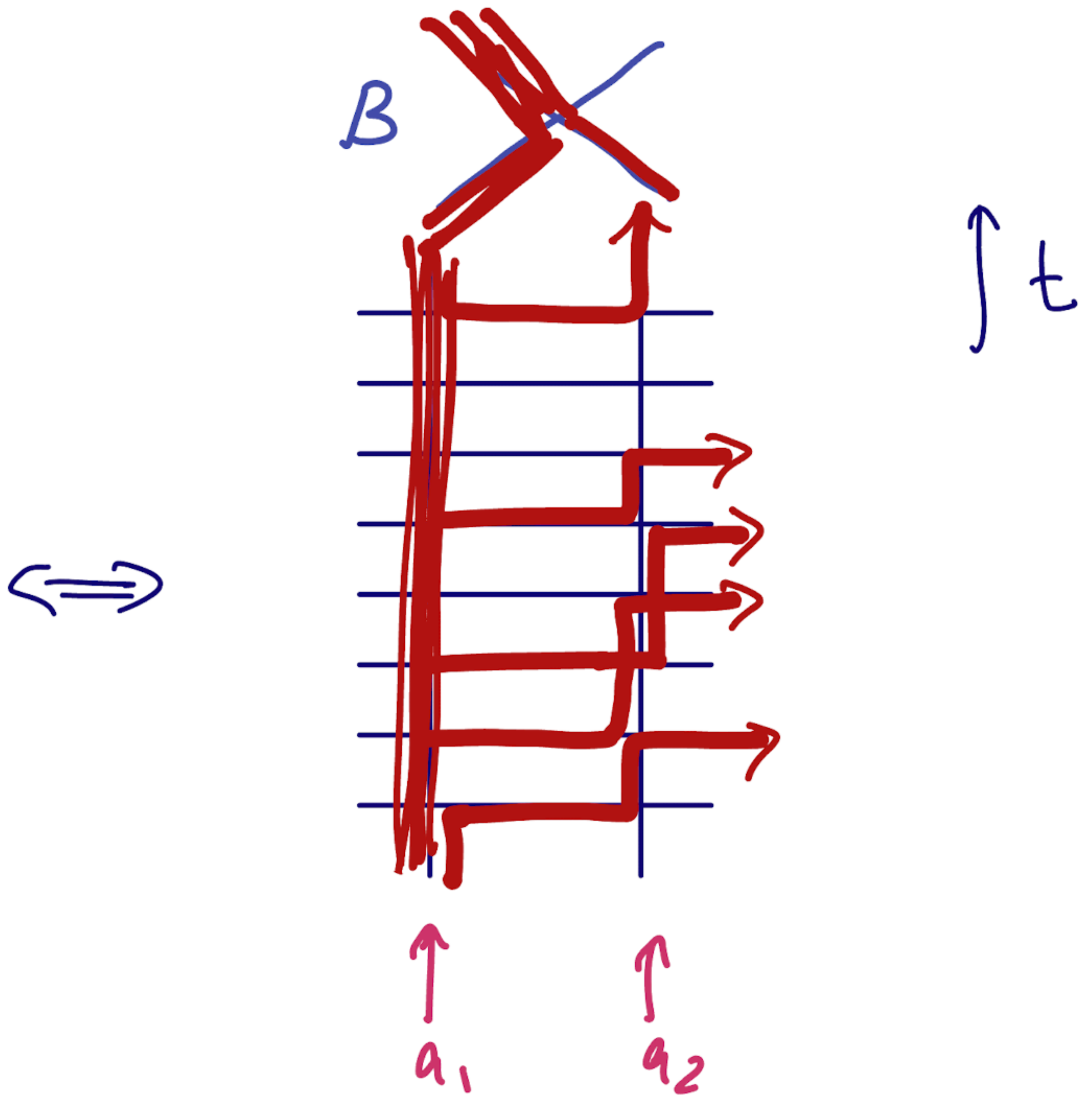
Rewriting history processes

Rewriting history processes

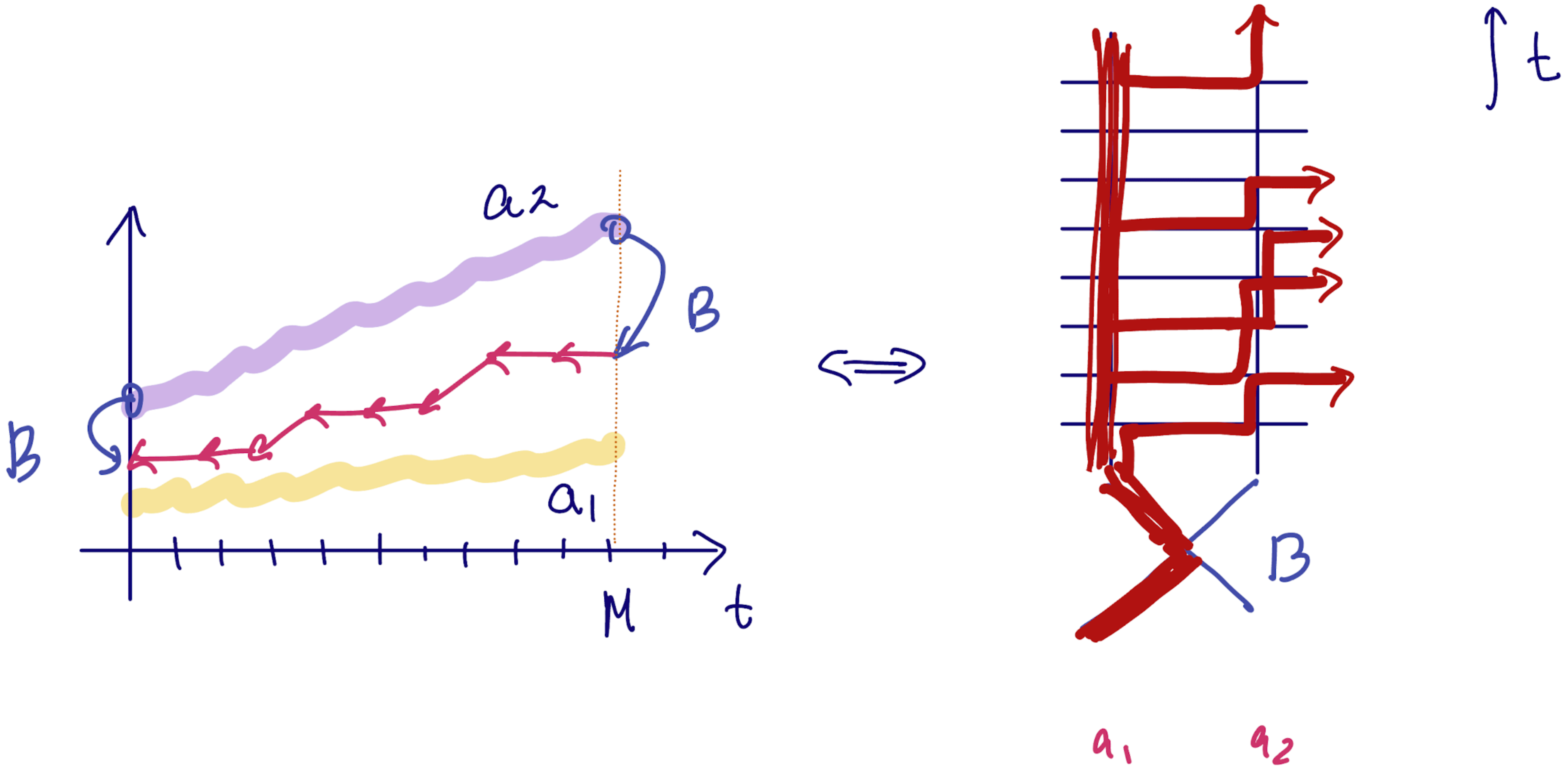


Rewriting history processes



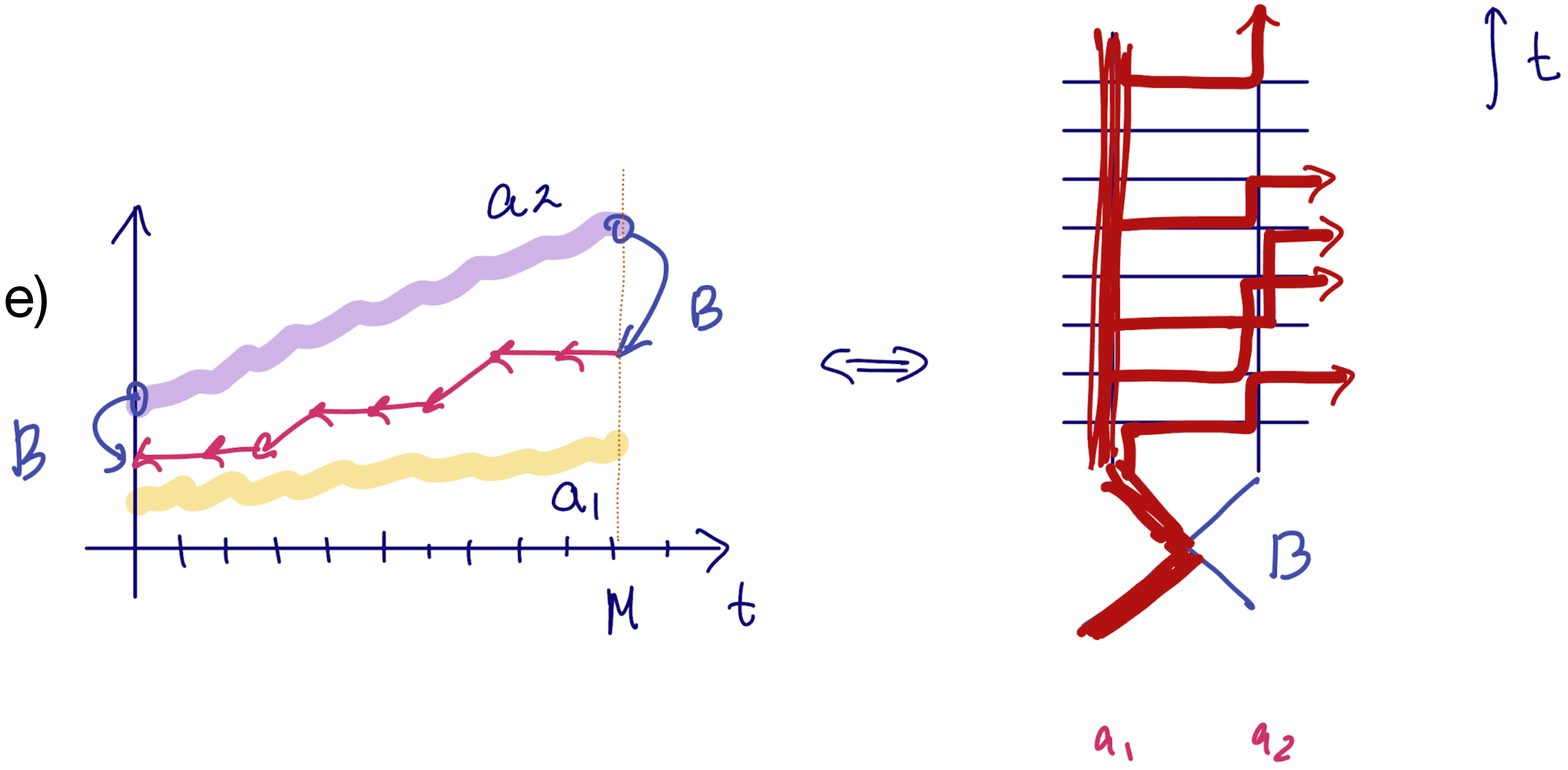


Rewriting history processes



Rewriting history processes

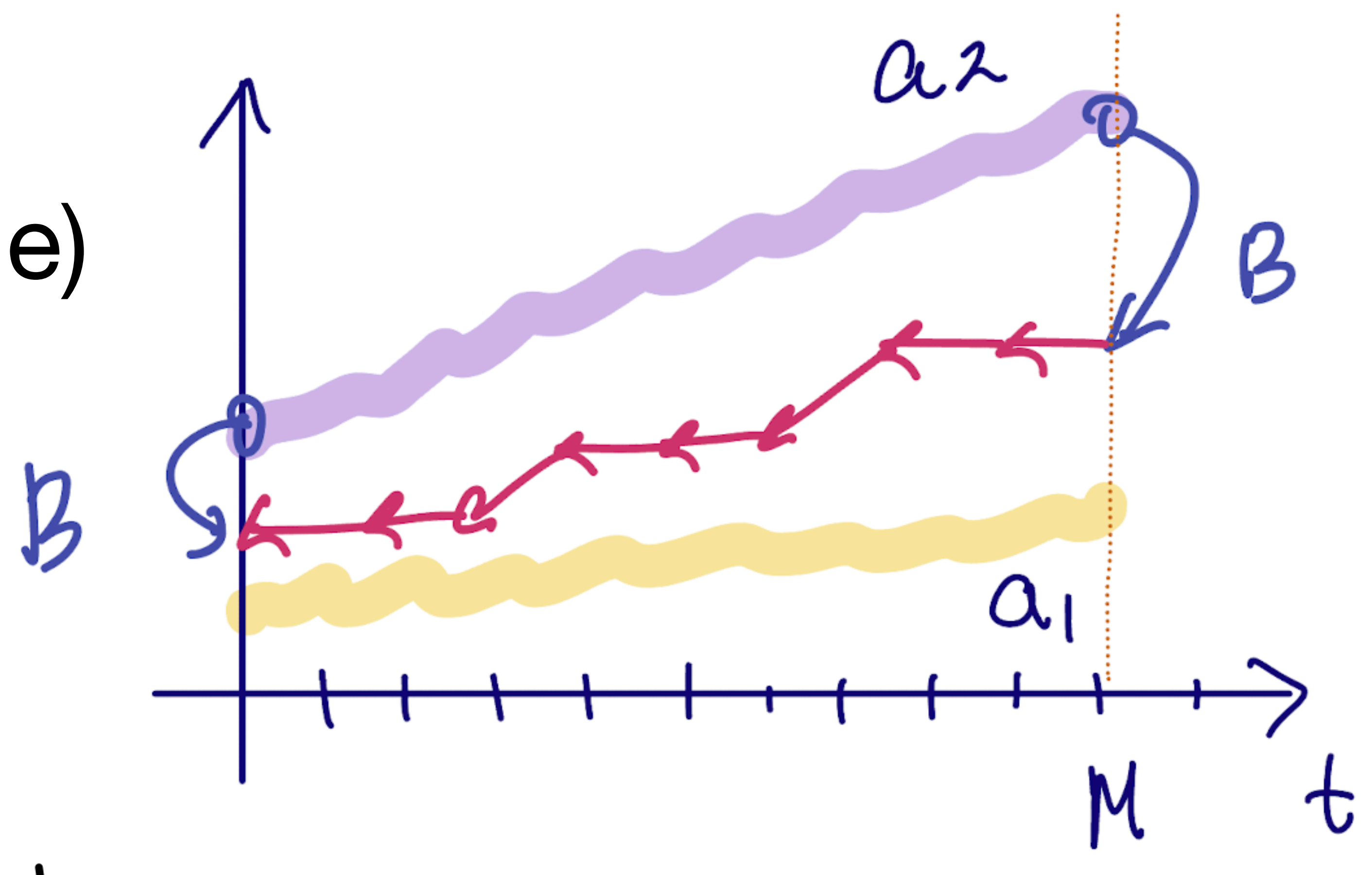
We get two processes for rewriting history:
- from future to past (in the figure)
- from past to future (discussed in next slide)

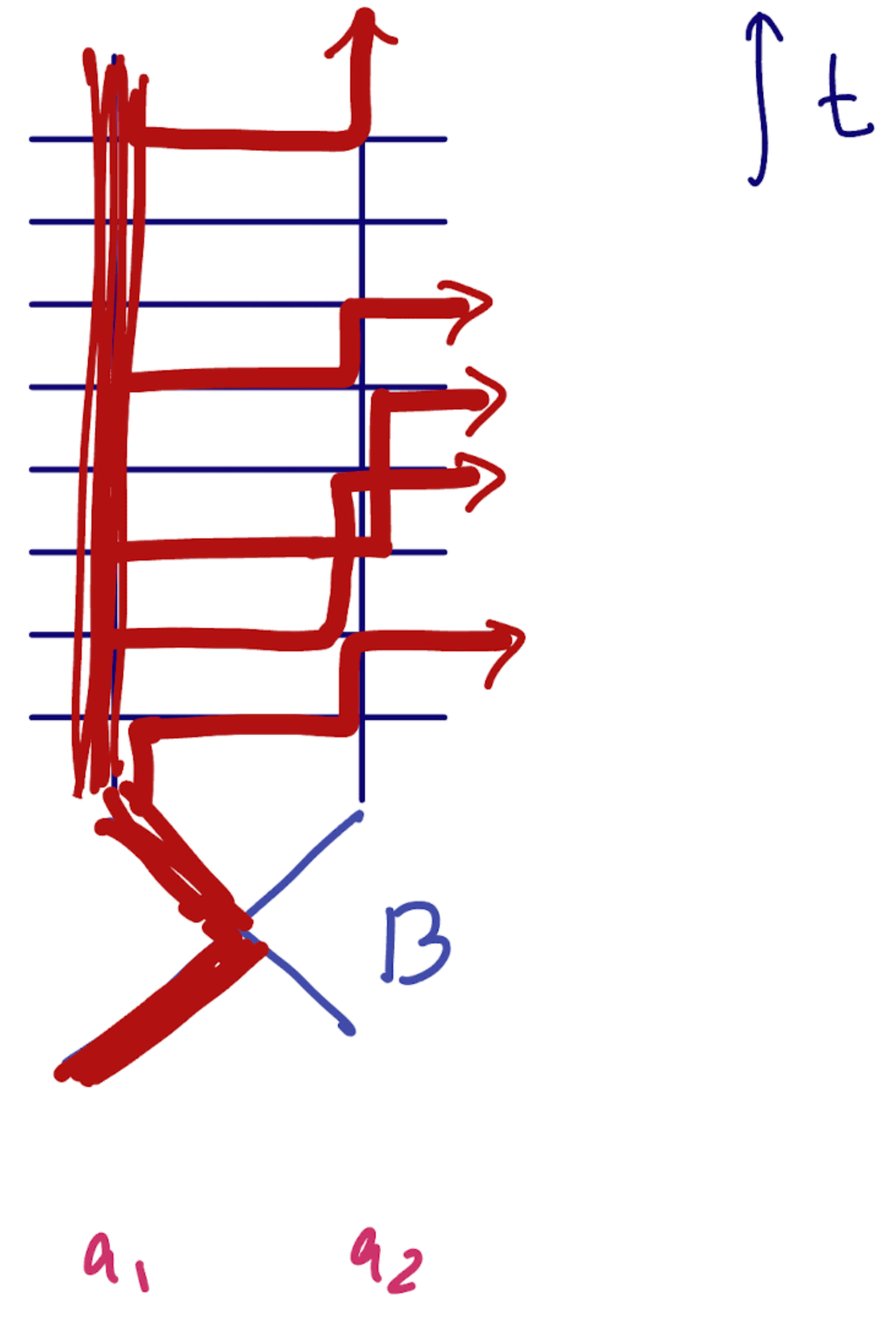


Rewriting history processes

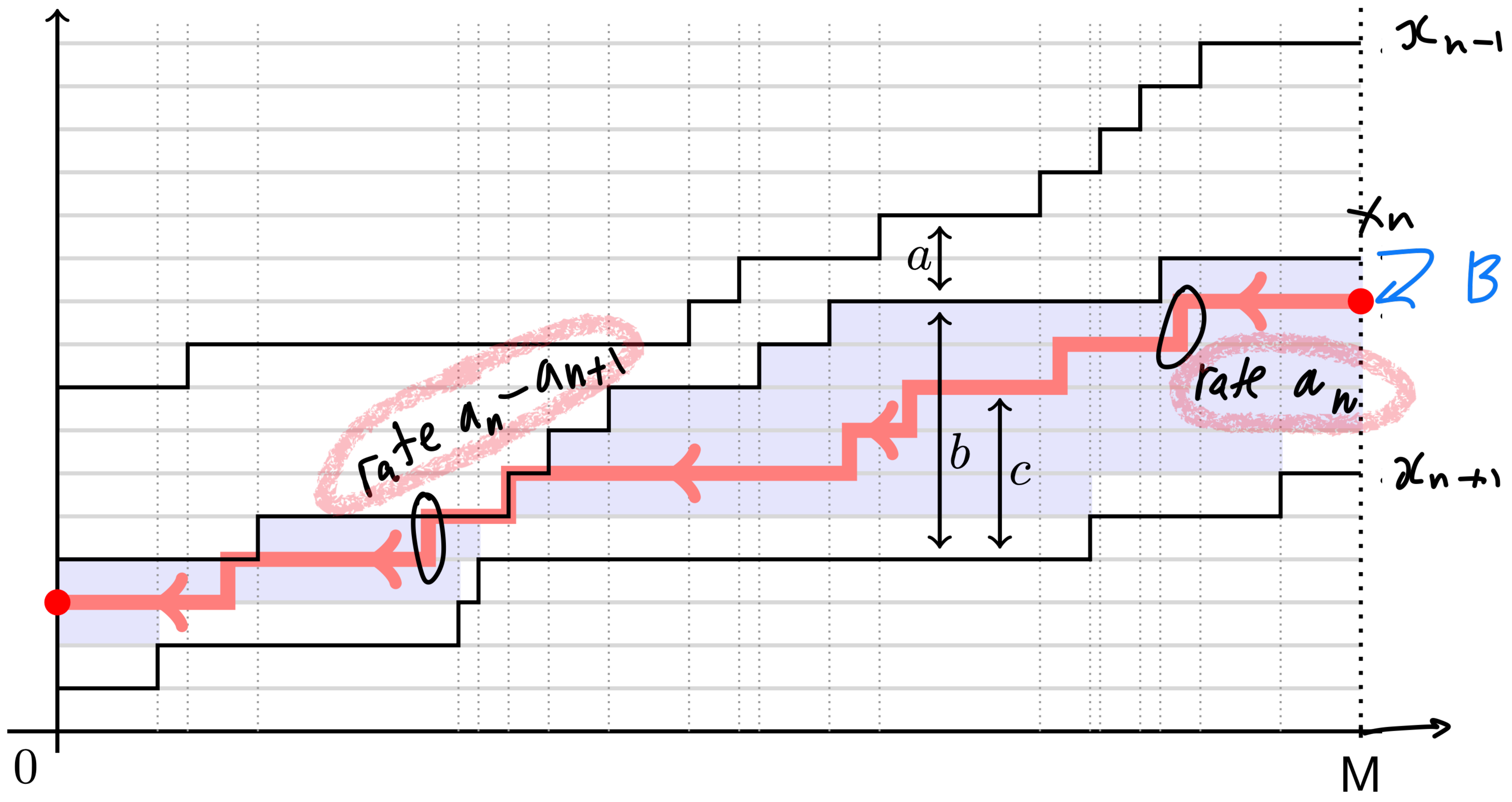
We get two processes for rewriting history:

- from future to past (in the figure)
- from past to future (discussed in next slide)





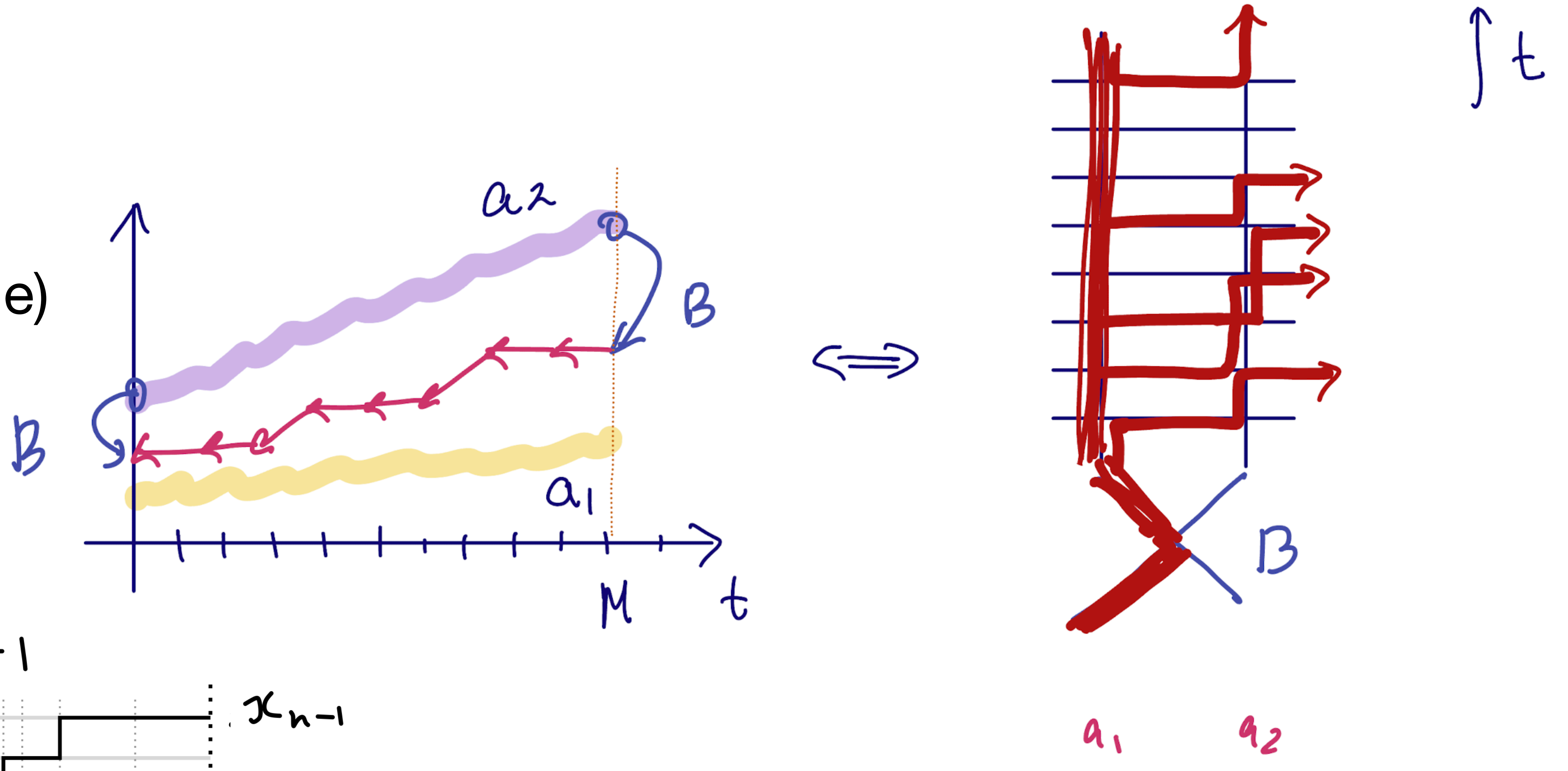
Cont. Time TASEP w. $a_n > a_{n+1}$

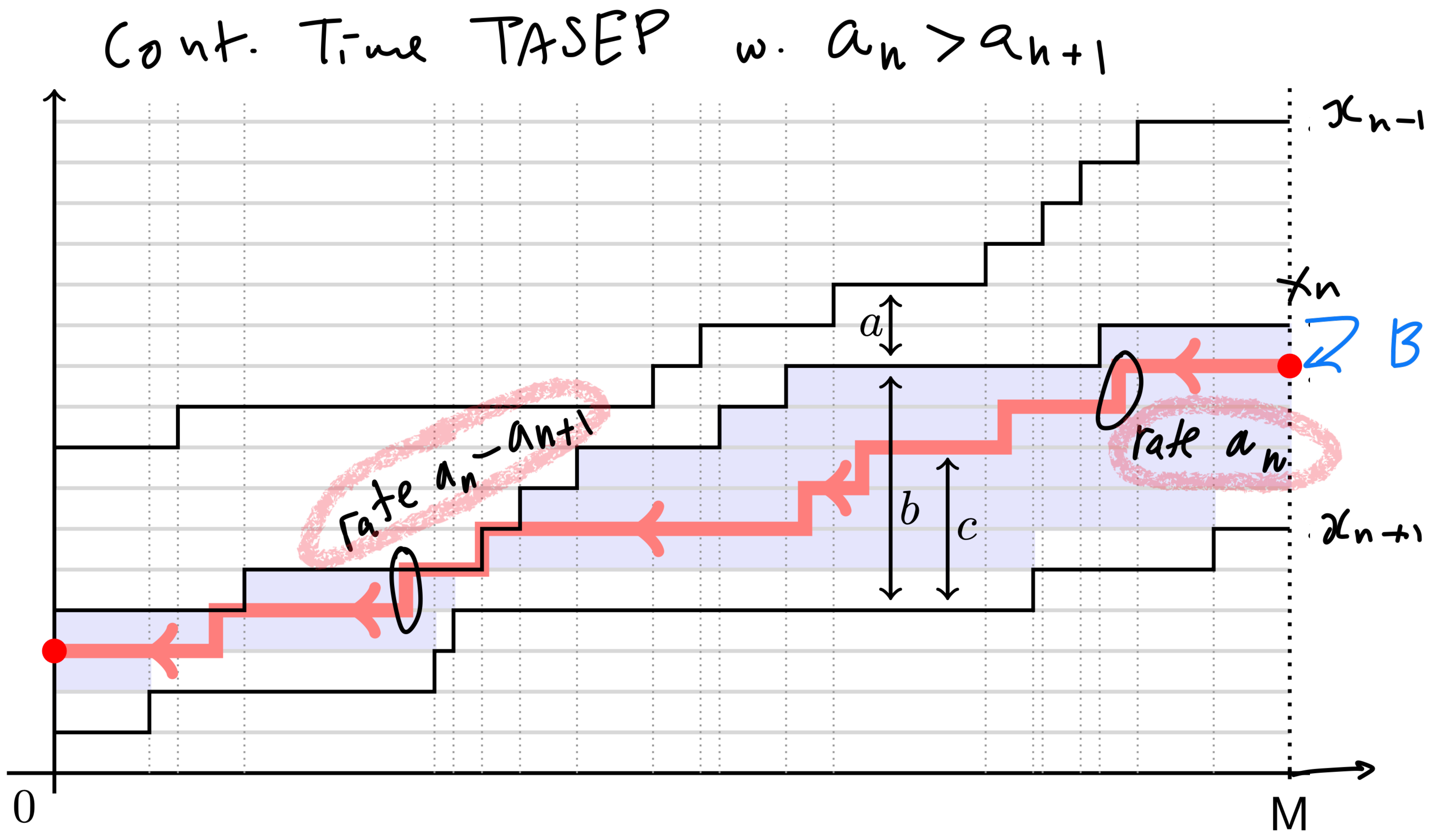


Rewriting history processes

We get two processes for rewriting history:

- from future to past (in the figure)
- from past to future (discussed in next slide)





Theorem [P.-Saenz 2022]. Move x_n back by geometric jump B. Then run a random walk x_n' in the chamber between x_n, x_{x+1}, in reverse time, with jump rates down $a_n - a_{n+1}\mathbf{1}_{b=c}$.

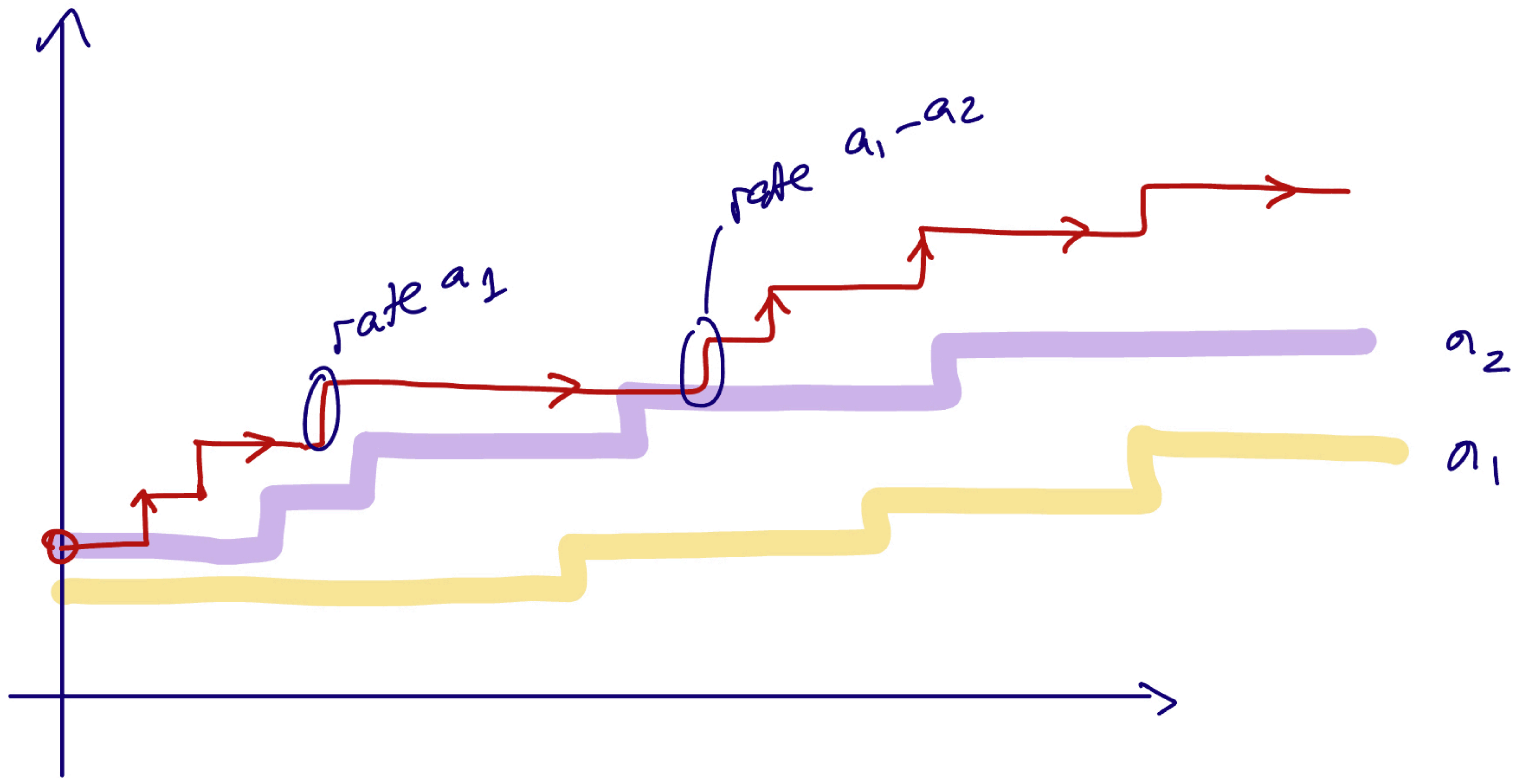
Then the new trajectories are distributed as a TASEP with speeds $(\ldots, a_{n-1}, a_{n+1}, a_n, a_{n+2}, \ldots)$

Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)

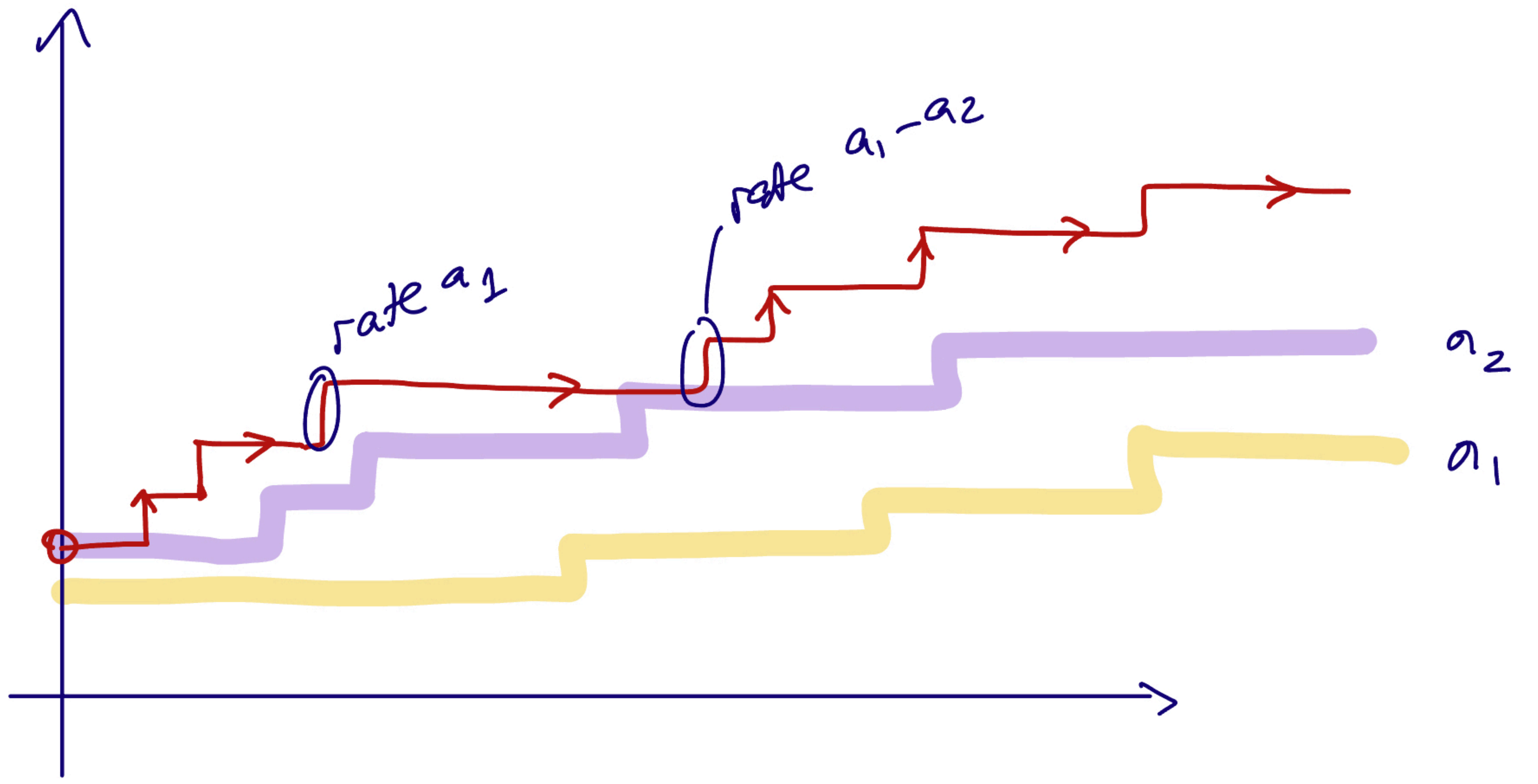


Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)



Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

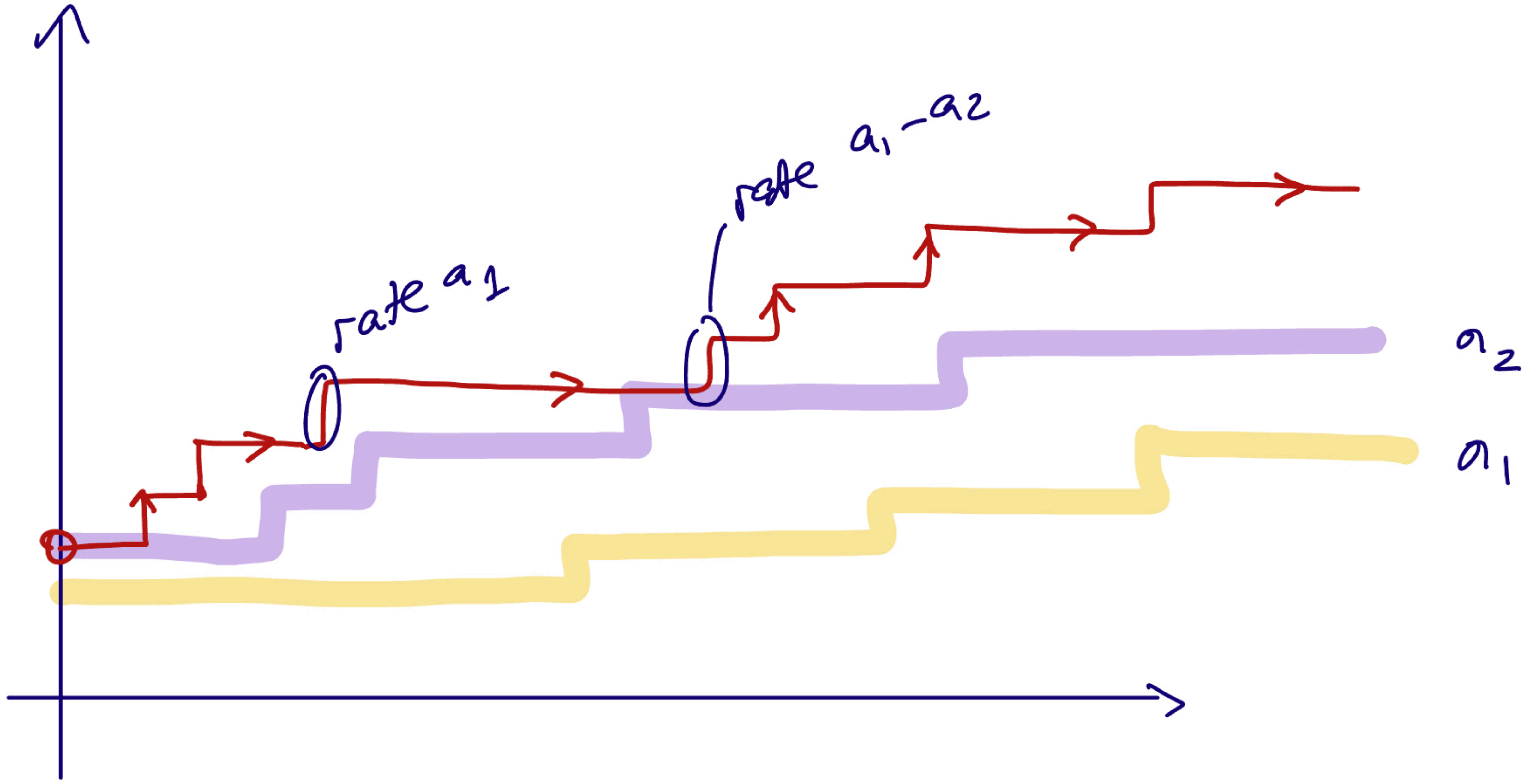
After time τ, we get Poisson walk of slope $\tau + 1$.

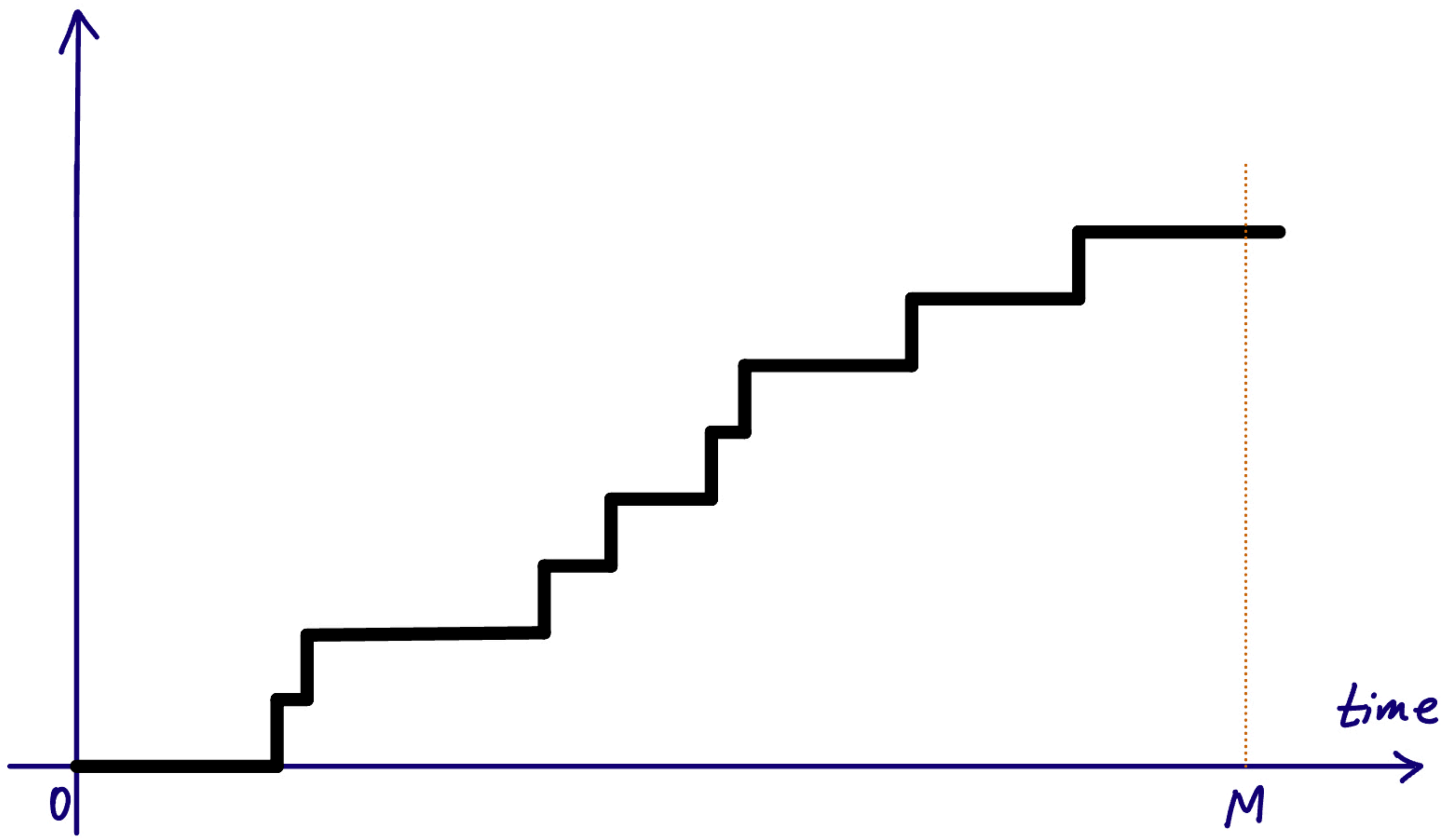
Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)





Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

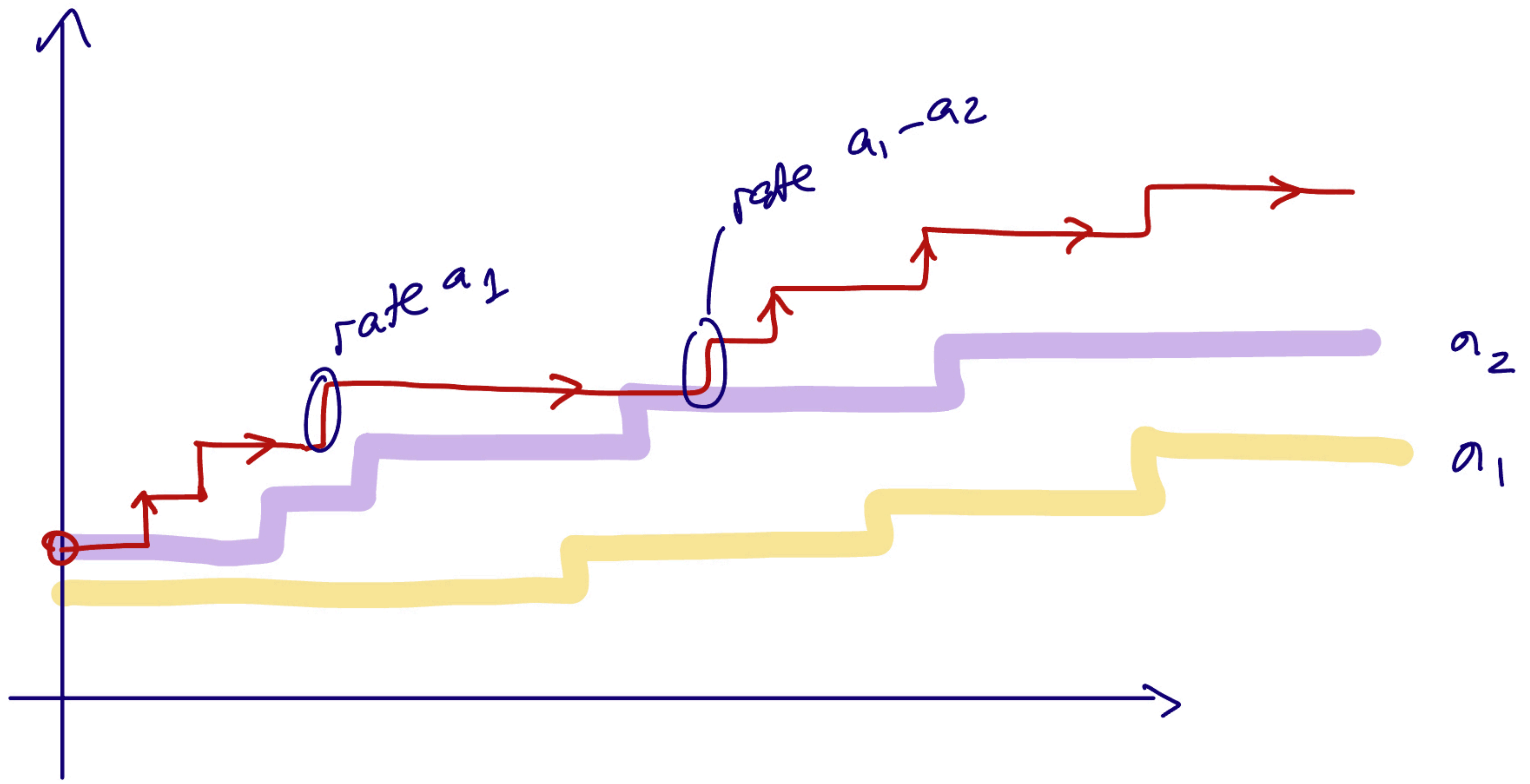
After time τ, we get Poisson walk of slope $\tau + 1$.

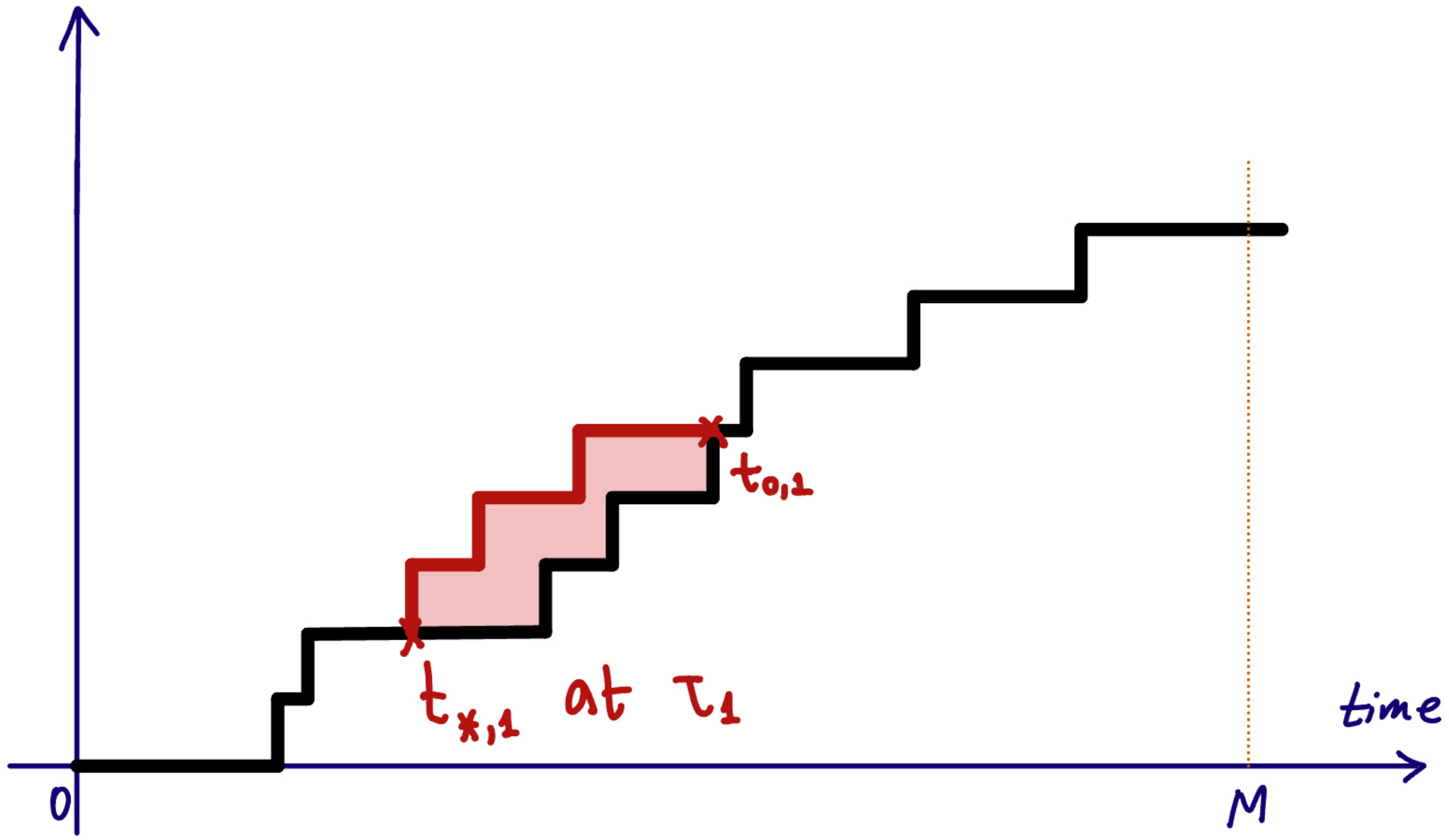
Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)





Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

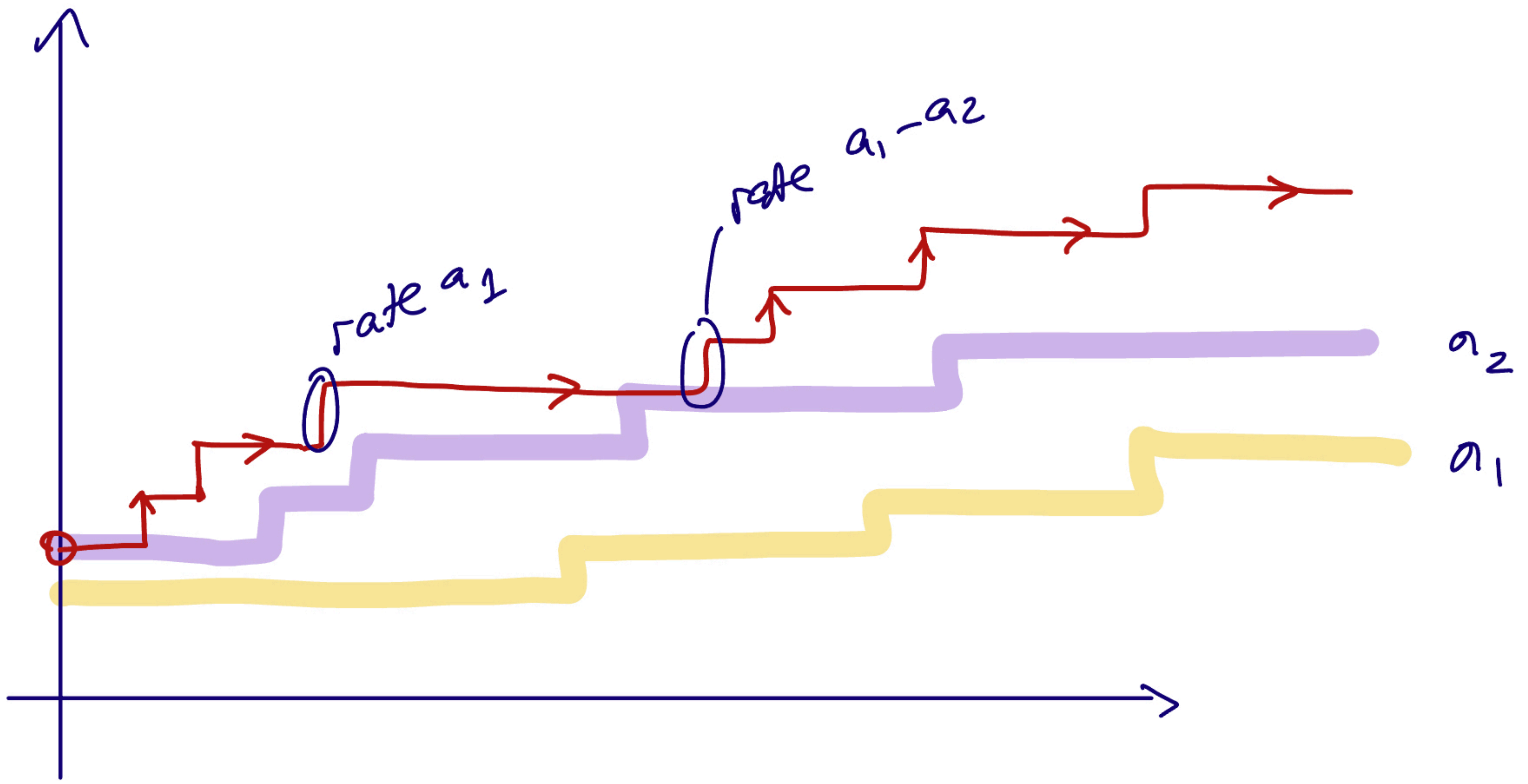
After time τ, we get Poisson walk of slope $\tau + 1$.

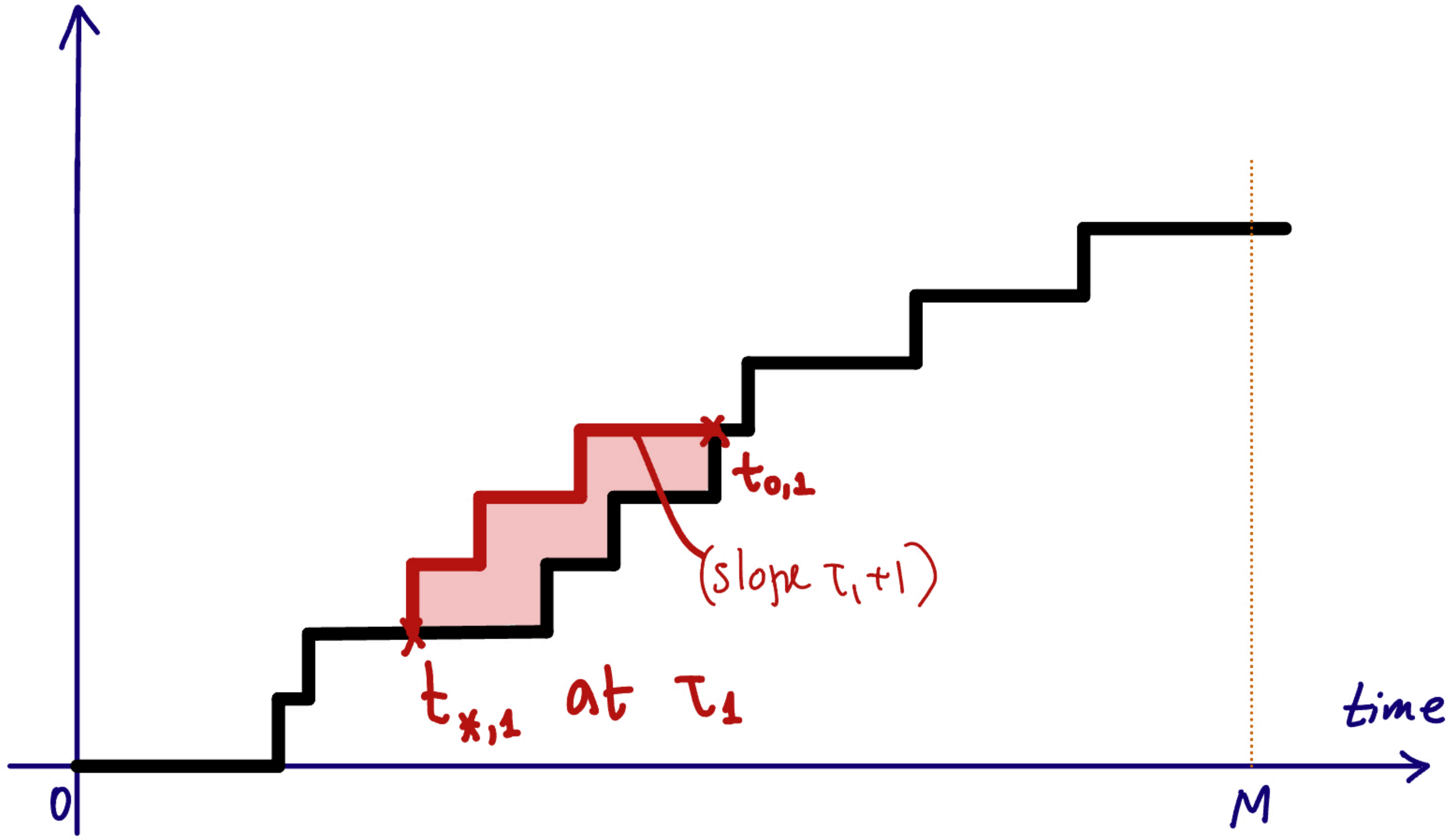
Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)





Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

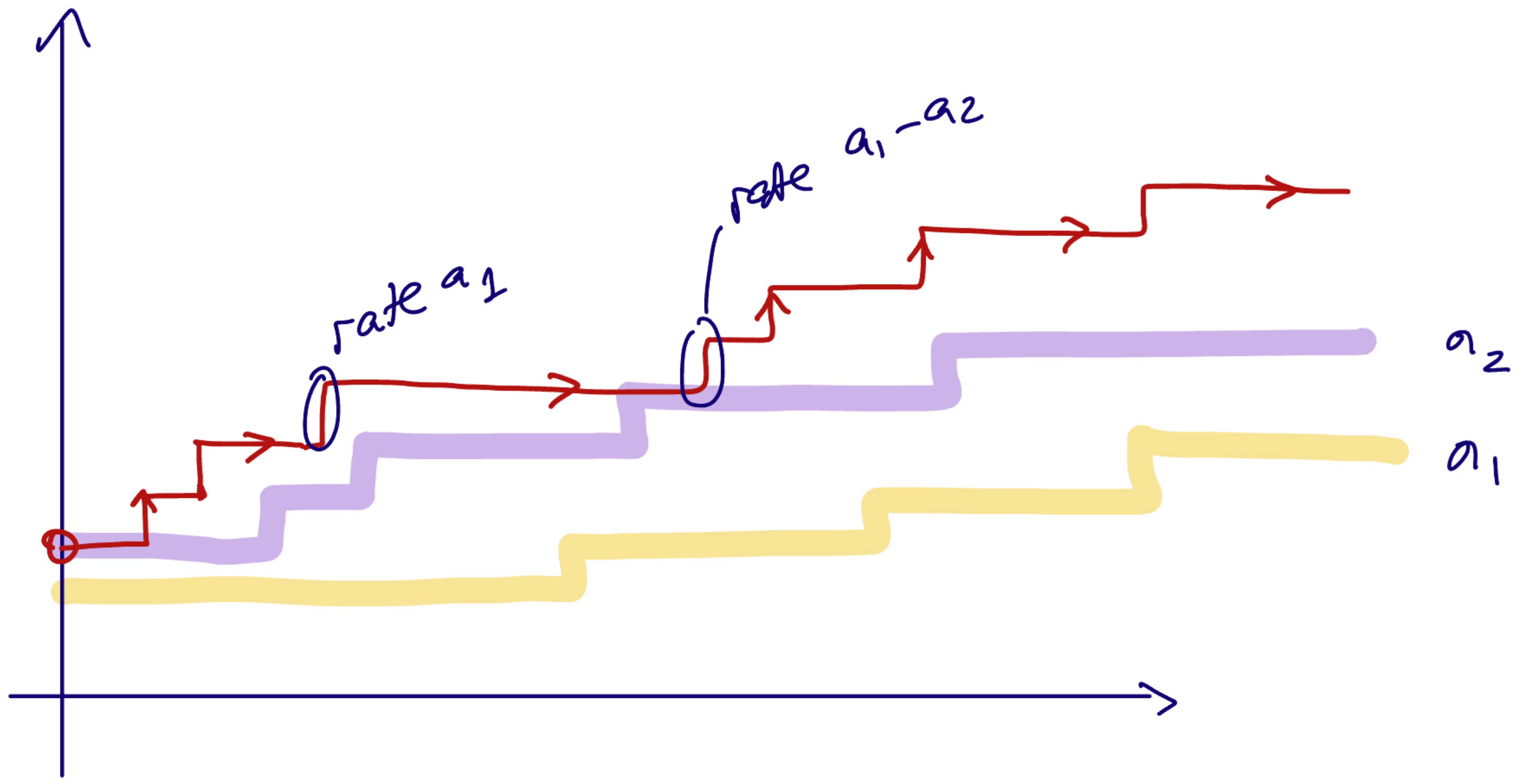
After time τ, we get Poisson walk of slope $\tau + 1$.

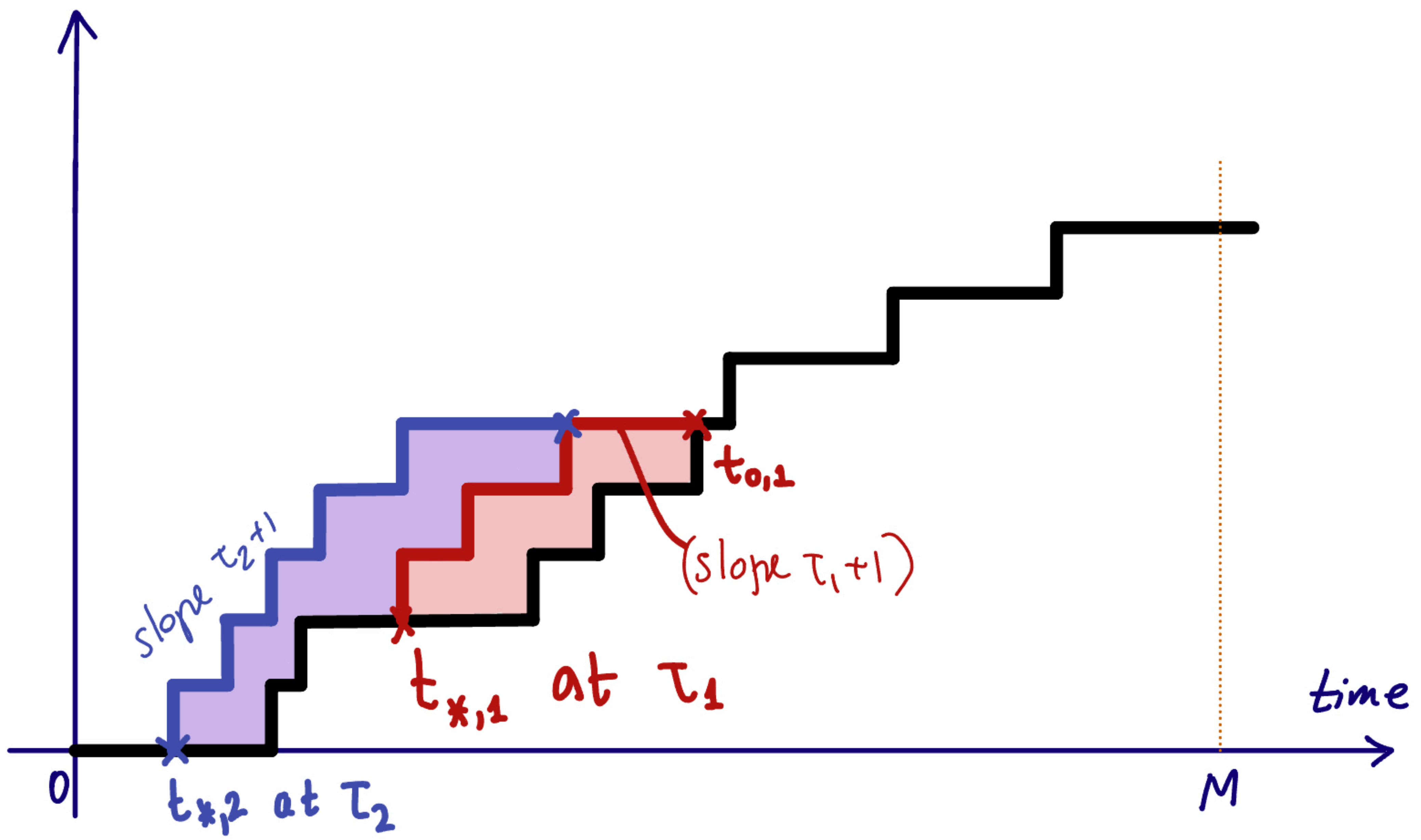
Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)





Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

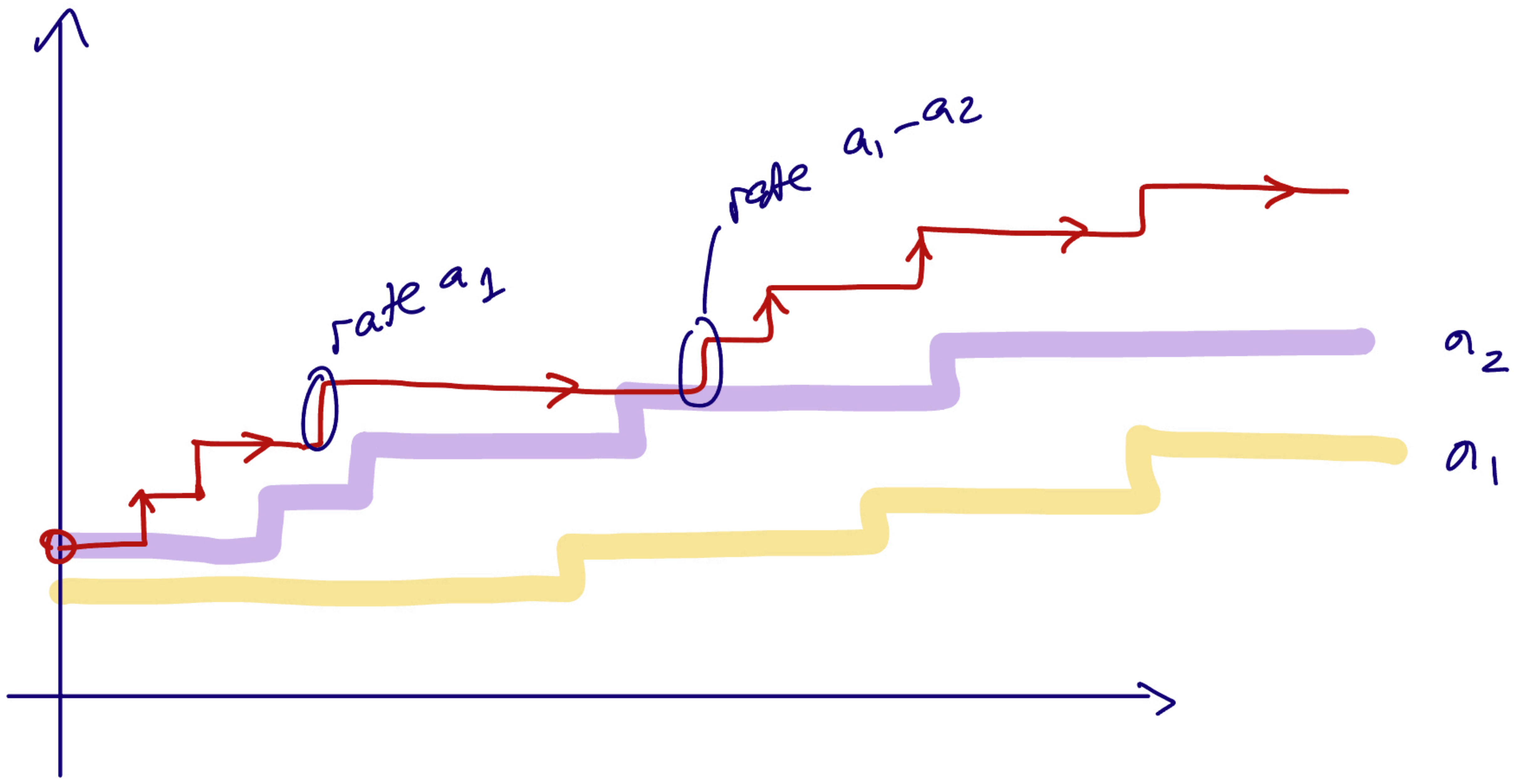
After time τ, we get Poisson walk of slope $\tau + 1$.

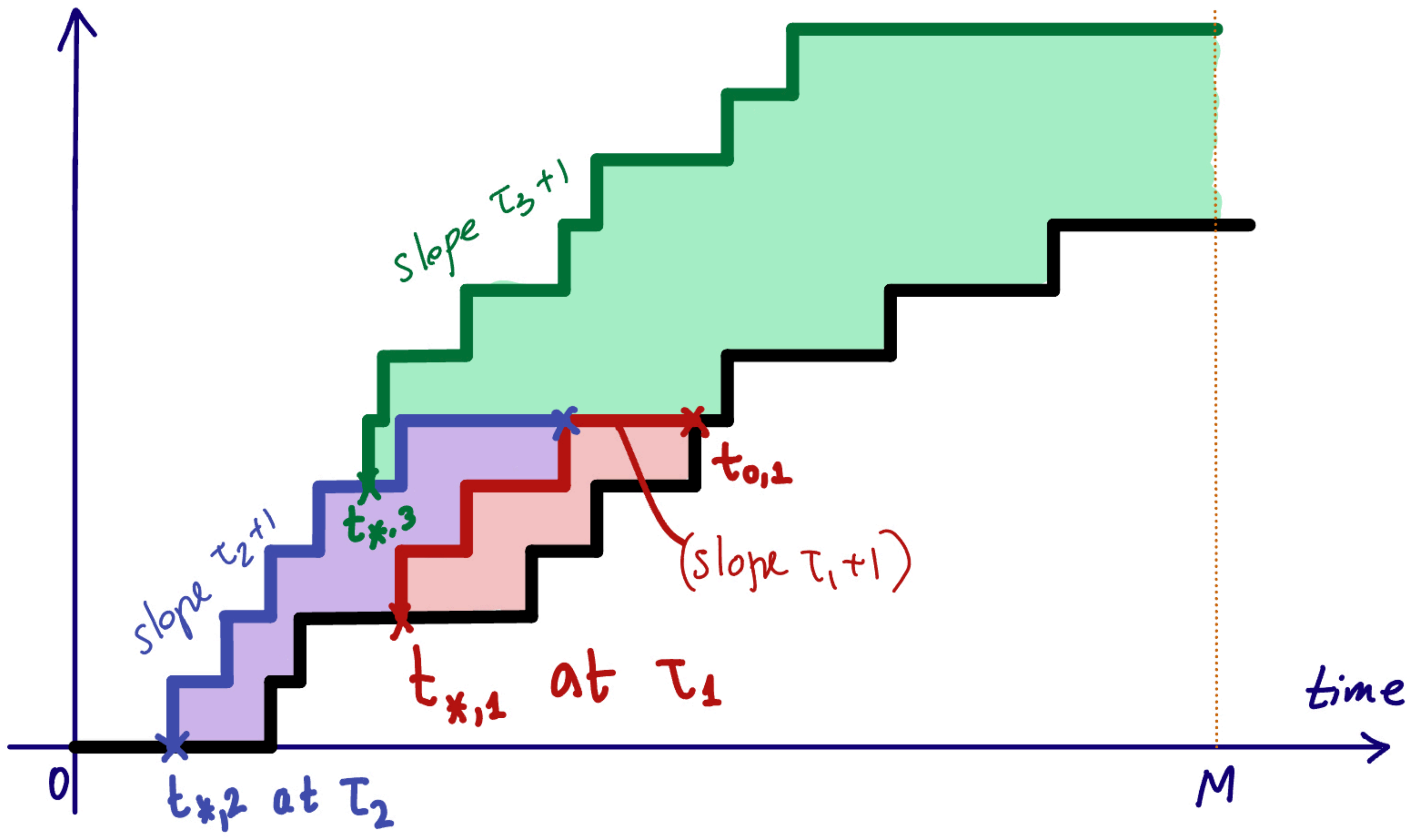
Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)





Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

After time τ, we get Poisson walk of slope $\tau + 1$.

Conclusions

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)
- Integrable (exactly solvable) systems possess Yang-Baxter equation and are solvable in the multiparameter setting

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)
- Integrable (exactly solvable) systems possess Yang-Baxter equation and are solvable in the multiparameter setting
- The multiparameter systems are in many ways symmetric in parameters

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)
- Integrable (exactly solvable) systems possess Yang-Baxter equation and are solvable in the multiparameter setting
- The multiparameter systems are in many ways symmetric in parameters
- Permuting parameters can be done in a *stochastic* way, that is, we can get nice couplings

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)
- Integrable (exactly solvable) systems possess Yang-Baxter equation and are solvable in the multiparameter setting
- The multiparameter systems are in many ways symmetric in parameters
- Permuting parameters can be done in a *stochastic* way, that is, we can get nice couplings
- We discover Lax equations for q-TASEP and TASEP (their role should be better explored)

Conclusions

- There are lots of interacting particle systems with interesting asymptotic behavior (Tracy-Widom fluctuations, KPZ universality, all that)
- Integrable (exactly solvable) systems possess Yang-Baxter equation and are solvable in the multiparameter setting
- The multiparameter systems are in many ways symmetric in parameters
- Permuting parameters can be done in a *stochastic* way, that is, we can get nice couplings
- We discover Lax equations for q-TASEP and TASEP (their role should be better explored)
- We also construct neat couplings for the whole trajectories, and get a new result about the Poisson process

Thank you for your attention!

For questions and remarks