Free-fermionic Schur functions

Slava Naprienko

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Solvable Lattice Seminar

Outline

I) Schur functions $s_\lambda(x)$ and supersymmetric Schur functions $s_\lambda(x/y)$

II) Factorial (double) (supersymmetric) Schur functions $s_\lambda(x \| a)$ and $s_\lambda(x/y \| a)$ and double Schur functions $\hat{s}_\lambda(x \| b)$

III) Free-fermionic Schur functions $s_\lambda(x \| a/b)$ and $s_\lambda(x/y \| a/b)$.

Takeaway

Lattice models are powerful:

* combinatorial (tableaux, Gelfand-Tsetlin patterns, Lusztig data etc.)
* lattice paths (LGV lemma)
* functional (YB equation, algebraic Bethe Ansatz)

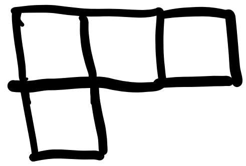
I. Classical Schur and supersymmetric Schur functions

Schur functions: $s_\lambda(x)$

supersymmetric Schur: $s_\lambda(x/y)$.

1. Tableaux.

Semistandard Young tableaux for Schur functions $s_\lambda(x)$:

Example shape  , $n=2$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array}$$

$$x_1^3x_2\qquad x_1^2x_2^2\qquad x_1x_2^3$$

$$S_{\begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ \\\cline{1-1}\end{array}}(x_1,x_2)=$$

$$=x_1^3x_2+x_1^2x_2^2+x_1x_2^3.$$

Super (or primed) tableaux for supersymmetric Schurs:

Fill with $1<1'<2<2'<\ldots<n<n'$

- rows and columns are weakly increasing
- no repeated symbol i in columns

– no repeated symbol i' in rows.

Ex

$\begin{ytableau} 1 & 1 \\ 1' \end{ytableau}$	$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}$	$\begin{ytableau} 1 & 1 \\ 2' \end{ytableau}$	$\begin{ytableau} 1 & 1' \\ 1' \end{ytableau}$
$x_1^2 y_1$	$\underline{x_1^2 x_2}$	$x_1^2 y_2$	$x_1 y_1^2$
$\begin{ytableau} 1 & 1' \\ 2 \end{ytableau}$	$\begin{ytableau} 1 & 1' \\ 2' \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 1 \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 1' \end{ytableau}$
$x_1 x_2 y_1$	$x_1 y_1 y_2$	$\underline{x_1^2 x_2}$	$x_1 x_2 y_1$
$\begin{ytableau} 1 & 2 \\ 2' \end{ytableau}$	$\begin{ytableau} 1' & 2 \\ 1' \end{ytableau}$	$\begin{ytableau} 1' & 2 \\ 2 \end{ytableau}$	$\begin{ytableau} 1' & 2 \\ 2' \end{ytableau}$
$x_1 x_2 y_2$	$x_2 y_1^2$	$x_2^2 y_1$	$x_2 y_1 y_2$
$\begin{ytableau} 1' & 2' \\ 1' \end{ytableau}$	$\begin{ytableau} 1' & 2' \\ 2 \end{ytableau}$	$\begin{ytableau} 1' & 2' \\ 2' \end{ytableau}$	
$y_1^2 y_2$	$x_2 y_1 y_2$	$y_1 y_2^2$	

2. Relations / Properties

- $S_\lambda(x/y)\big|_{y=0} = S_\lambda(x)$
- $S_\lambda(x/y) = S_{\lambda'}(y/x)$
- $S_\lambda(x/y)\big|_{x=0} = S_{\lambda'}(y)$
- $S_{\lambda/\mu}(x/y) =$

$$= \sum_{\nu} S_{\nu/\mu}(x) S_{\lambda'/\nu'}(y).$$

- $s_\lambda(x|y)\Big|_{\substack{x_1=t\\ y_1=-t}}$ does not depend on t.

- $s_\lambda(x|y)\Big|_{x_n=0} = s_\lambda(x'|y)$

- $s_\lambda(x|y)\Big|_{y_n=0} = s_\lambda(x|y')$

Factorization $\lambda=(n^n+\tau)\cup\eta$:

- $s_\lambda(x|y)=s_\tau(x)s_{\eta'}(y)\prod_{i,j}(x_i+y_j)$

3. Jacobi-Trudi identity

If $h_r(x|y) = s_{(r)}(x|y)$

$h_r(x) = s_{(r)}(x)$, then

- $s_\lambda(x|y) =$

$$= \det\left(h_{\lambda_i + i - j}(x|y)\right)$$

- $s_\lambda(x) =$

$$= \det\left(h_{\lambda_i + i - j}(x)\right)$$

The dual identity,
Giambelli identity,
Ribbon formula:

- $s_\lambda(x/y) = \det(s_\#(x/y))$

- $s_\lambda(x) = \det(s_\#(x))$

4. Explicit formulas

- **Weyl formula** for $s_\lambda(x)$:

$$A_\alpha = \det\left(x_i^{\alpha_j}\right),$$

$\delta = (n-1, n-2, \dots, 0)$. Then

$$s_\lambda(x) = \frac{A_{\lambda+\delta}}{A_\delta} = \frac{A_{\lambda+\delta}}{\prod_{i<j}(x_i - x_j)} =$$

$$= \frac{\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \sigma(x^\delta \cdot x^\lambda)}{\Delta(x)}.$$

- Sergeev-Pragacz formula

$$S_\lambda(x/y) = \frac{\displaystyle\sum_{\sigma \in S_n^x \times S_n^y} \operatorname{sgn}(\sigma) \cdot \sigma\Big(x^\delta \cdot y^\delta \prod_{(i,j)\in\lambda} (x_i + y_j)\Big)}{\Delta(x)\Delta(y)}$$

5. Cauchy identities

Schur + Schur:

- $$\sum_{\lambda} s_\lambda(x) s_\lambda(z) = \prod_{i,j} \frac{1}{1 - x_i z_j}$$

Schur + Schur':

- $$\sum_{\lambda} s_\lambda(x) s_{\lambda'}(z) = \prod_{i,j} (1 + x_i z_j)$$

Super+Schur:

- $$\sum_{\lambda} s_\lambda(x/y)s_\lambda(z) = \prod_{i,j} \frac{1+y_i z_j}{1-x_i z_j}$$

Super+super:

- $$\sum_{\lambda} s_\lambda(x/y)s_\lambda(z/w) = \prod_{i,j} \frac{1+x_i w_j}{1-x_i z_j}\frac{1+y_i z_j}{1-y_i w_j}.$$

II. Factorial Schur and factorial supersymmetric Schur functions.

$x = (x_1, \dots, x_n)$ $\quad a = (a_i)_{i \in \mathbb{Z}}$

$y = (y_1, \dots, y_n)$

factorial Schur: $s_\lambda(x \,\|\, a)$

factorial super Schur: $s_\lambda(x/y \,\|\, a)$.

1. Tableaux

Semistandard tableaux for factorial Schur f-ns:

Fill shape λ with

$1 < 2 < \ldots < n$ s.t.

- weakly increasing in rows
- strictly increasing in columns

$\text{content}(\alpha) = j - i$.

collect $(x_{T(\alpha)} - c(\alpha))$ for each box α with entry $T(\alpha)$.

Ex Shape $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$, $n=2$.

$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2 \\ \cline{1-1}\end{array}$ $(x_1-a_0)(x_1-a_{-1})(x_1-a_{-2})$
(x_2-a_1)

$\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2 \\ \cline{1-1}\end{array}$ $(x_1-a_0)(x_1-a_{-1})(x_2-a_{-2})$
(x_2-a_1)

$\begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 2 \\ \cline{1-1}\end{array}$ $(x_1-a_0)(x_2-a_{-1})(x_2-a_{-2})$
(x_2-a_1)

Ex $\begin{array}{|c|c|}\hline 1&1\\ \hline\end{array}$ $(x_1-a_0)(x_1-a_{-1})$

$\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}$ $(x_1-a_0)(x_2-a_{-1})$

$\begin{array}{|c|c|}\hline 2&2\\ \hline\end{array}$ $(x_2-a_0)(x_2-a_{-1})$

Supertableaux for $S_\lambda(x/y \| a)$.

$$1 < 1' < 2 < 2' < \dots < n < n'$$

as before, Collect

$(x_{\pi(\alpha)} - a_{c(\alpha)})$ for unprimed entries

$(y_{\pi(\alpha)} + a_{c(\alpha)})$ for primed entries,

<u>Ex</u> $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1' \\ \cline{1-1}\end{array}$ $\quad (x_1 - a_0)(x_1 - a_{-1})$
$(y_1 - a_1)$

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad (x_1 - a_0)(x_1 - a_{-1})$
$(y_2 - a_1)$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2' & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad (x_1-a_0)(x_1-a_{-1})(y_2-a_1)$$

$$\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 1' & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad (x_1-a_0)(y_1-a_{-1})(y_1-a_1)$$

... And so on.

2. Relations / Properties

- $s_\lambda(x/y\,||\,0) = s_\lambda(x/y)$
- $s_\lambda(x\,||\,0) = s_\lambda(x)$.

- $S_\lambda(x/y\|a)\big|_{y=-a} = S_\lambda(x\|a)$

- $S_\lambda(x/y\|a) = S_{\lambda'}(y/x\|a')$,

where $a'_i = -a_{-i+1}$.

- $S_\lambda(x/y\|a)\big|_{x=-a'} = S_{\lambda'}(y\|a)$

- $S_{\lambda/\mu}(x/y\|a)$ $\qquad\qquad \mathbb{Z}$

$$= \sum_{\nu} \tilde{S}_{\nu/\mu}(x \| a)\, \tilde{S}_{\lambda'/\nu'}(y \| a')$$

- $S_\lambda(x/y \| a)\Big|_{\substack{x_1 = t \\ y_1 = -t}}$ does not depend on t

- $S_\lambda(x/y \| a)\Big|_{x_n = -a'_n} = S_\lambda(x'/y \| a)$

- $S_\lambda(x/y \| a)\Big|_{y_n = -a_n} = S_\lambda(x/y' \| a)$

Factorization $\lambda = (n^n + \tau) \cup \eta$:

$$s_\lambda(x/y\|a) = s_\tau(x\|(a_{i+n}))\, s_{\eta'}(y) \prod_{i,j}(x_i + y_j)$$

3. Jacobi-Trudi identity

If $h_r(x/y\|a) = s_{(r)}(x/y\|a)$

$h_r(x\|a) = s_{(r)}(x\|a)$, then

let $\tau^s(a_i) = a_{i-s}$.

- $S_\lambda(x/y\|a) =$
$$= \det\left(h_{\lambda_i+i+j}(x/y\|\tau^{-j+1}a)\right).$$

- $S_\lambda(x\|a) =$
$$= \det\left(h_{\lambda_i+i-j}(x\|\tau^{-j+1}a)\right)$$

The dual identity,
Giambelli identity,
Ribbon formula are similar.

4. Explicit formulas

- Weyl formula for $s_\lambda(x)$:

Let $(x|a)^0 = 1$, and

$(x|a)^r = (x - a_n) \dots (x - a_{n-r+1})$

$$A_\alpha = \det\left((x_i|a)^{\alpha_j}\right)$$

$\delta = (n-1, n-2, \dots, 0)$. Then

$$s_\lambda(x) = \frac{A_{\lambda+\delta}}{A_\delta} = \frac{A_{\lambda+\delta}}{\prod_{i<j}(x_i - x_j)} =$$

$$= \frac{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \sigma(f_\lambda(x||a))}{\Delta(x)}.$$

- <u>Sergeev-Pragacz formula</u>

$S_\lambda(x/y||a) =$

$$\frac{\sum_{\sigma \in S_n^x \times S_n^y} \operatorname{sgn}(\sigma) \cdot \sigma\left(f_\lambda(x/y||a)\right)}{\Delta(x)\Delta(y)}$$

5. Cauchy identities.

A problem! We need dual Schur functions $\hat{s}_\lambda(x \| a)$. could be defined by

- Determinant (Weyl formula)

$$\frac{\det\left(\dfrac{x_i^{\lambda_j+j-1}(1-a_{n-\lambda_j-j})\cdots(1-a_{1-\lambda_j-j})}{(1-x_i a_0)(1-x_i a_{-1})\cdots(1-x_i a_{-r+1})}\right)}{\Delta(x)}$$

- tableaux with rational weights.

Then we have Cauchy identity:

Schur + dual:

- $$\sum_{\lambda} s_\lambda(x\|a)\hat{s}_\lambda(z\|a) = \prod_{i,j} \frac{1-a_i y_j}{1-x_i y_j}$$

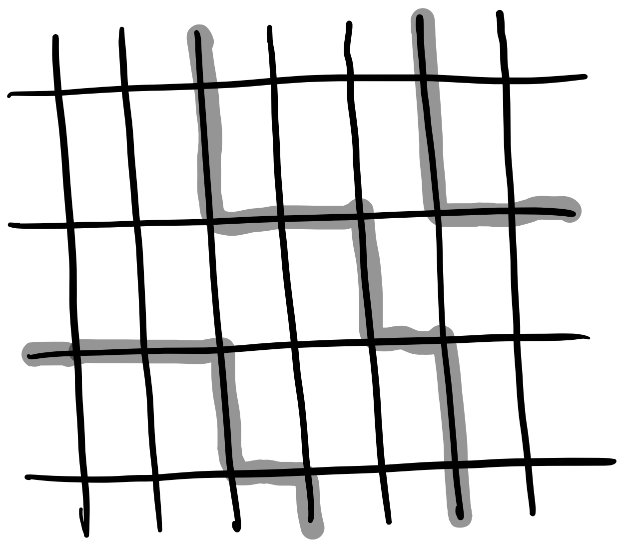
Super + dual:

- $$\sum_{\lambda} s_\lambda(x/y\|a)\hat{s}_\lambda(z\|a) = \prod_{i,j} \frac{1+y_i z_j}{1-x_i z_j}.$$

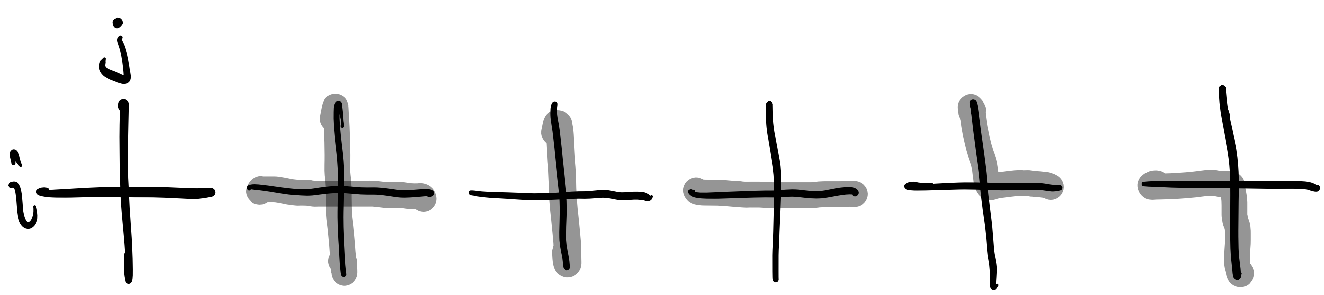
But what are these dual Schur functions?

III. Free-fermionic Schur functions

Recall the six-vertex model:



Let $x = (x_1, \dots, x_n)$, $a = (a_i)_{i \in \mathbb{Z}}$
$y = (y_1, \dots, y_n)$, $b = (b_i)_{i \in \mathbb{Z}}$.



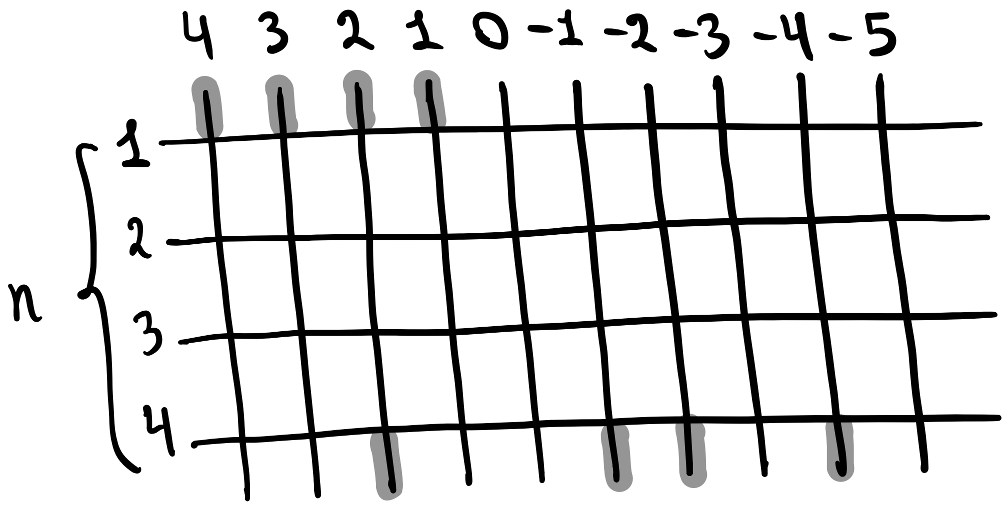
$1 - x_i b_j$ $\quad y_i + a_j$ $\quad 1 + y_i b_j$ $\quad x_i - a_j$ $\quad 1 - a_j b_j$ $\quad x_i + y_i$

These weights are free-fermionic:

$$a_1a_2+b_1b_2=c_1c_2,$$

and integrable (Yang-Baxter equations the rows and columns).

For $\alpha=(\alpha_1,\dots,\alpha_k)$, consider G^n_α:



Let $\delta=(n,n-1,\ldots,1)$. Then for a partition λ, define free-fermionic Schur functions:

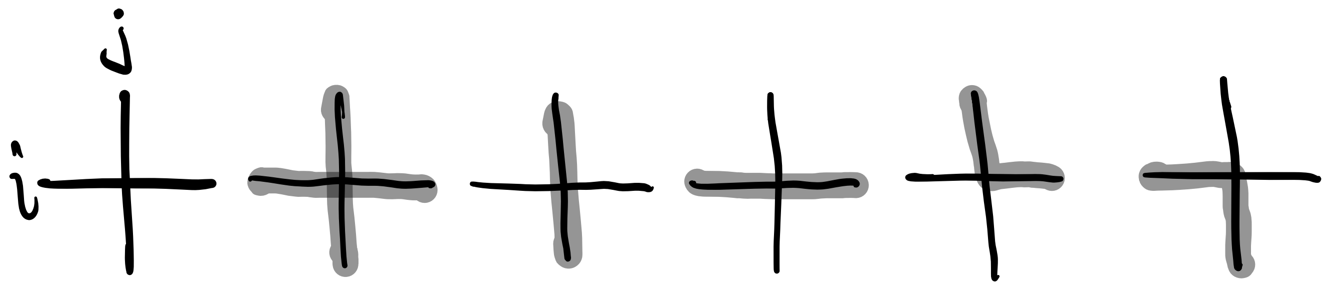
$$\boxed{S_\lambda(x/y\,\|\,a/b)=\frac{G^n_{\lambda+\delta}}{G^n_\delta}}$$

Theorem (N.) [Specializations]

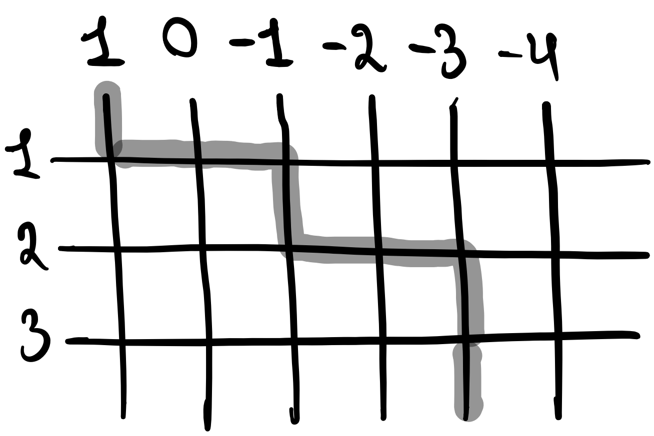
- $S_\lambda(x/0\,\|\,0/0)=S_\lambda(x)$ classical Schur
- $S_\lambda(x/y\,\|\,0/0)=S_\lambda(x/y)$ supersymmetric
- $S_\lambda(x/(-a)\,\|\,a/0)=S_\lambda(x\|a)$ factorial
- $S_\lambda(x/y\,\|\,a/0)=S_\lambda(x/y\|a)$ factorial supersymmetric
- $S_\lambda(x/0\,\|\,0/b)=\hat{S}_\lambda(x\|b)$ dual Schur

Proof of special case (sketch)

Let $b=0$. Then



$1-x_ib_j$	y_i+a_j	$1+y_ib_j$	x_i-a_j	$1-a_jb_j$	x_i+y_i
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
1	y_i+a_j	1	x_i-a_j	1	x_i+y_i



$\boxed{1}\boxed{1'}\boxed{2}\boxed{2'}$

Note that $x_i+y_j =$

$=(x_i-a_j)+(y_i+a_j)$.

„Bijection" with supertableaux.

1. Hence, we have a <u>tableaux</u> description of $s_\lambda(x/y \| a/b)$.

2. <u>Relations</u> / <u>Properties</u>

Let $s_\lambda(x \| a/b) = s_\lambda(x/y \| a/b)\big|_{y=-a}$.

<u>Proposition</u> [Factorization]

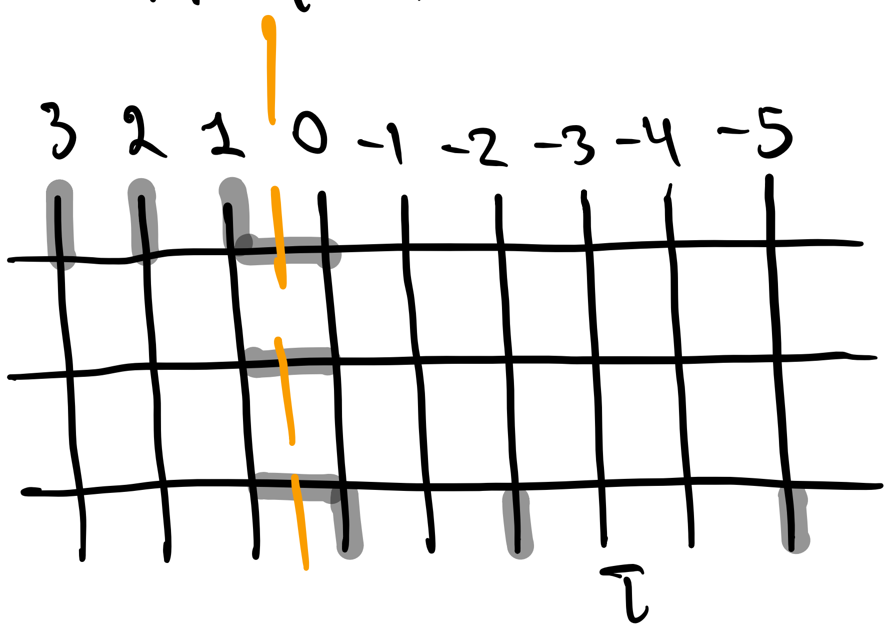
Let $\lambda = (n^n + \tau) \cup \eta$. Then

$$s_\lambda(x/y \| a/b) =$$

$$= s_\tau(x \| a_{i+n}/b_{i+n})\, s_{\eta'}(y \| a'/b') \prod_{i,j} (x_i + y_j)$$

Proof (sketch) (case $\eta=0$). $n=3$, $\lambda=(6,4,3)$

$$\lambda=(3,3,3)+(3,1,0)$$



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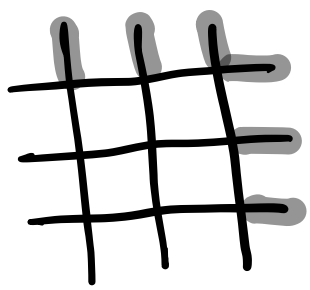
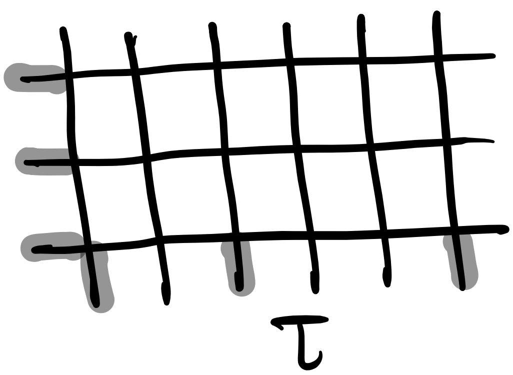
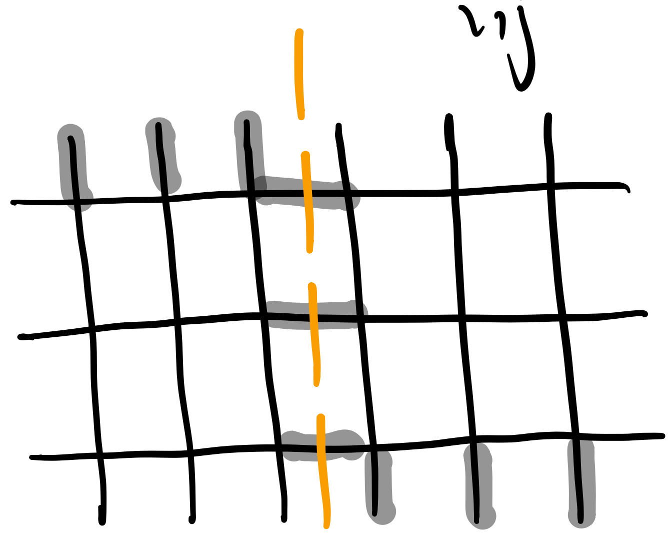
$=$  $\times$ 

$=$

$$= \prod_{i,j}(x_i+y_j)\cdot S_\tau(x\,||\,a/b).$$

Special case:

$$s_{(n^n)}(x/y\,||\,a/b)=\prod_{i,j}(x_i+y_j)$$



3. Jacobi-Trudi identity

Let $h_r(x/y\,||\,a/b)=s_{(r)}(x/y\,||\,a/b)$.

$e_r(x/y\,||\,a/b)=s_{(1^r)}(x/y\,||\,a/b)$.

Thm. (N.) [Determinant expressions]

$$s_\lambda(x/y\,\|\,a/b) =$$

$$= \det\left(h_{\lambda_i+i-j}(x/y\,\|\,\tau^{-j+1}a/\tau^{-j+1}b) \right)$$

$$= \det\left(e_{\lambda'_i-i+j}(x/y\,\|\,\tau^{j-1}a/\tau^{j-1}b) \right)$$

$$= \det\left(s_{(\alpha_i|\beta_j)}(x/y\,\|\,a/b) \right)$$

$$= \det\left(s_{[\alpha_i|\beta_j]}(x/y\,\|\,a/b) \right).$$

4. Cauchy identity.

Recall that

$$\hat{S}_\lambda(z\|a) = S_\lambda(z/0\|0/a).$$

Then we had

$$\sum_\lambda S_\lambda(x/y\|a/0)S_\lambda(z/0\|0/a) = \prod_{i,j}\frac{1+y_iz_j}{1-x_iz_j}$$

Thm (N.) [Cauchy identities]

Full + degeneration:

$$\sum_\lambda S_\lambda(x/y\,||\,a/b)\,S_\lambda(z/(-b)\,||\,b/a) =$$

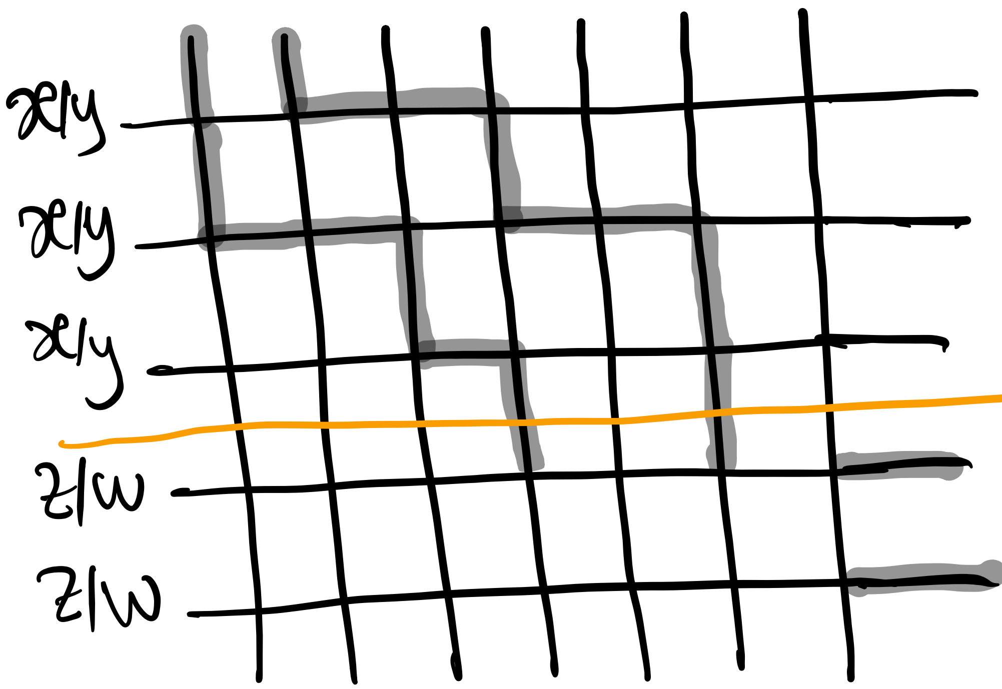
$$= \prod_{i,j} \frac{1+z_iy_j}{1-z_ix_j}\,\frac{1-b_ix_j}{1+b_iy_j}$$

$\mathbb{Z}$

Full + Full:

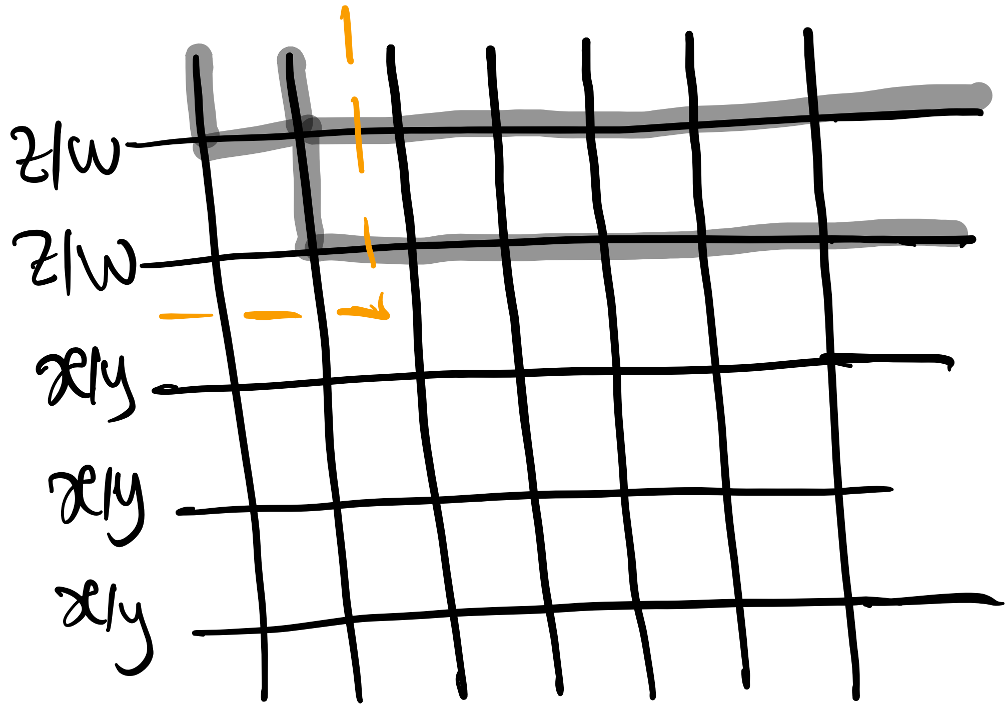
$$\boxed{\sum_\lambda S_\lambda(x/y\,||\,a/b)\,S_\lambda(z/w\,||\,b/a) = \prod \frac{1+z_iy_j}{1-z_ix_j}\,\frac{1+w_ix_j}{1-w_iy_j}.}$$

Proof (sketch)



$$= \sum_{\substack{\lambda \\ |\lambda_1| \le M}} s_\lambda(x/y\,||\,a/b)\, f_\lambda(z/w\,||\,a/b)$$

On the other hand, we use the Yang-Baxter equation:



And compute explicitly the DWBC.

5. Explicit expressions.

Thm (N.) [Weyl formula]

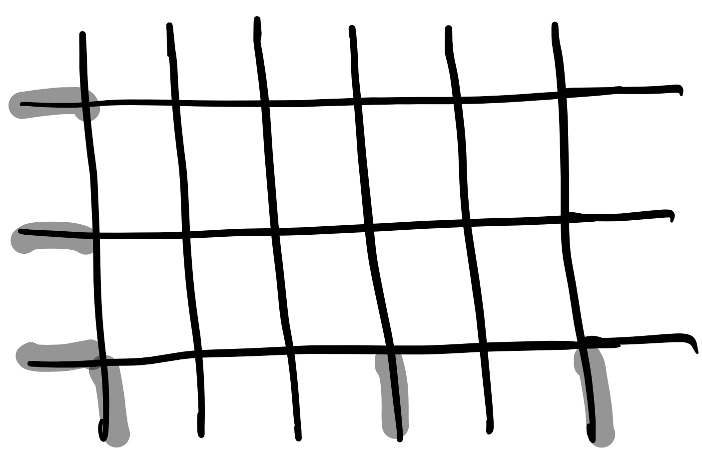
Let $(x\,|\,a/b)^{\circ} = 1$, and

$$(x|a/b)^r = \frac{(x-a_0)\dots(x-a_{-r+1})}{(y-b_0)\dots(y-b_{-r+1})},$$

Let $A_\alpha = \det\left((x_i|a/b)^{\alpha_j}\right)$.

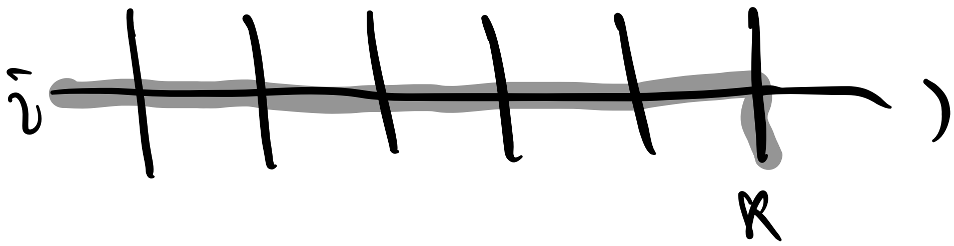
Then $s_\lambda(x\|a/b) = \dfrac{A_{\lambda+\delta}}{A_\delta}$,

Proof



We use the YB equation to express the partition

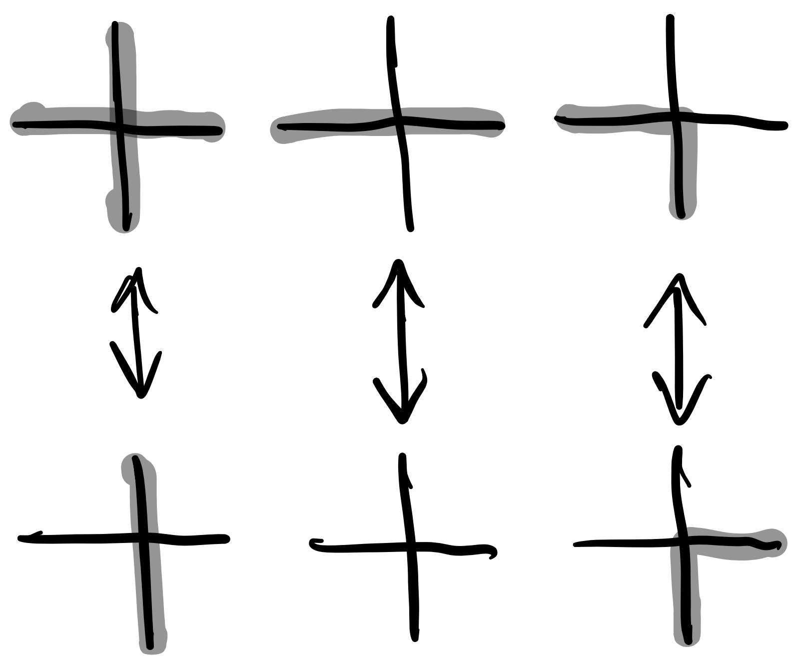
Function in terms of 1-row:



Which equals to $(x_i|a/b)^k$.

Bonus: Lattice Involutions

1



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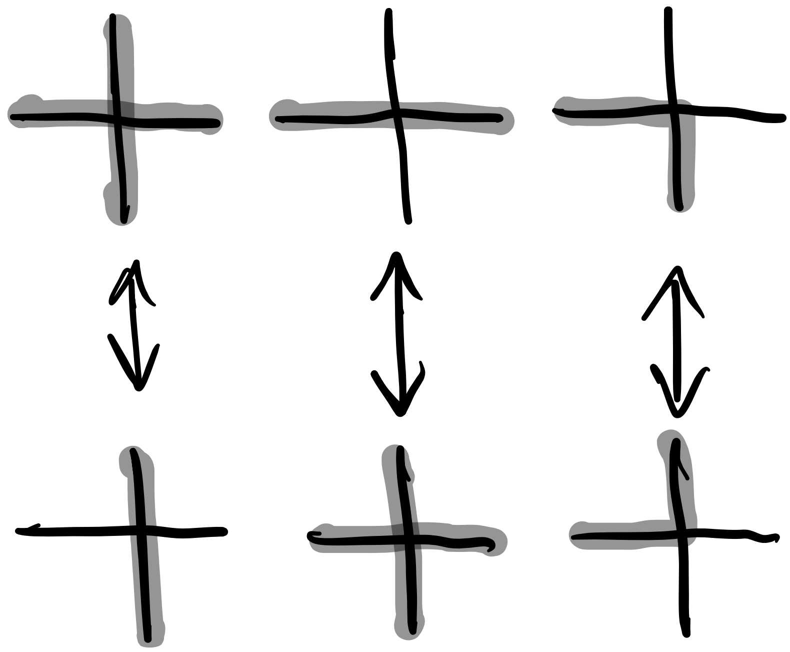

Thank you!

P.S. The symmetric function catalog by Per Alexandersson was very useful!