Bijective proof of Cauchy and Littlewood Identities for q-Whittaker polynomials

Solvable Lattice Models Seminar

Matteo Mucciconi — based on a joint work with Takashi Imamura and Tomohiro Sasamoto

Cauchy Identities

$$\sum_{\lambda} b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) = \prod_{k\geq 0} \prod_{i,j} \frac{1 - t x_i y_j q^k}{1 - x_i y_j q^k}$$

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- The Cauchy Identity can be used to define $P_\lambda(x;q,t)$ [Macdonald's book]
- There are combinatorial definitions of Macdonald polynomials: tableaux expansion [Haglund-Haiman-Loher], Vertex models [Cantini-De Gier-Wheeler, Borodin-Wheeler, Garbali-Wheeler,...], alcove walks [Ram-Yip],...

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    - $t = 0$ YES : [Imamura-M-Sasamoto'21]

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  - $q = t$ YES : Cauchy Identity for Schur polynomials
  - $t = 0$ YES : [Imamura-M-Sasamoto'21]
  - $q, t \neq 0$???

Plan

- We review the RSK correspondence and prove the CI for Schur polynomials
- We highlight major features and symmetries (Greene invariants, crystals, …) of the RSK
- Then, we generalize these features to prove the Cauchy Identity for q-Whittaker polynomials
- From the bijective proof we deduce many more identities (Littlewood identities, correspondence skew Schur/q-Whittaker)

Cauchy Identity for Schur polynomials

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$$\sum_{\lambda} \sum_{P,Q \in \mathrm{SSYT}(\lambda)} x^P y^Q = \sum_{M \in \mathbb{M}_{n \times n}} \prod_{i,j=1}^{n} (x_i y_j)^{M_{i,j}}$$

Bijective proof: $(P, Q) \overset{\text{RSK}}{\longleftrightarrow} M$

RSK correspondence

$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

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$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow ?, ?$$

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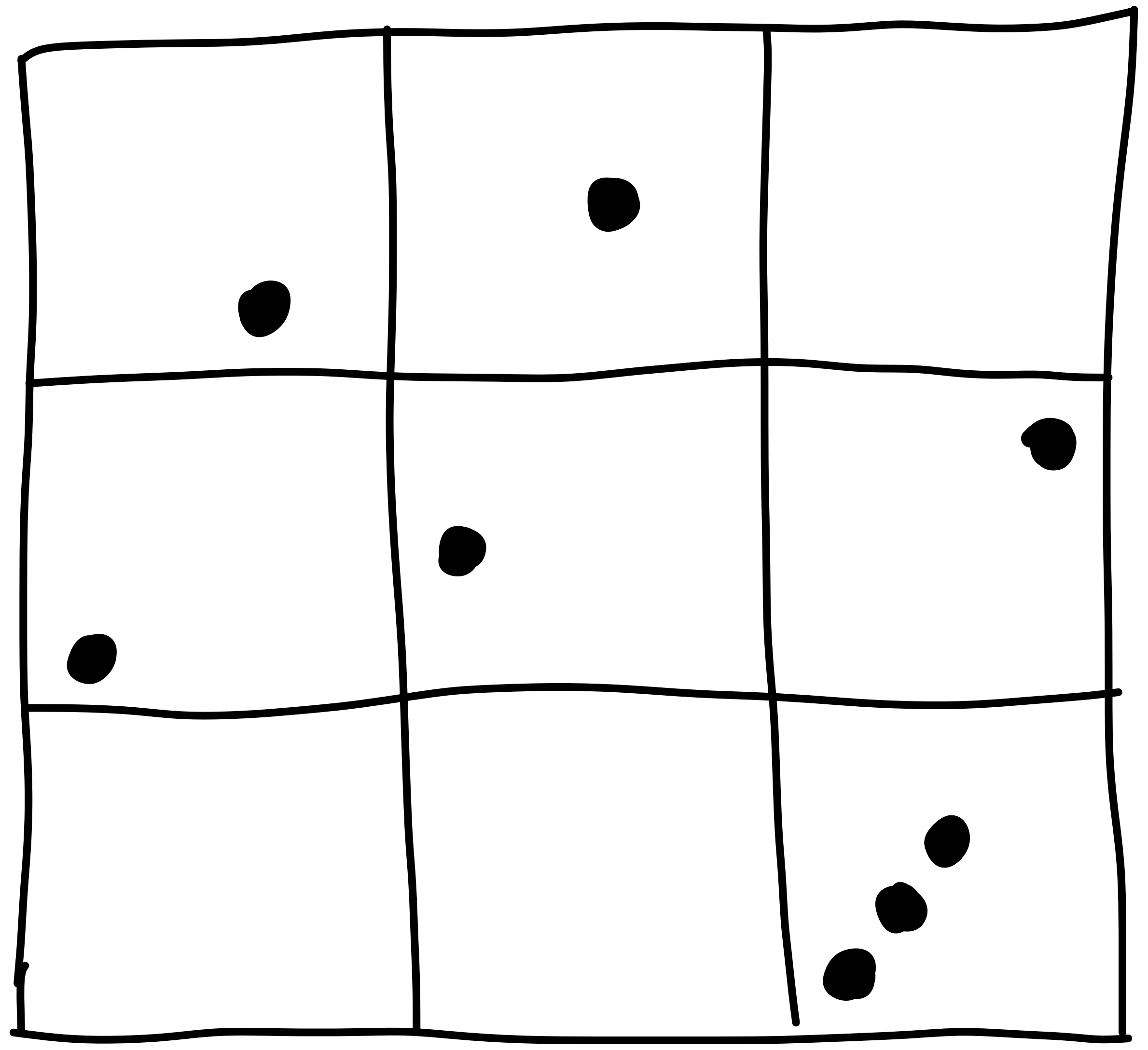

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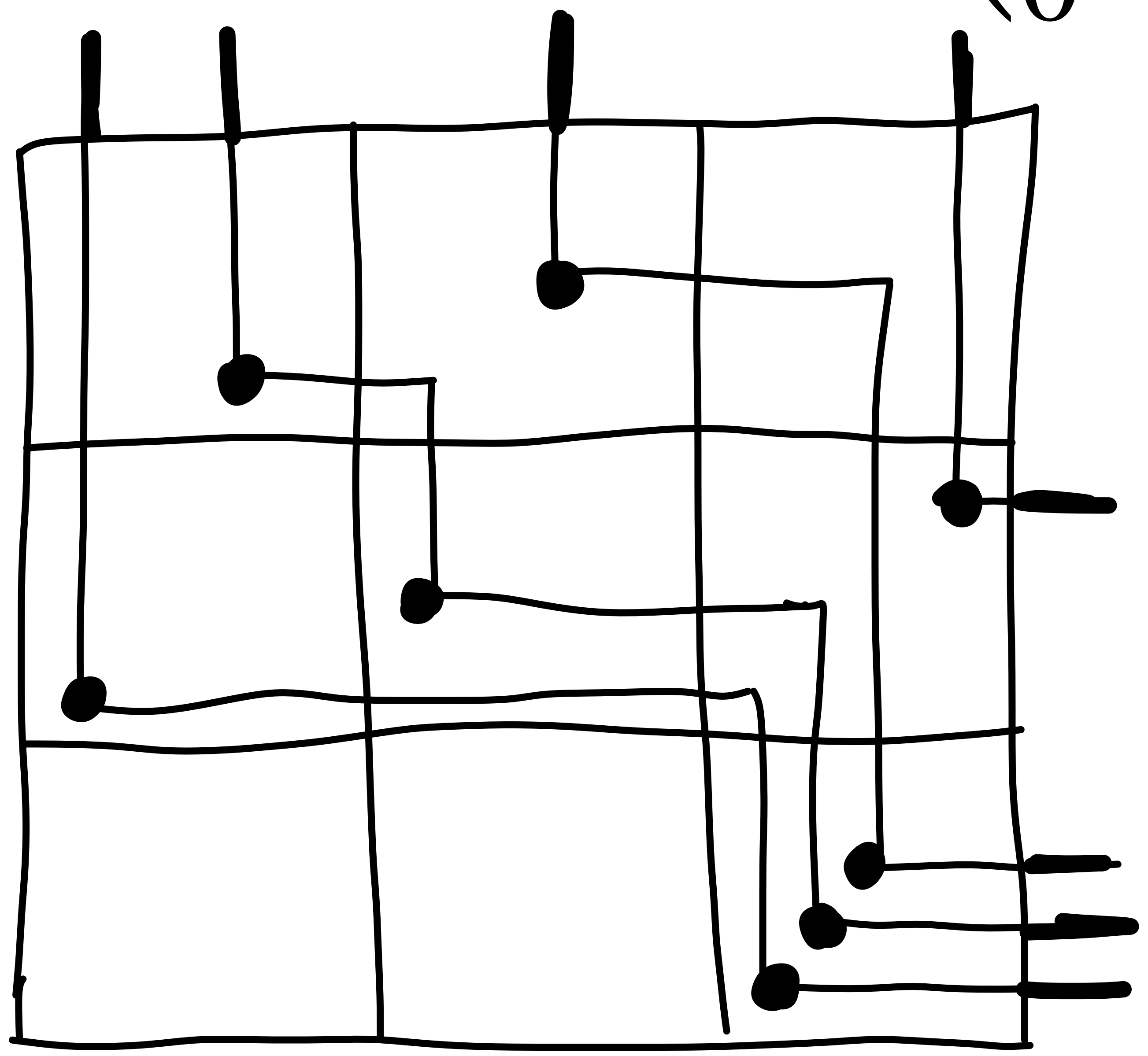

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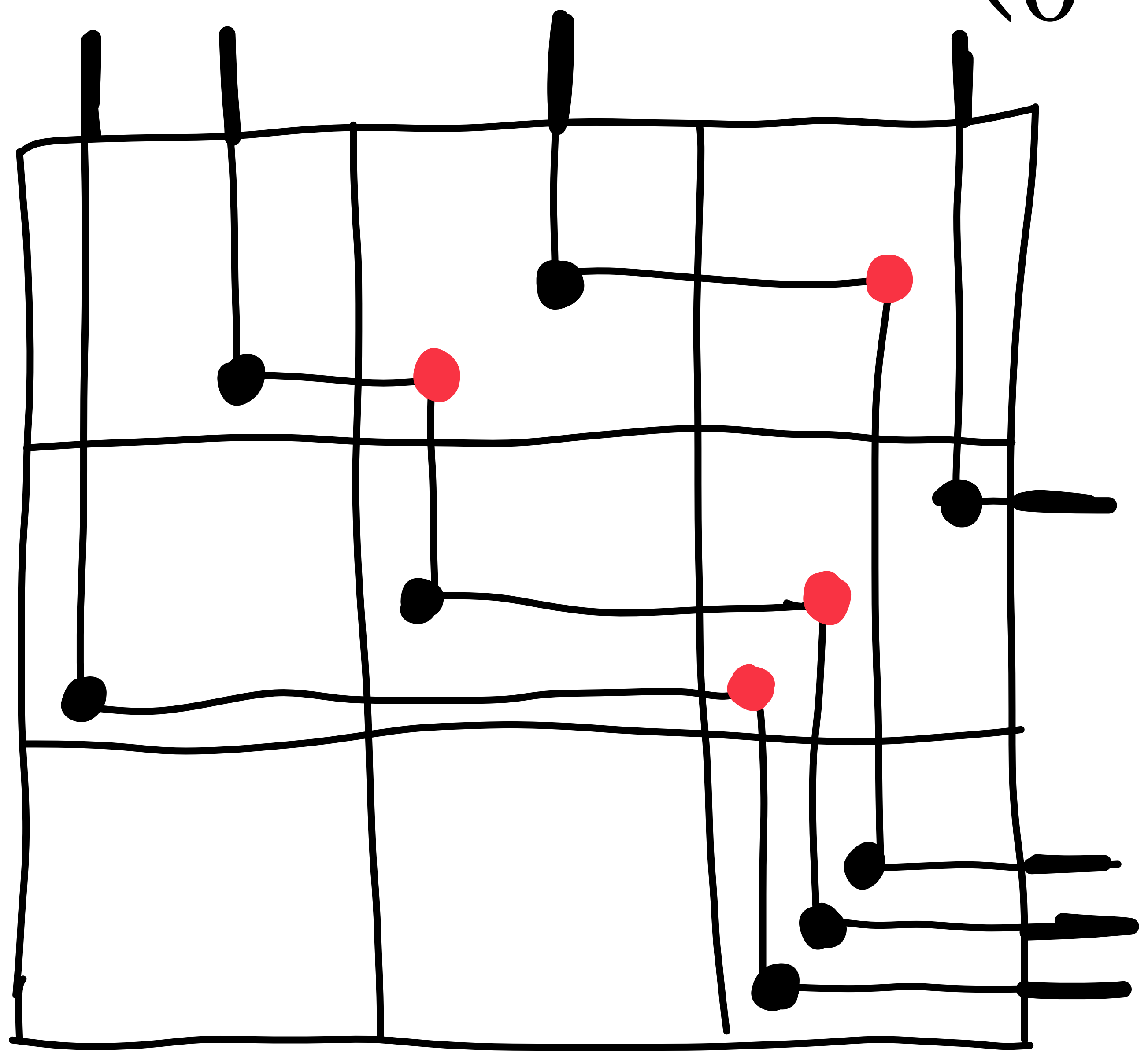

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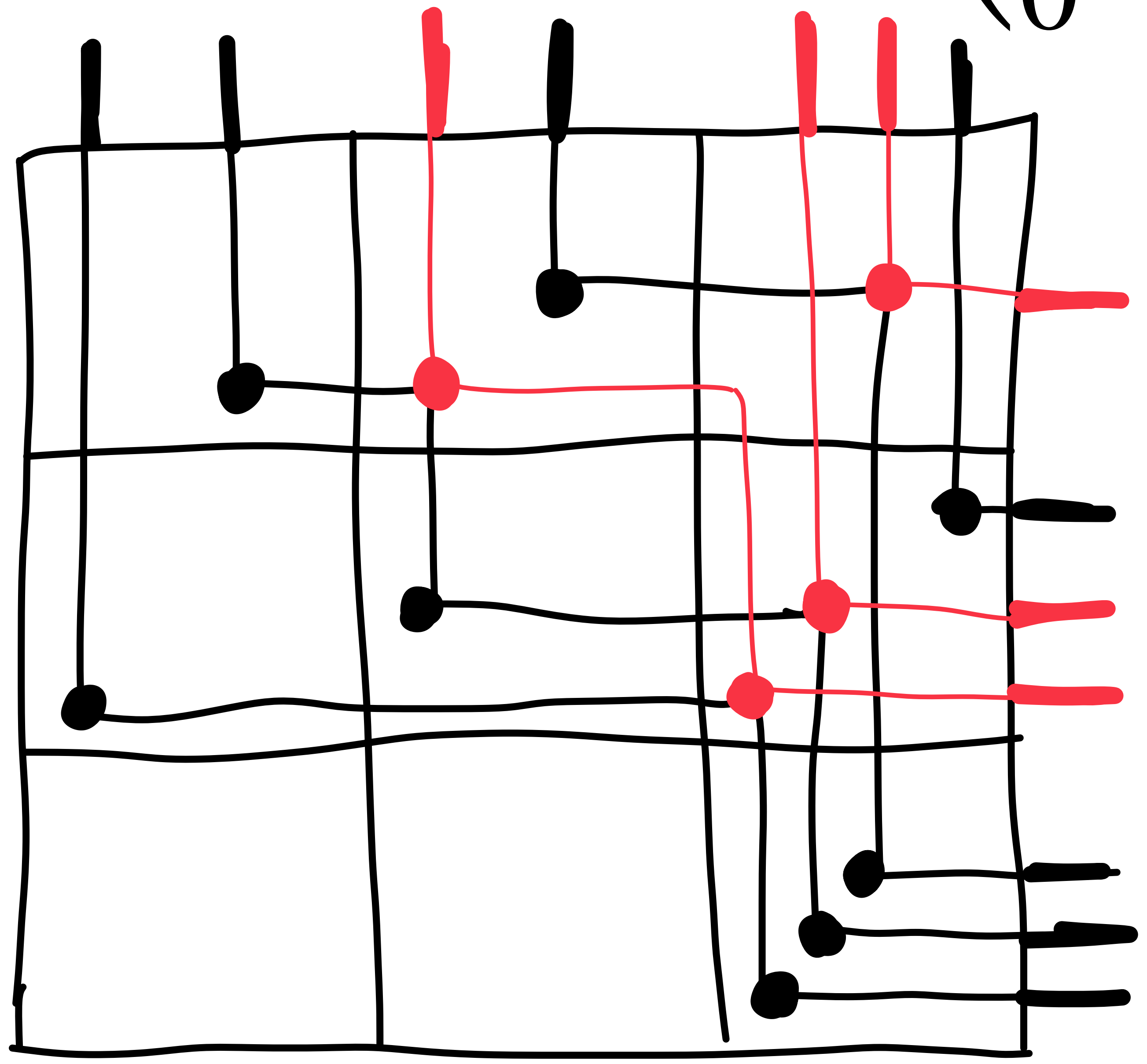

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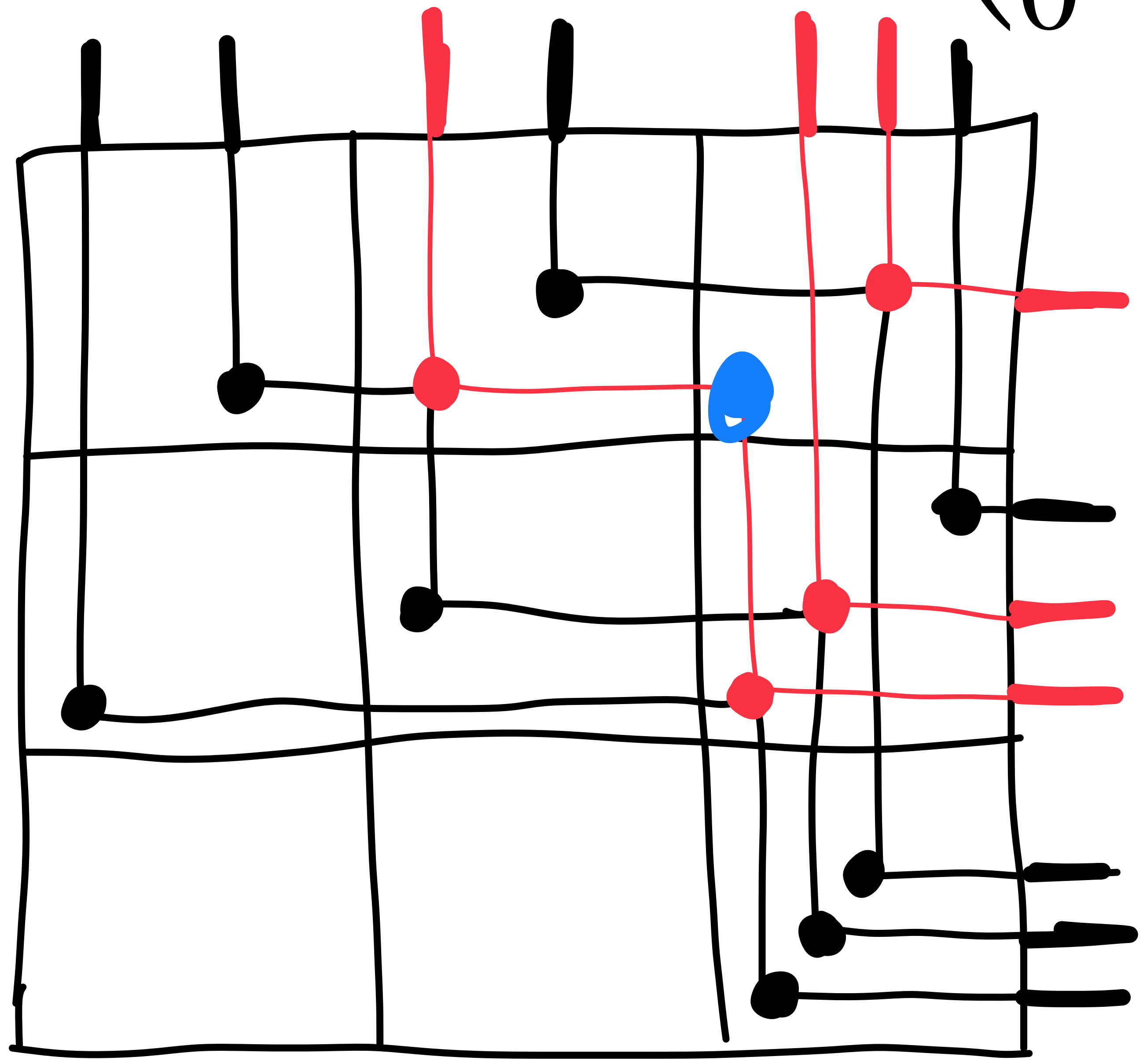

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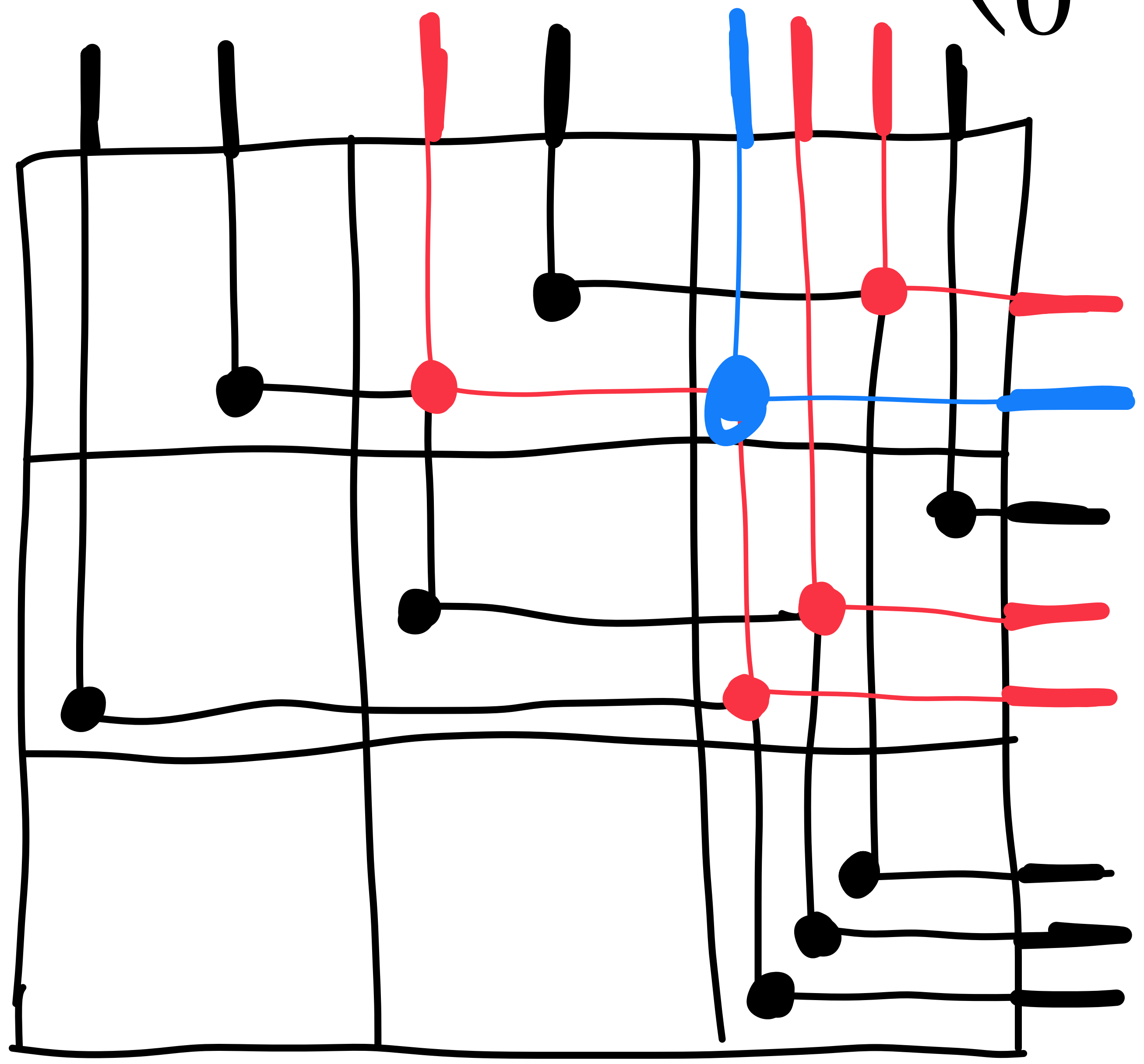

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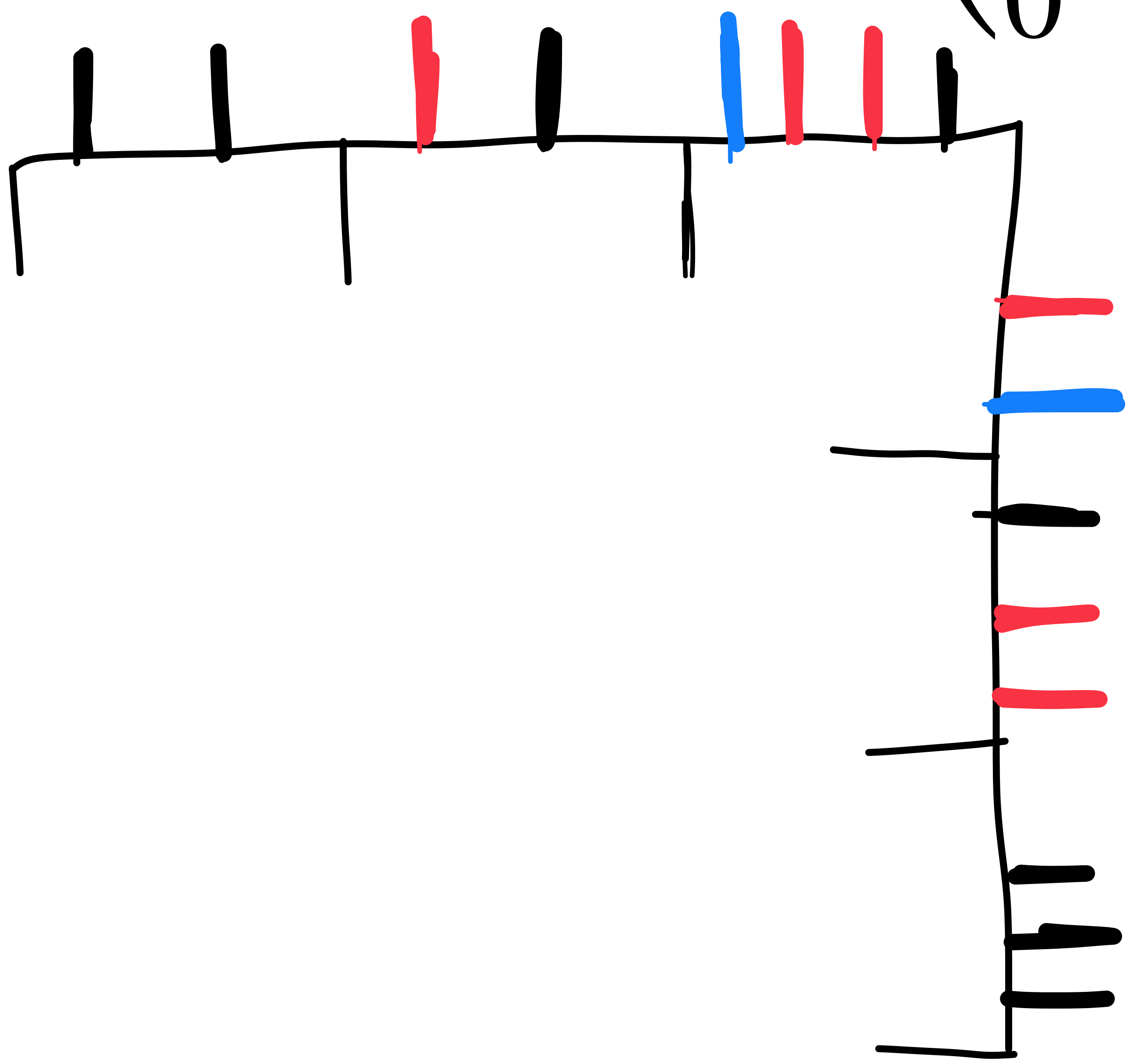

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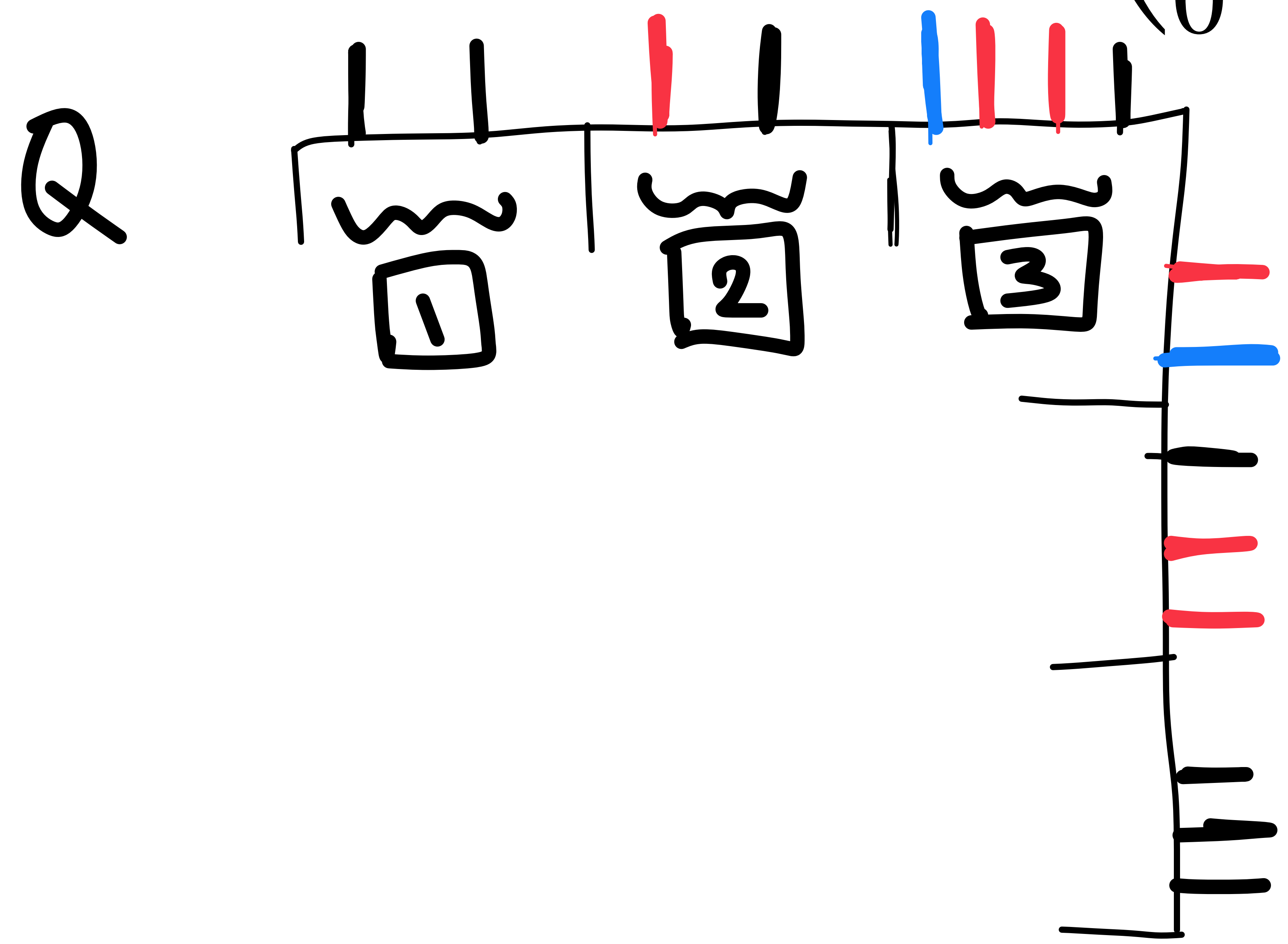

1st row

2nd row

3rd row

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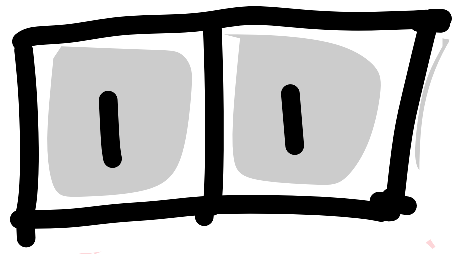

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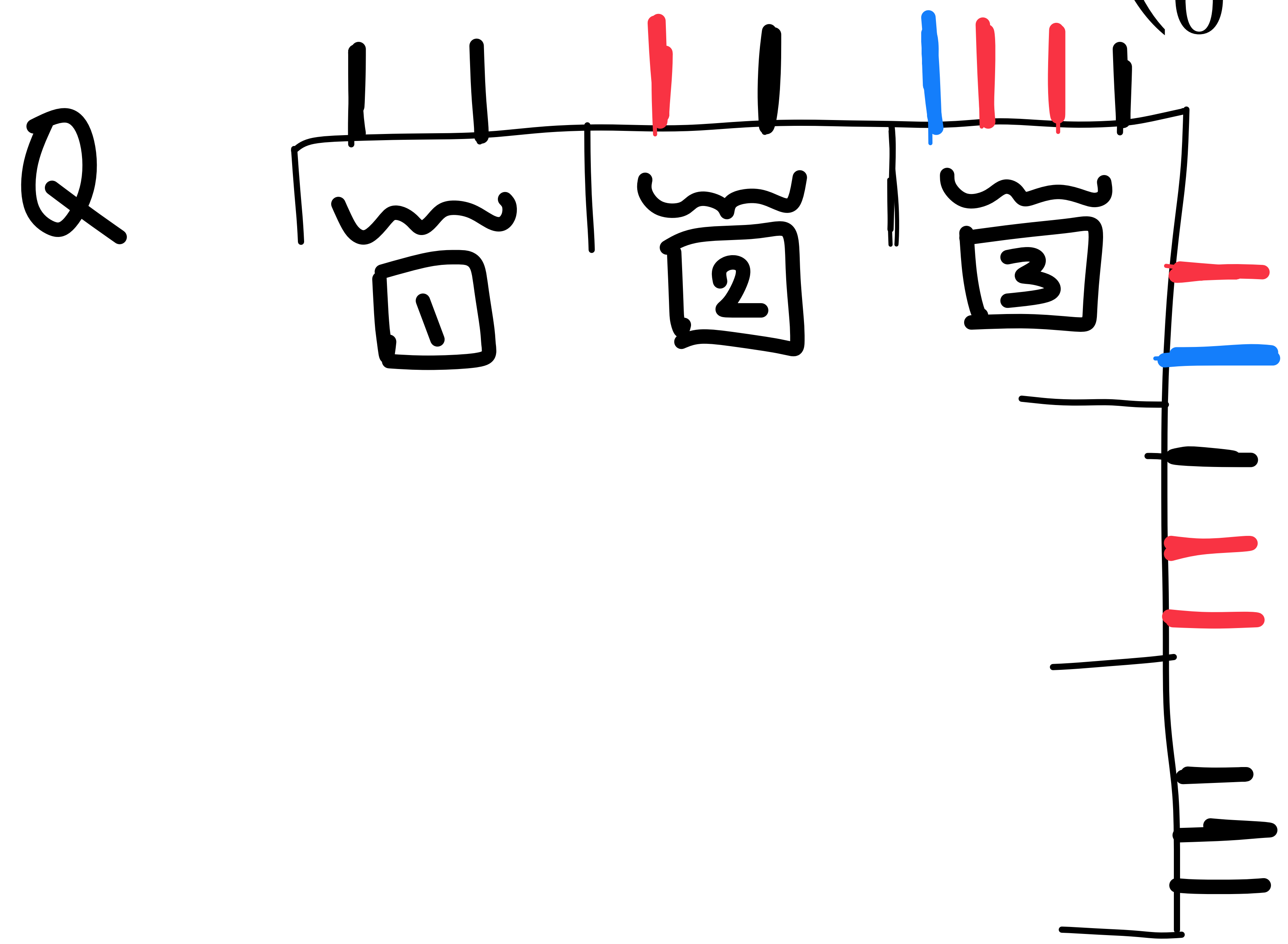

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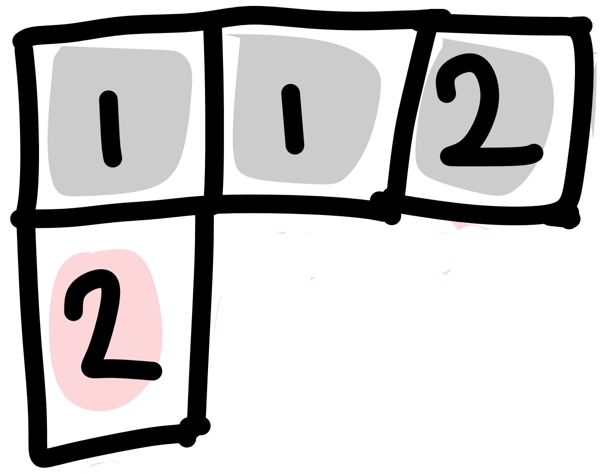

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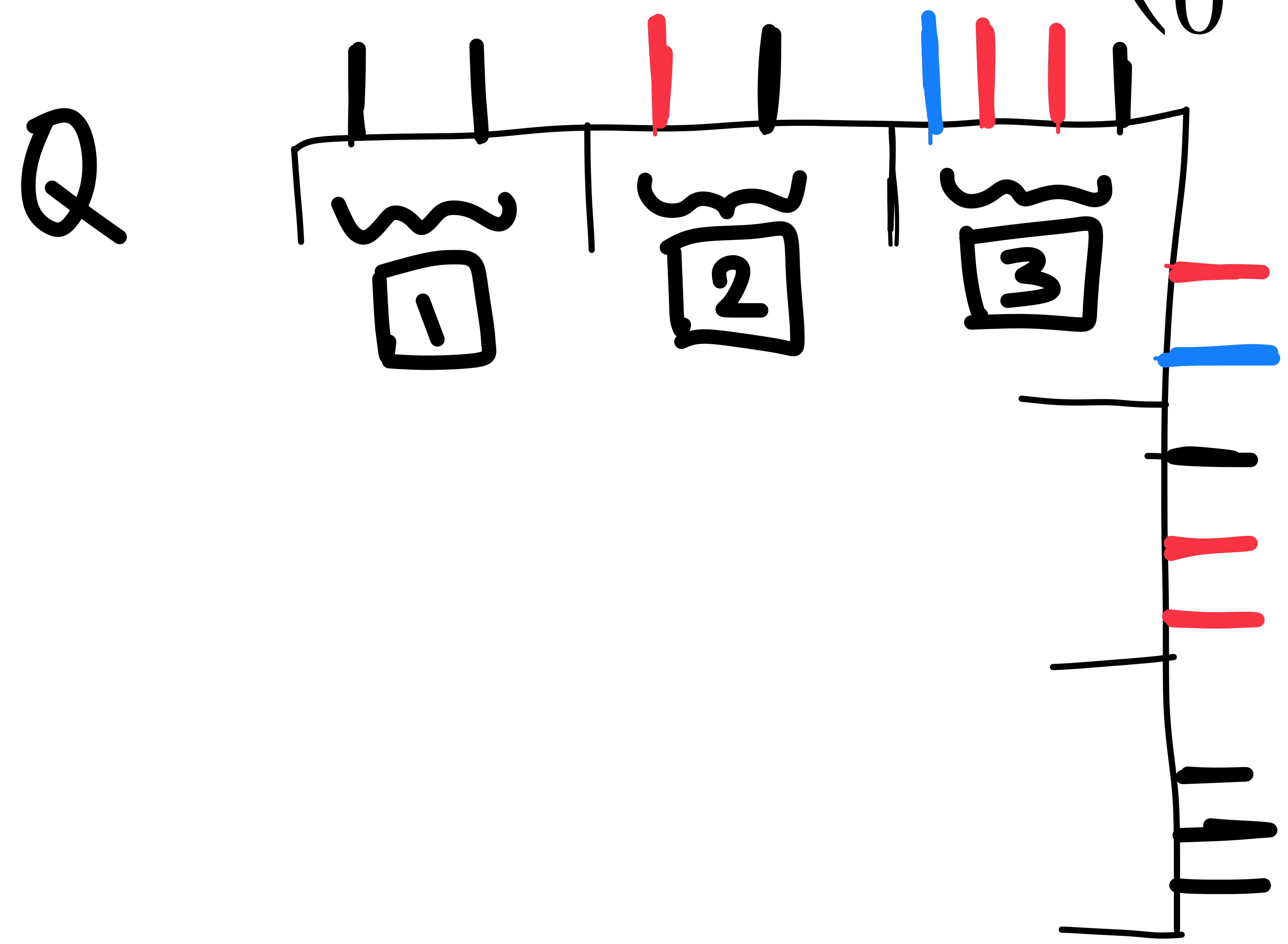

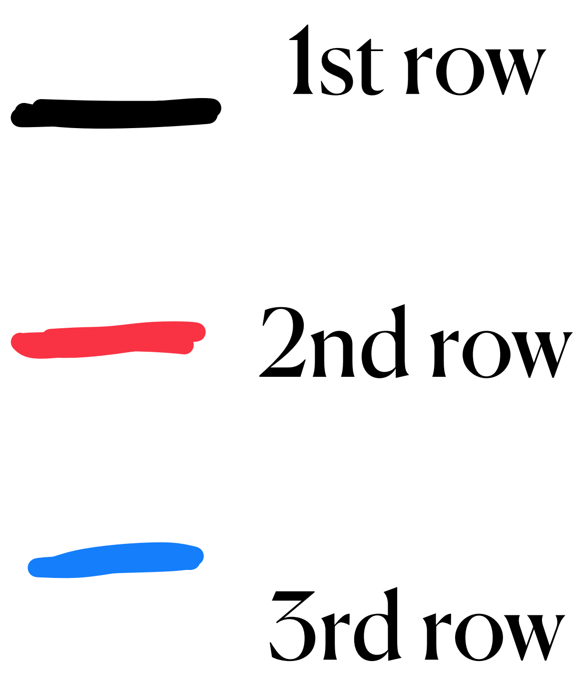

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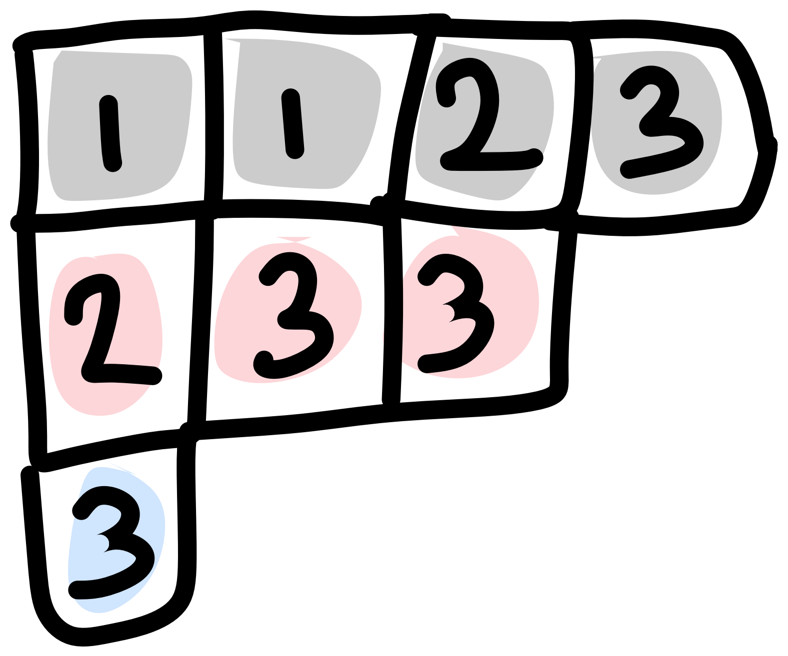

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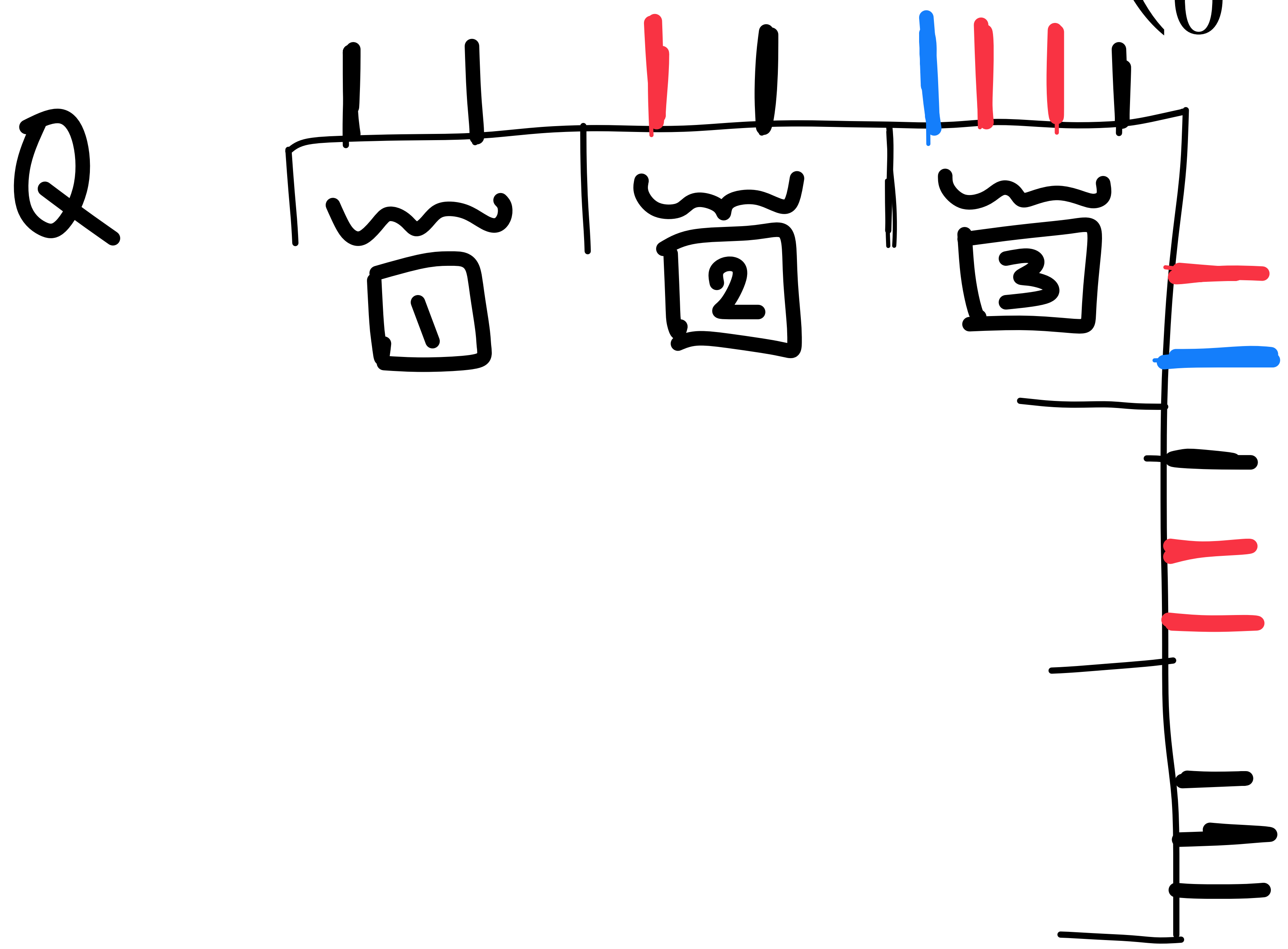

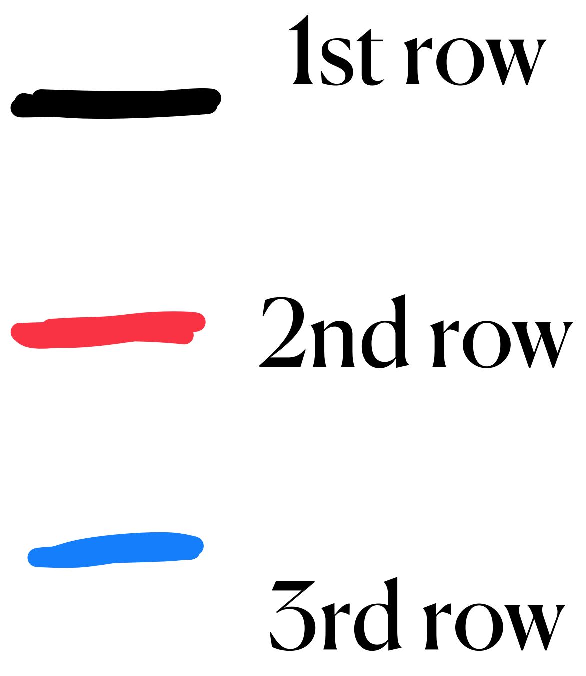

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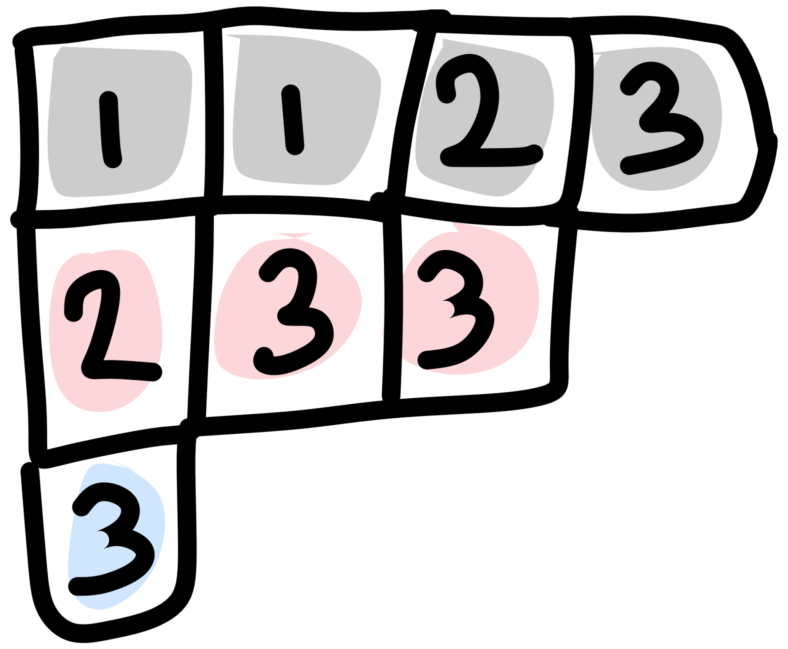

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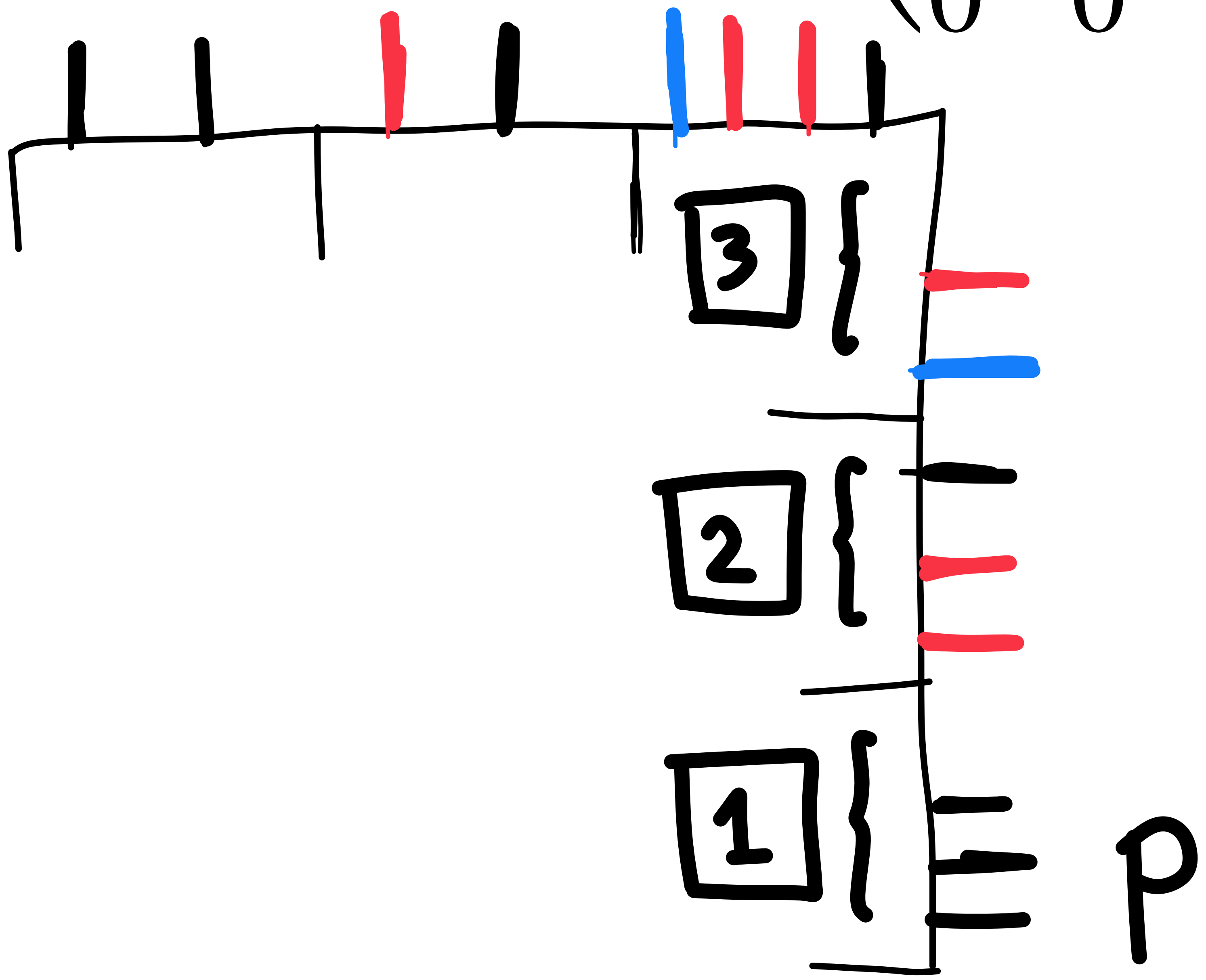

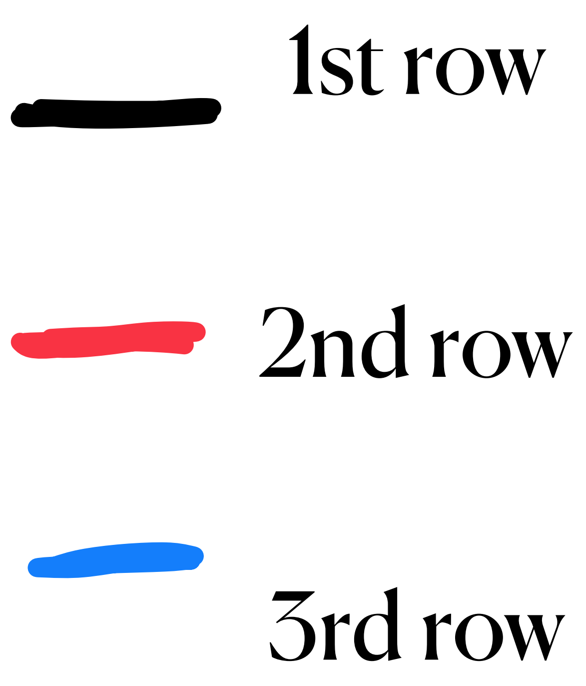

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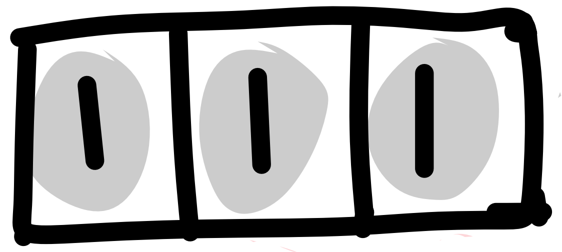

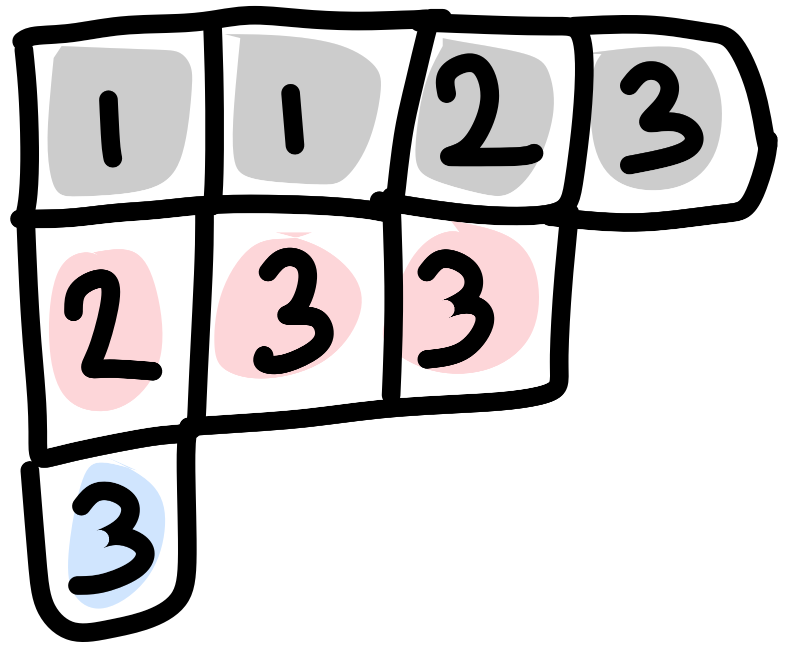

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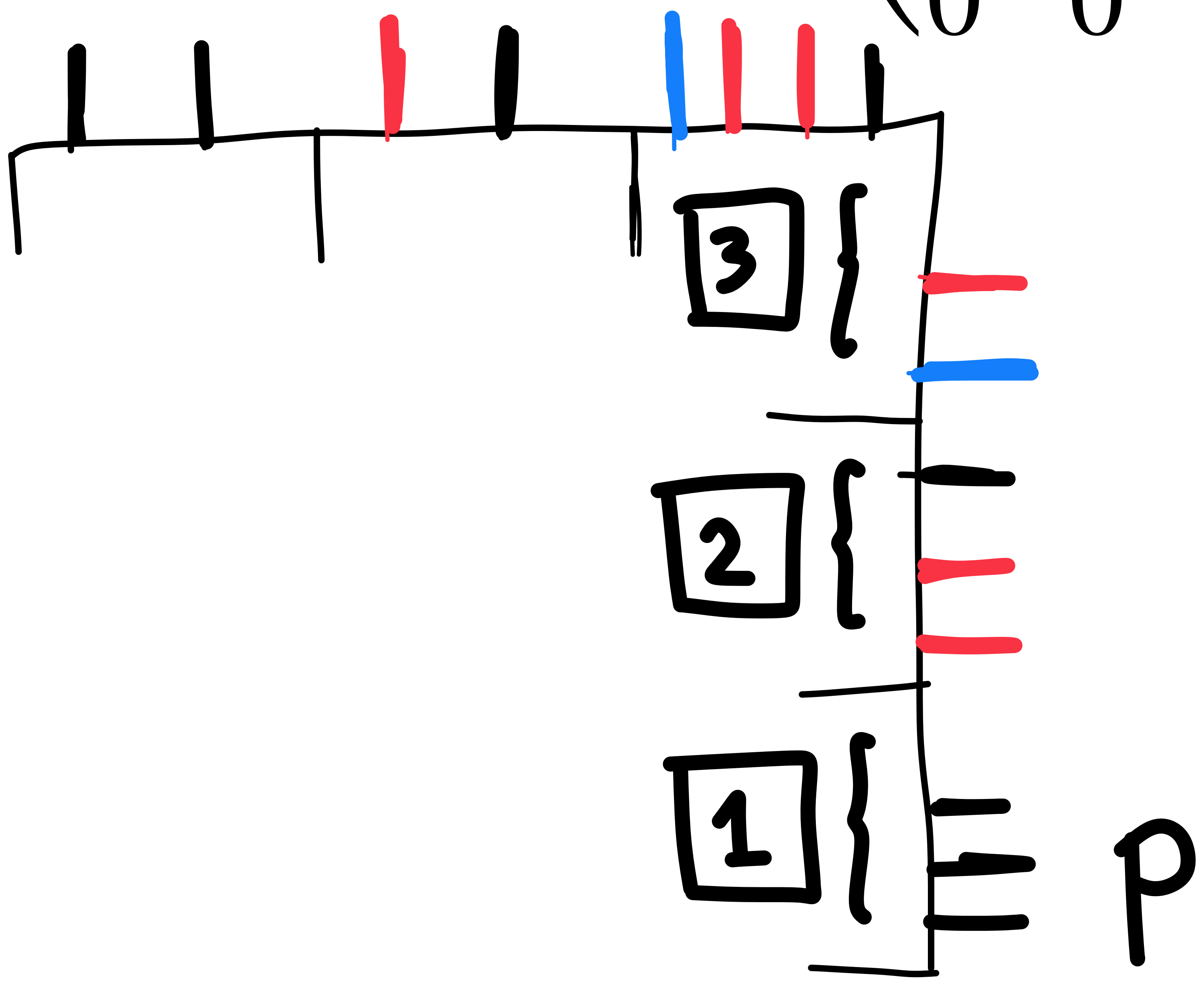

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1	1	1	2
2	2		

1	1	2	3
2	3	3	
3			

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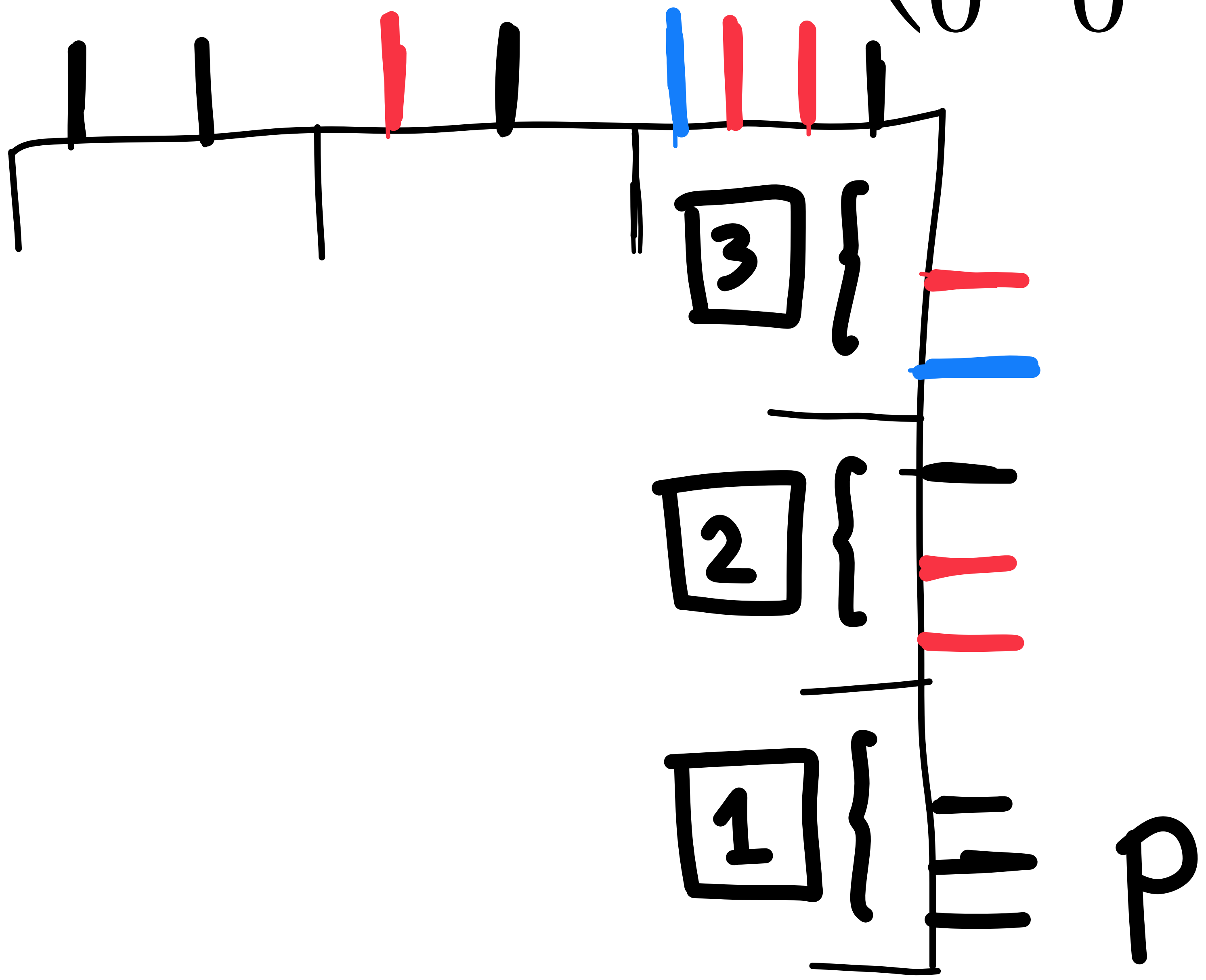

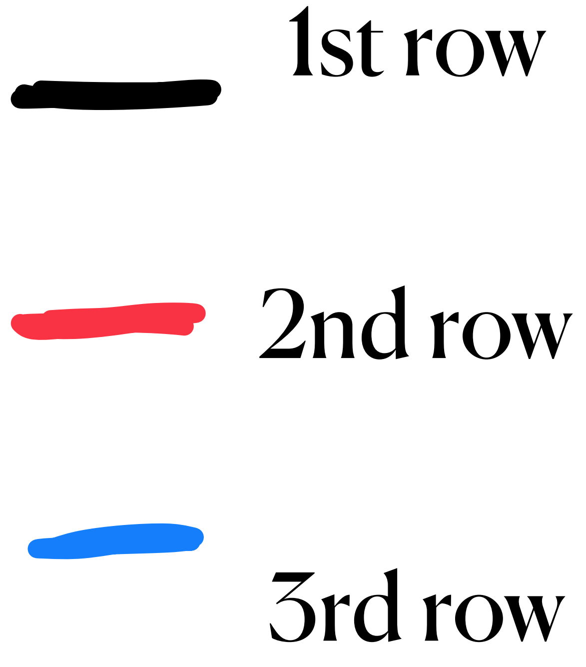

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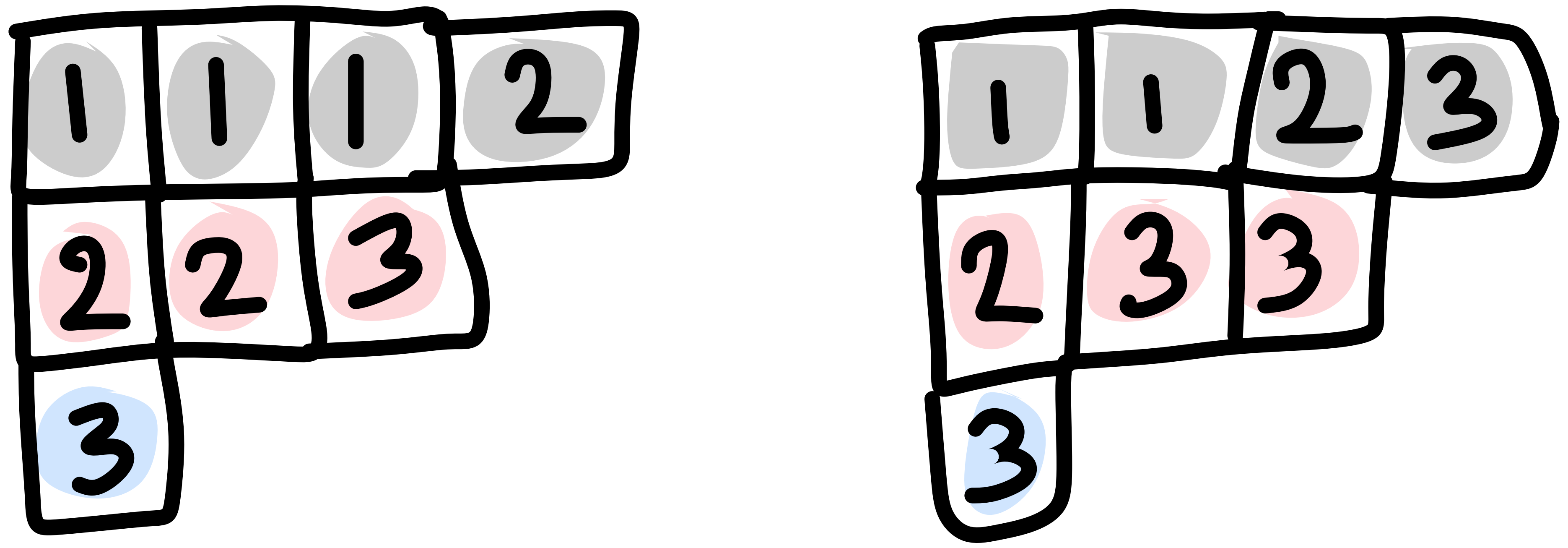

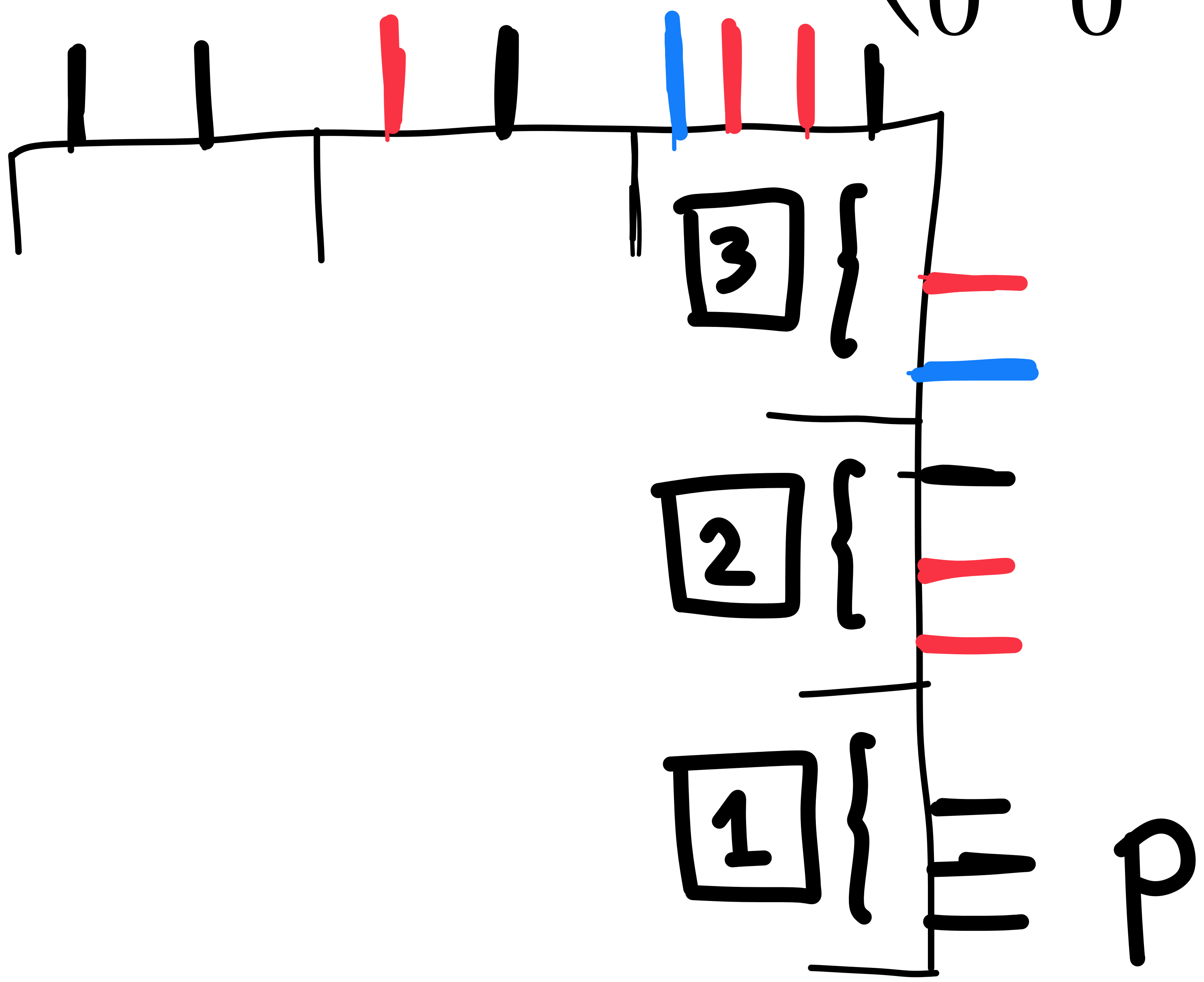

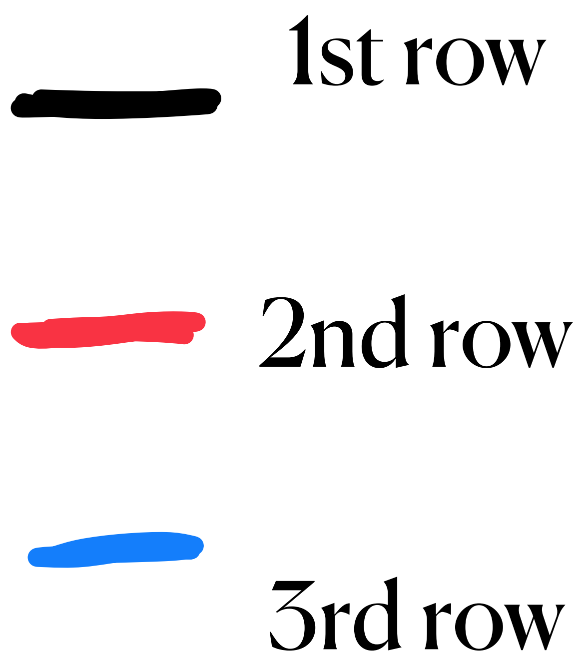

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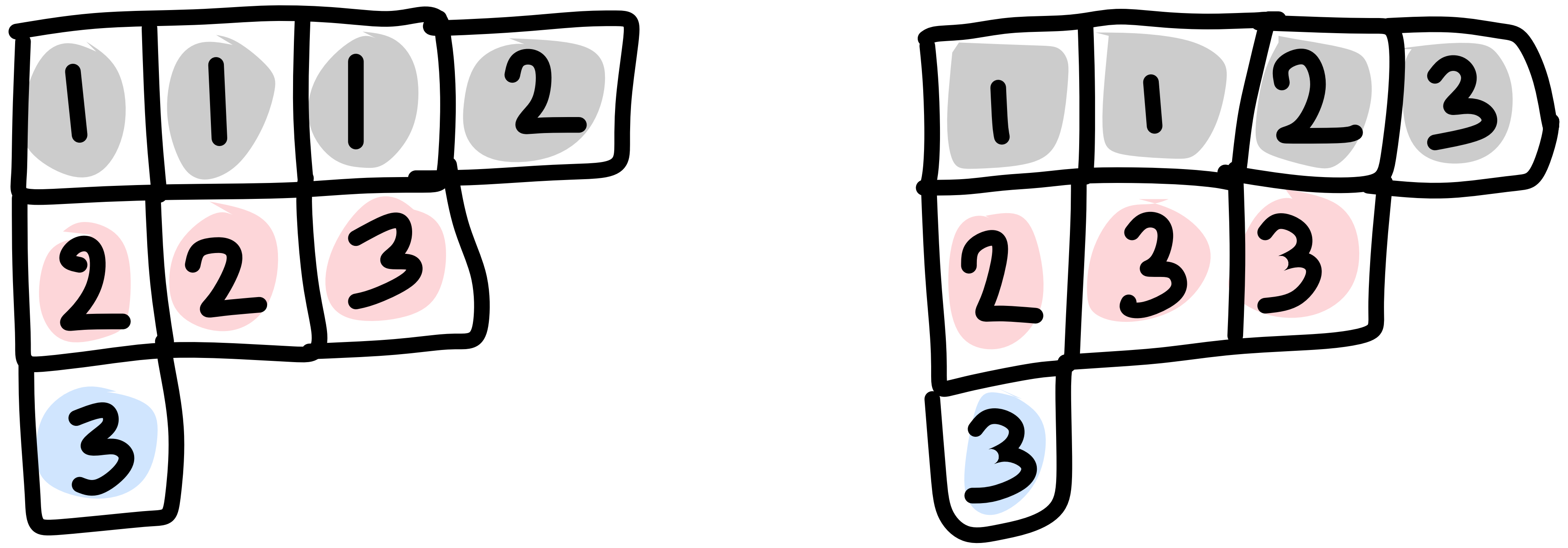

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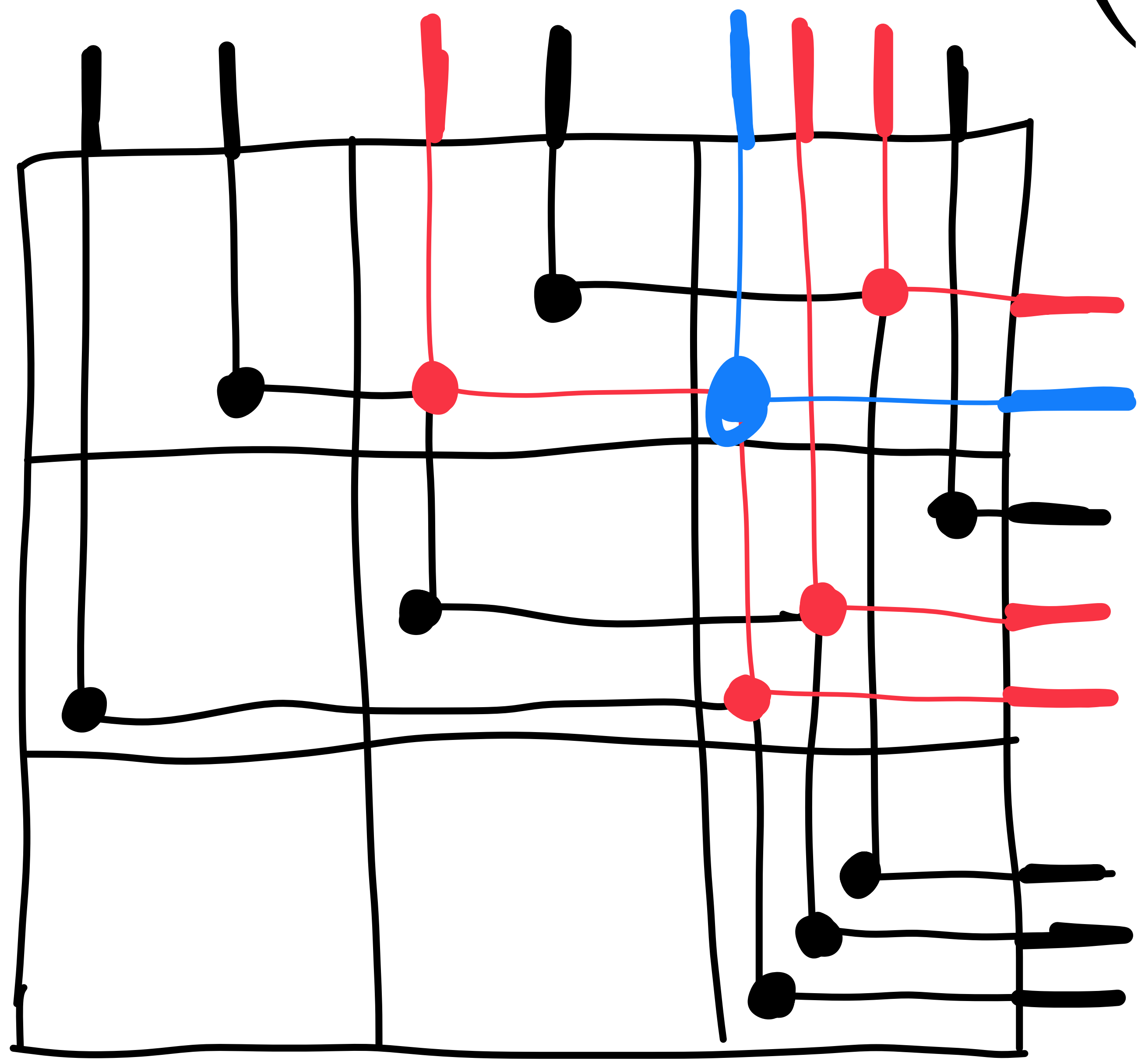

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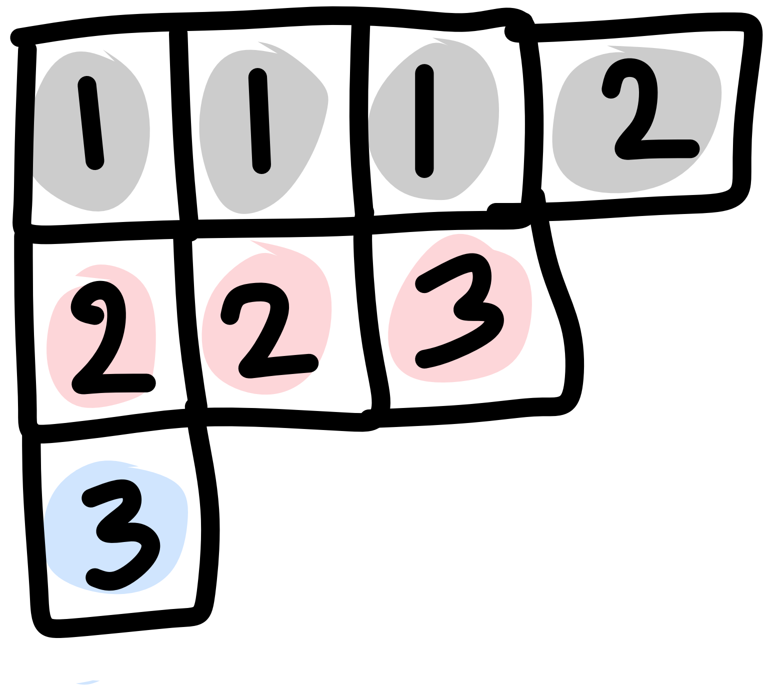
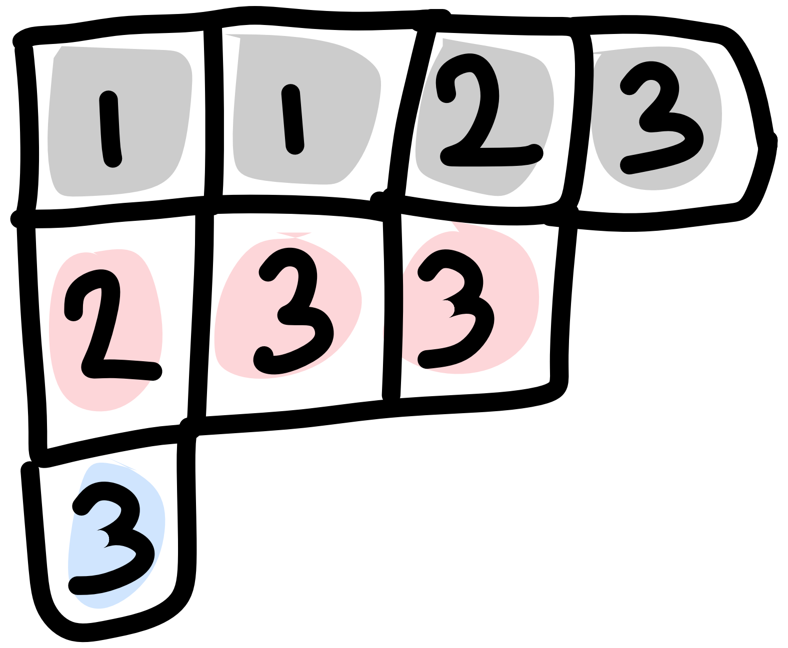
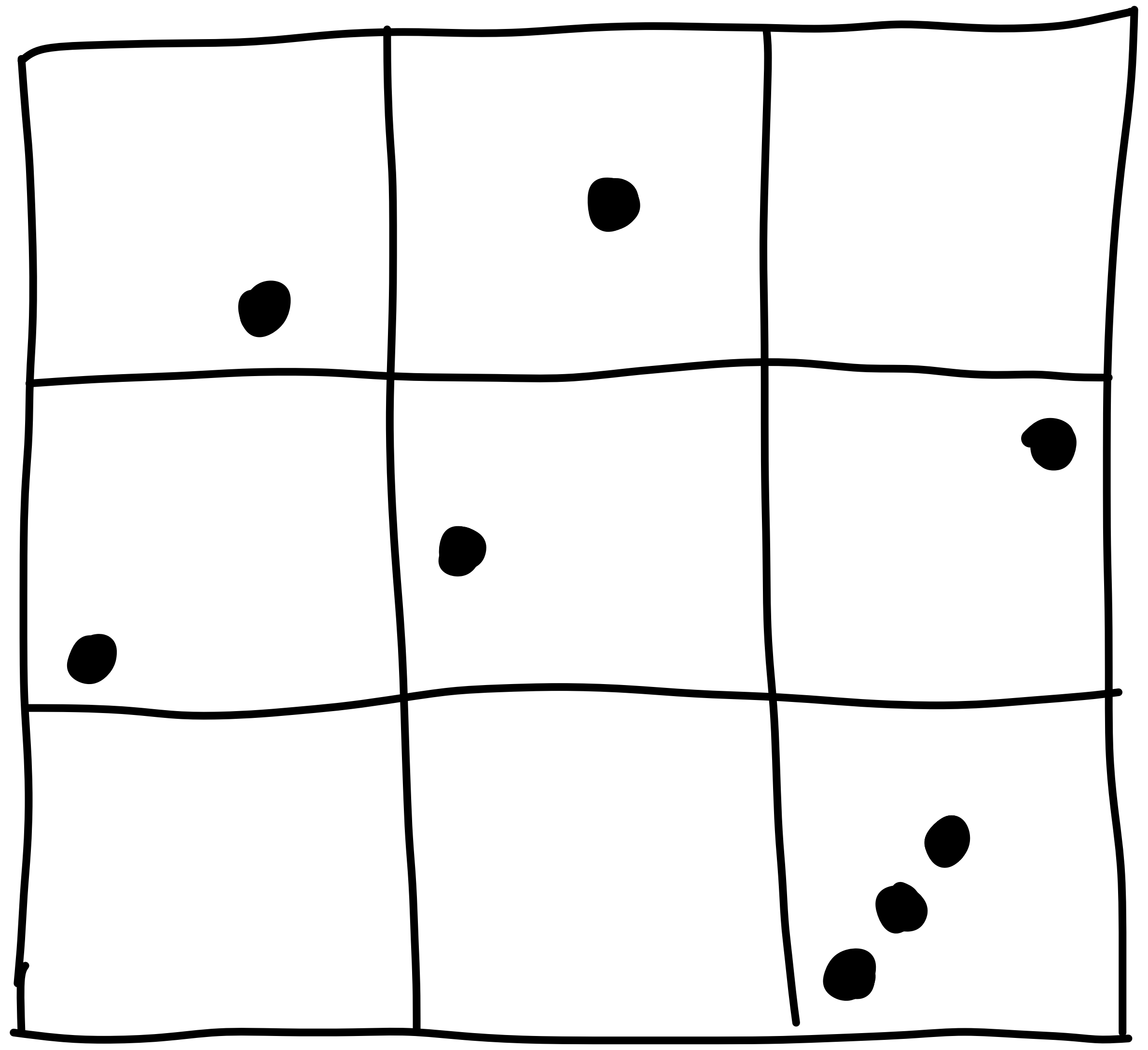
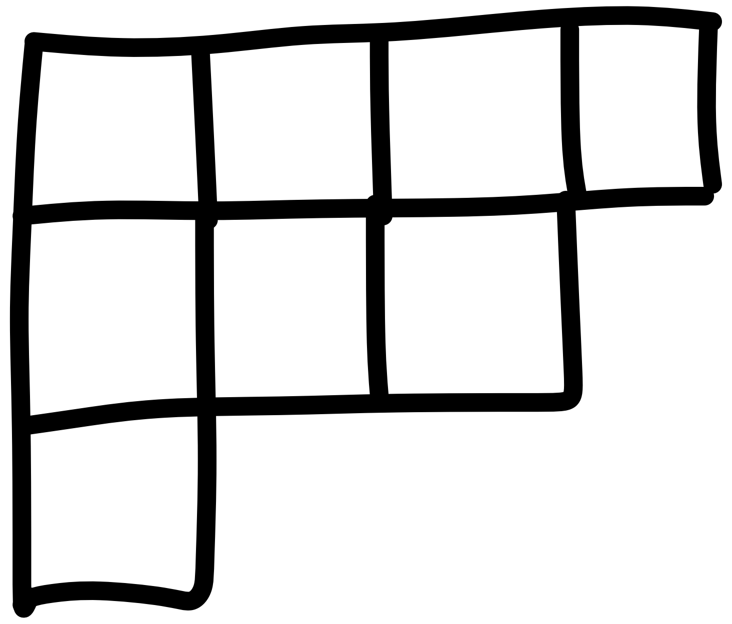






$\lambda =$



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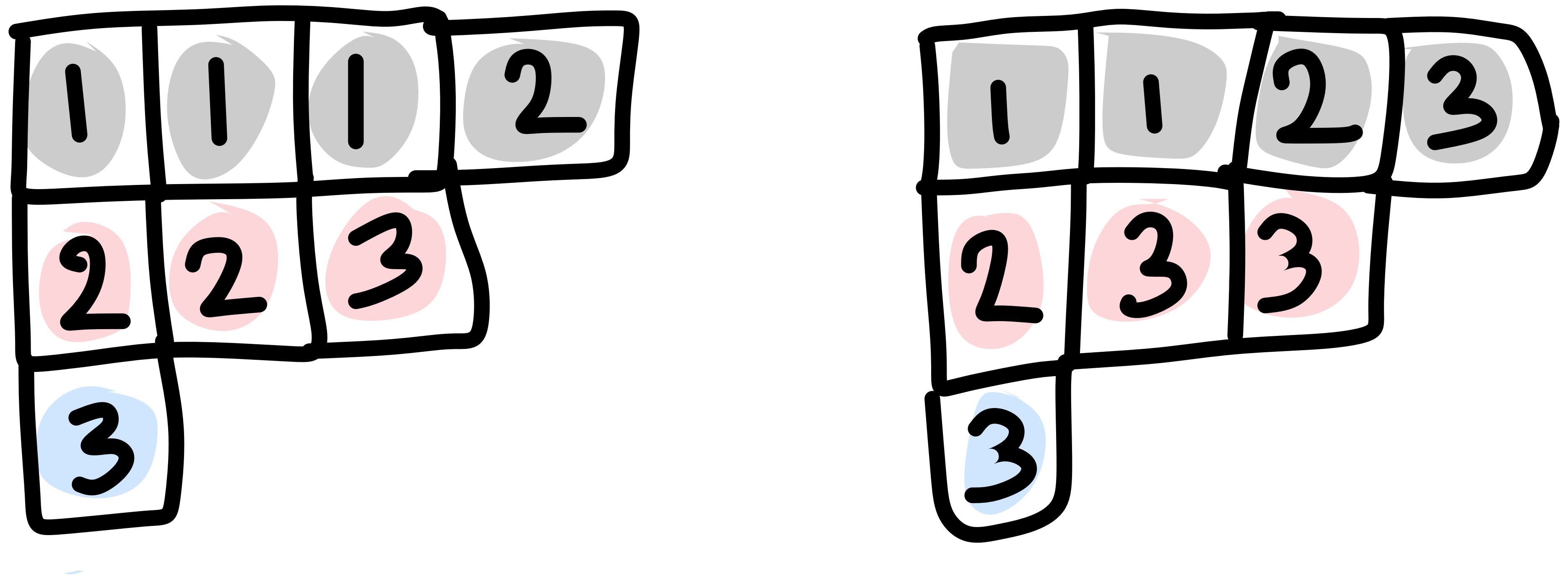
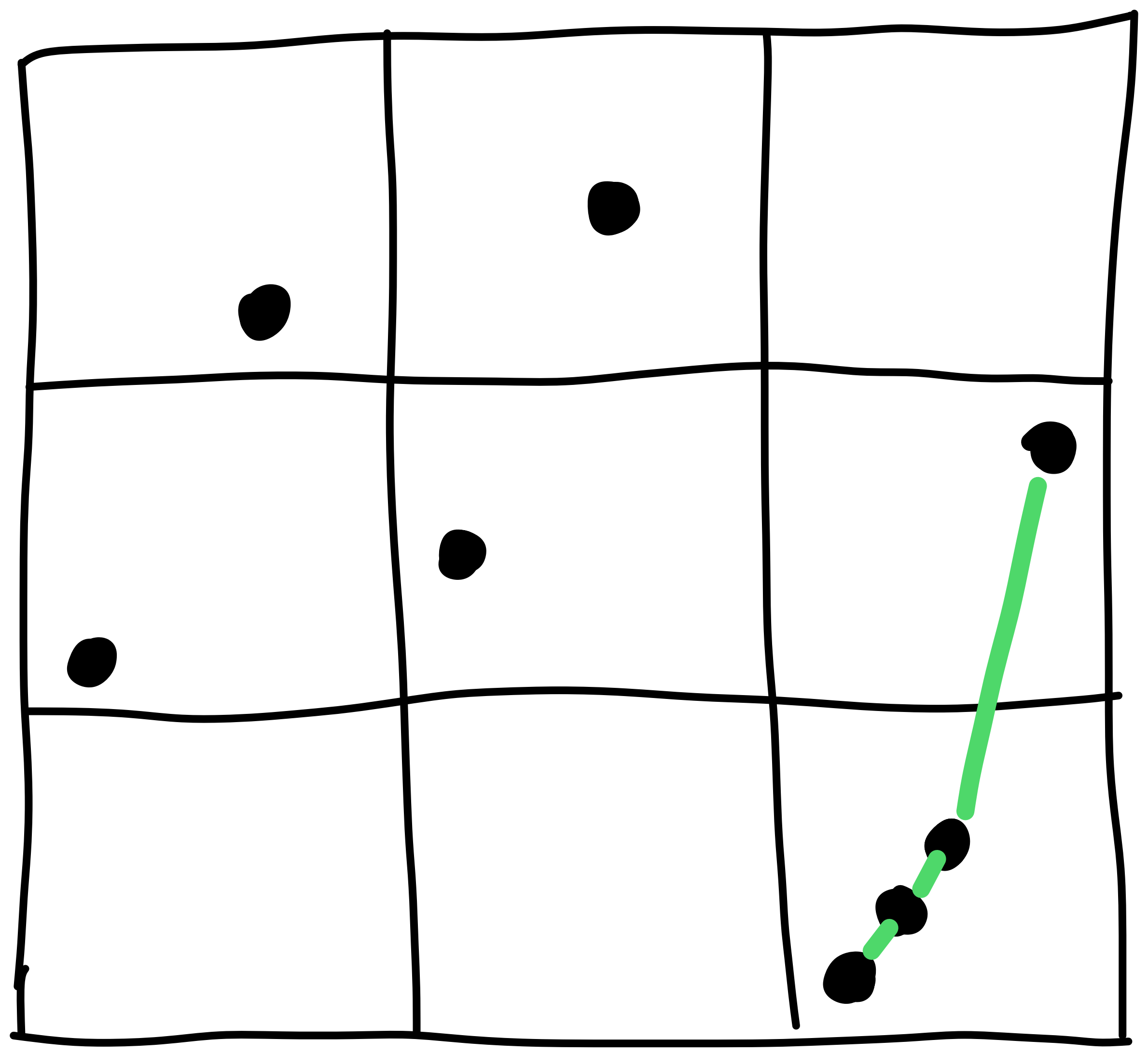


$$\lambda_1 = \mathrm{LIS}_1$$

Example



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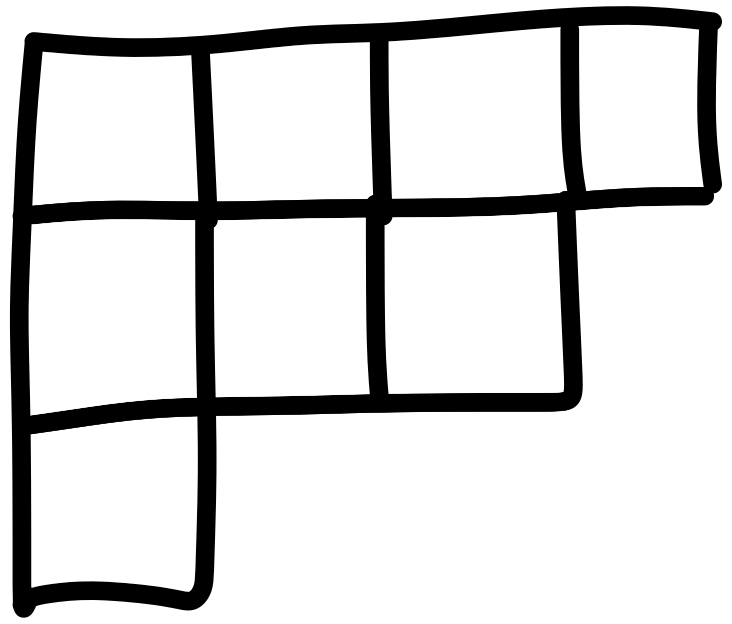

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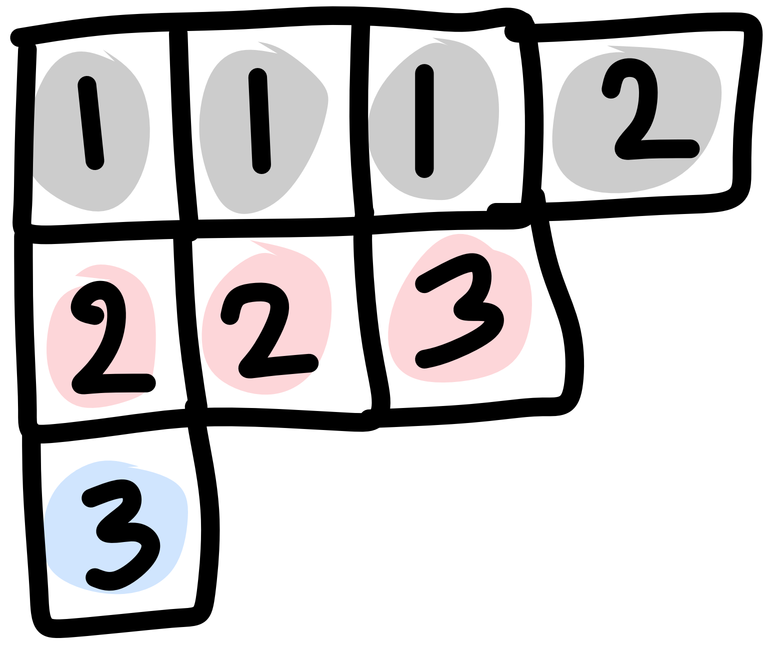

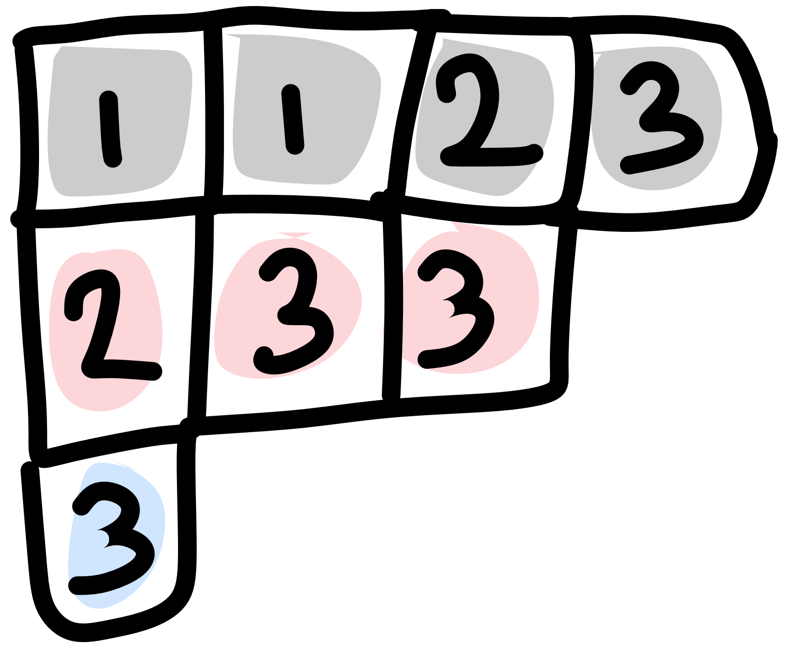

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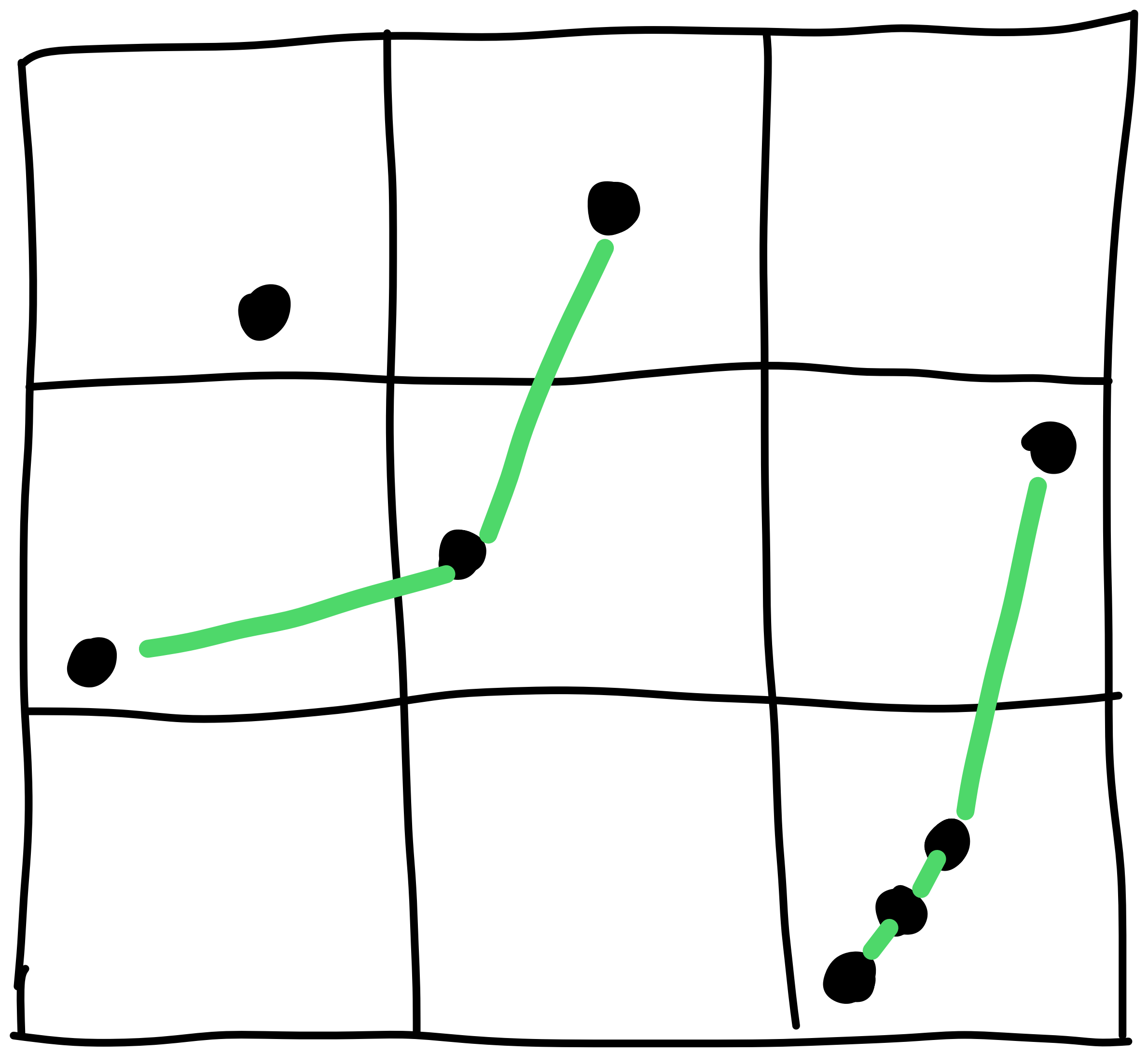
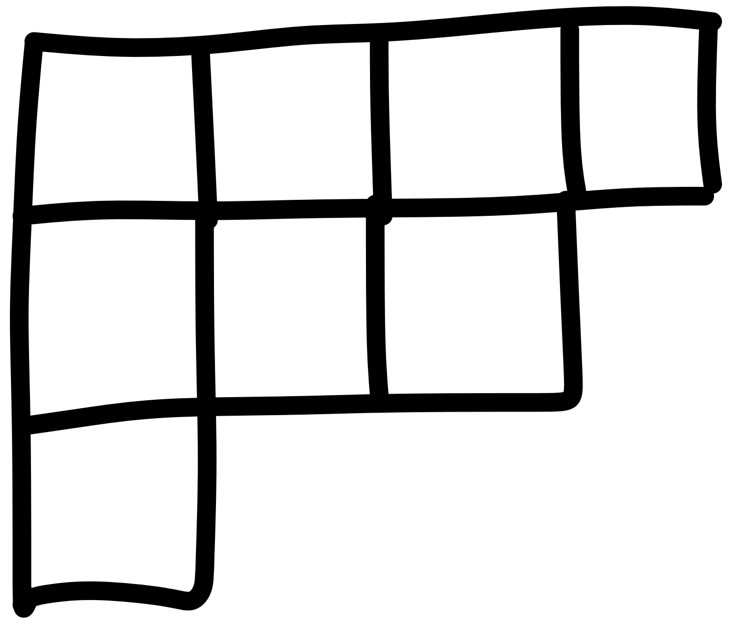


$$\lambda_1 = \mathrm{LIS}_1$$

$$\lambda_1 + \lambda_2 = \mathrm{LIS}_2$$

$\lambda =$ 

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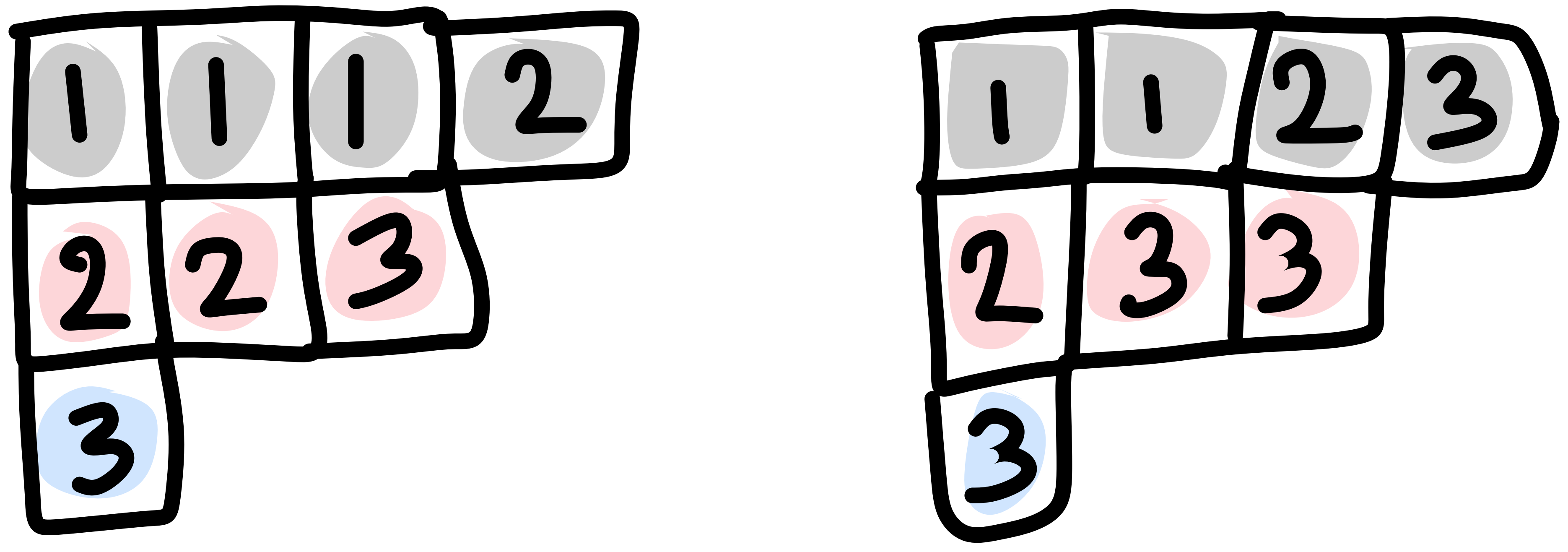

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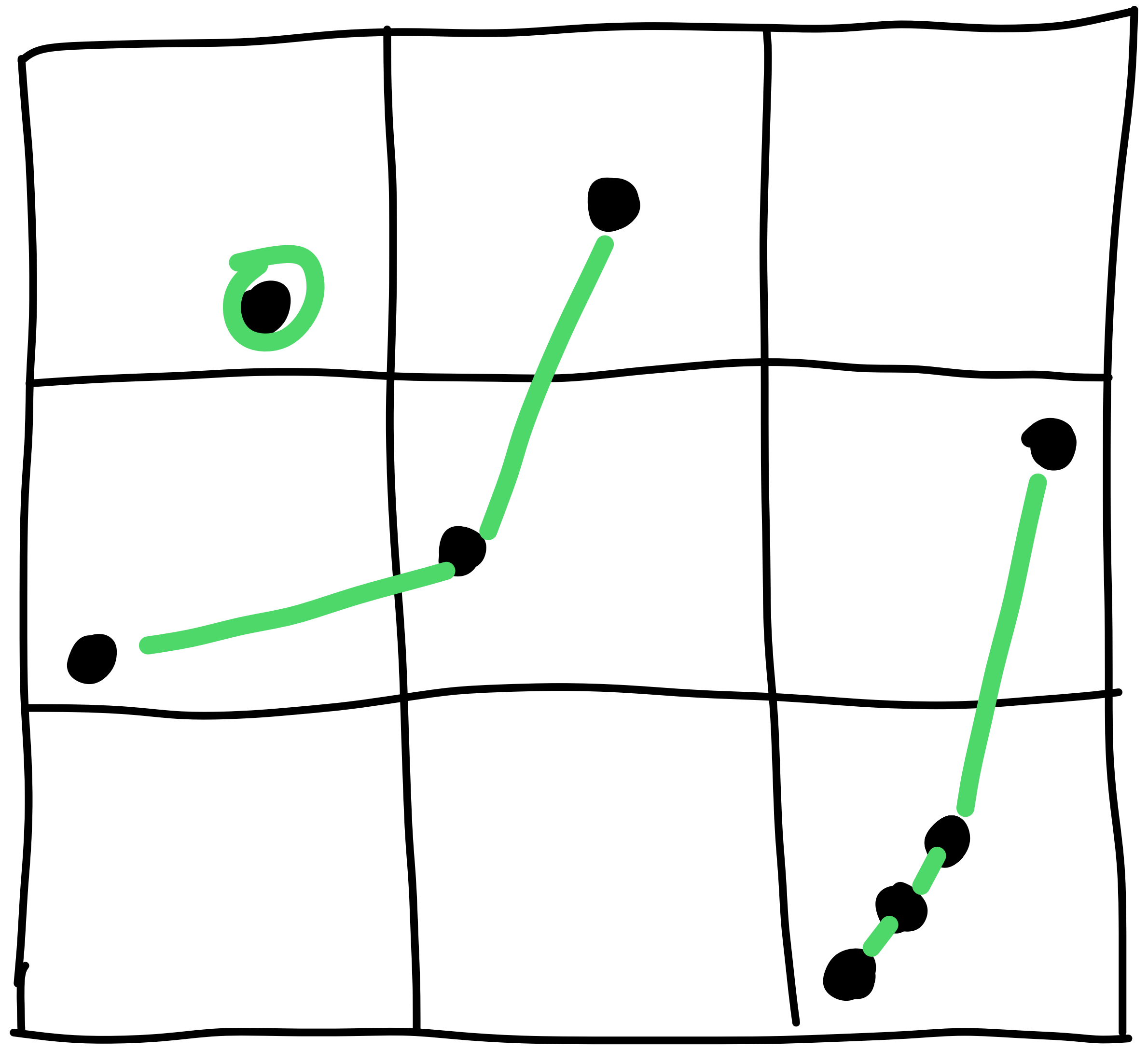

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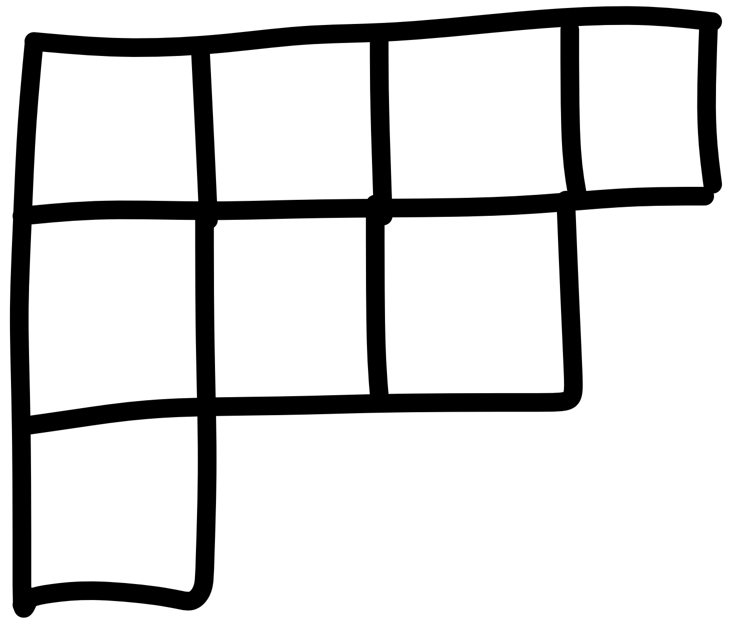
$$\lambda_1 + \lambda_2 = \mathrm{LIS}_2$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{LIS}_3$$

$\lambda =$ 

RSK correspondence via crystals

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EXAMPLE (on tableaux): $i = 1$

			1	2
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2	3	3		

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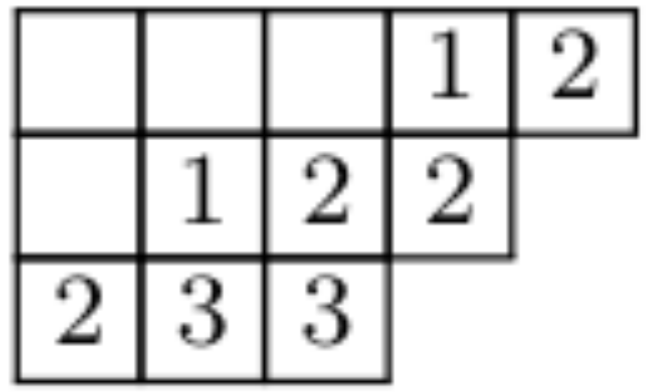

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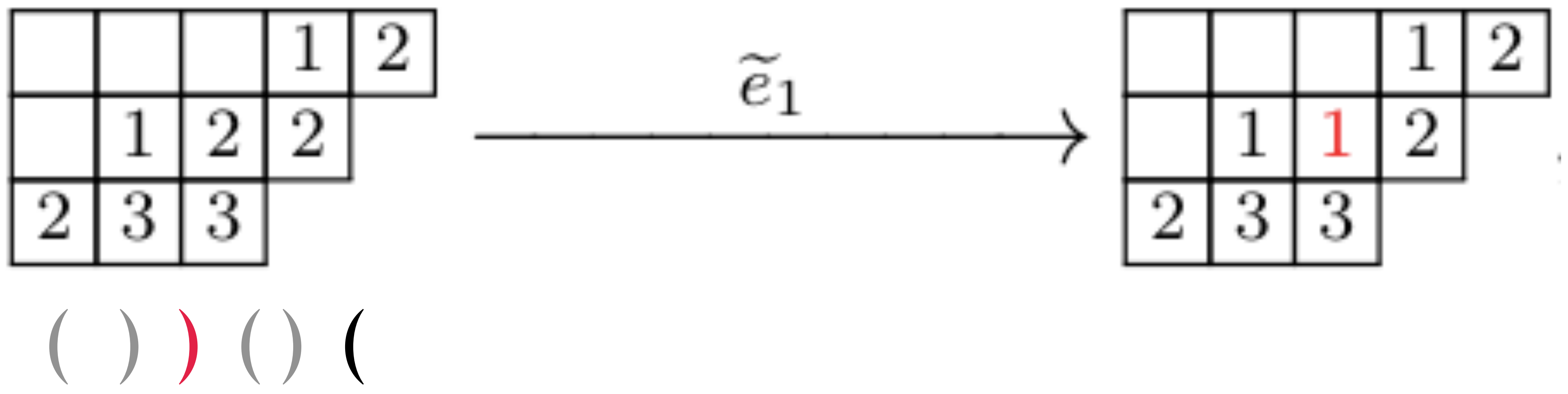

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$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{\tilde{e}_1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$(\quad (\quad)))($

RSK correspondence via crystals

- Define bi-crystal structures:

On tableaux

$$\tilde{E}_i^{(1)} : (P, Q) \mapsto (\tilde{e}_i(P), Q)$$

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Operators $\tilde{F}_i^{(1)}$, $\tilde{F}_i^{(2)}$ defined in the same way

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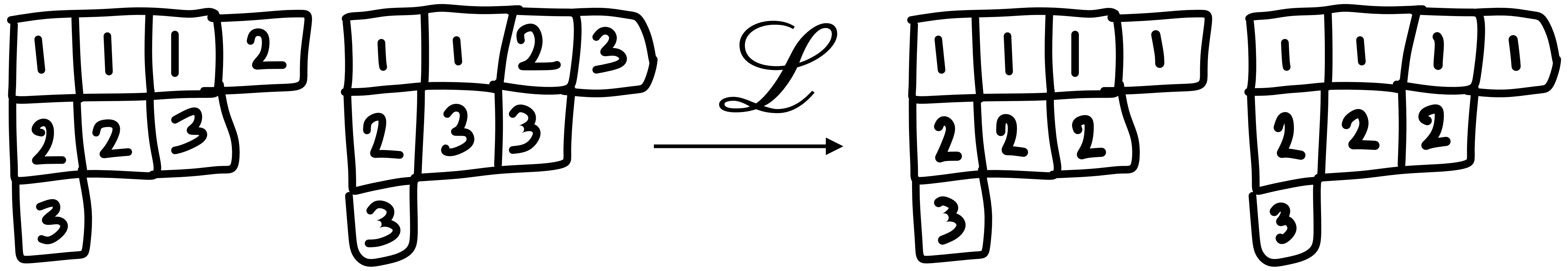

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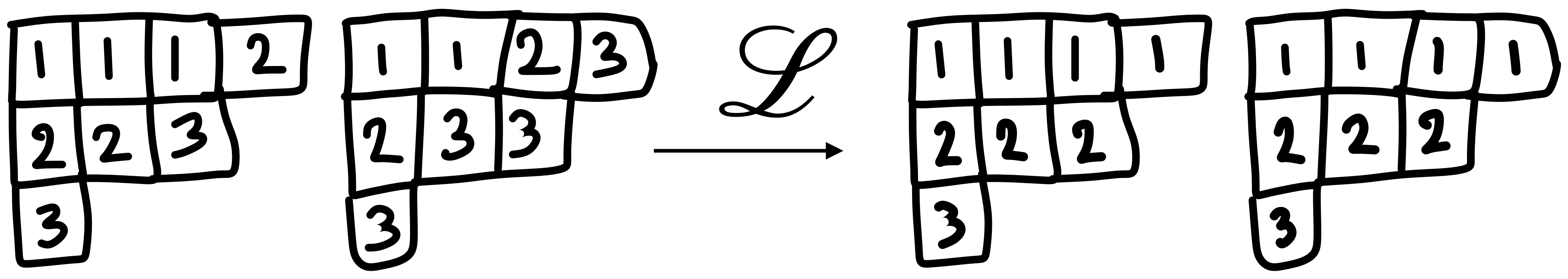

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{\mathscr{L}} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

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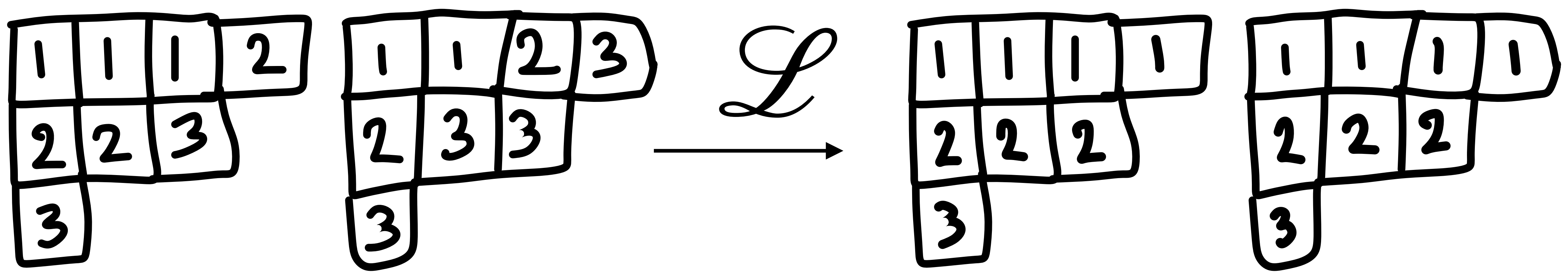

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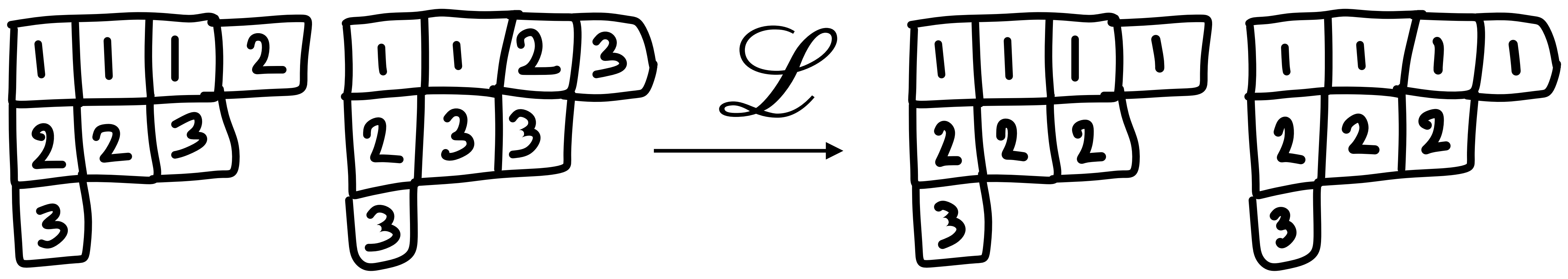
$\mathscr{L}$: Leading map

RSK correspondence via crystals

- We can define the **RSK** : $M \leftrightarrow (P, Q)$ imposing:
    1. **RSK** commutes with the bi-crystal structure
    2. **RSK** sends highest weight matrices to highest weight tableaux

Fact: iterating the action of $\tilde{E}_i^{(1)}$, $\tilde{E}_i^{(2)}$ we can drastically simplify the pair (P, Q) or the matrix M



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$\mathscr{L}$: Leading map

Cauchy Identity for q-Whittaker polynomials

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$$\sum_{\mu} b_{\mu}(q) P_{\mu}(x;q) P_{\mu}(y;q) = \prod_{k \geq 0} \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j q^k}$$

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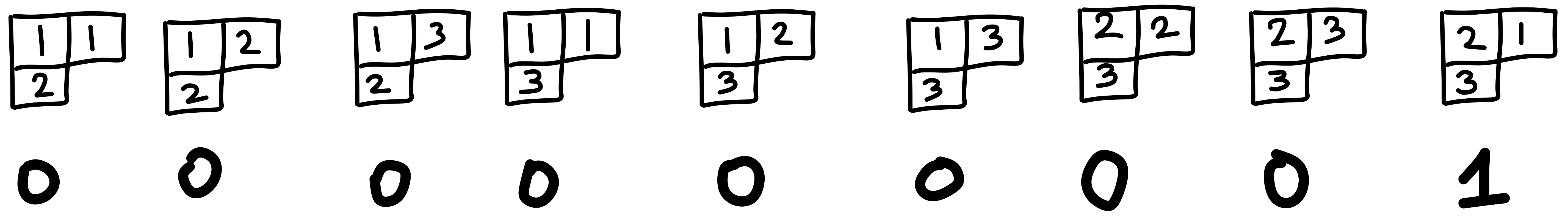
$\mathscr{H}$ = intrinsic energy (def. later)



$$\mathscr{P}_\mu(x;q) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + q x_1 x_2 x_3$$

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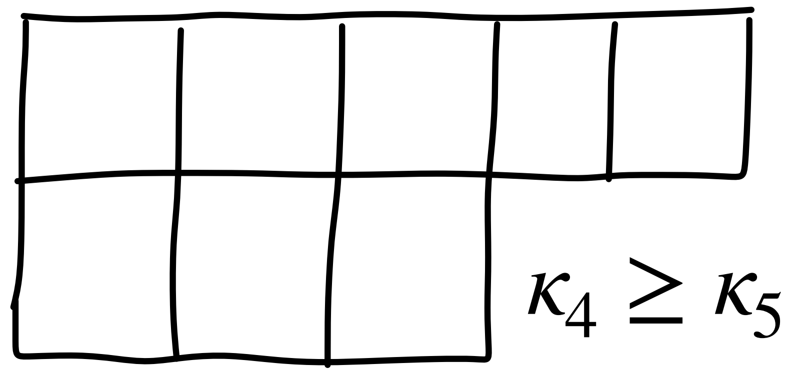
$\mu =$



$\kappa_4 \geq \kappa_5$

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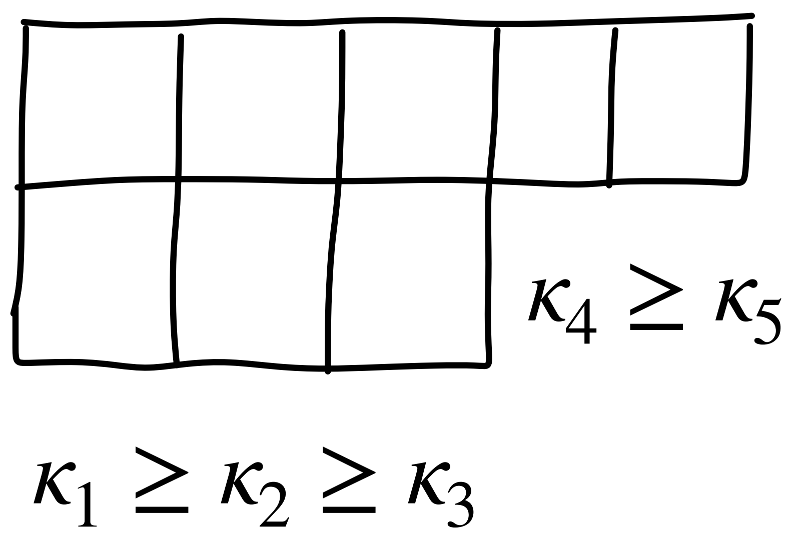

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Bijective proof: $M \overset{\Upsilon}{\longleftrightarrow} (V, W; \kappa)$: $\sum_{i,j=1}^{n}\sum_{k>0} k M_{i,j}^k = |\kappa| + \mathscr{H}(V) + \mathscr{H}(W)$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

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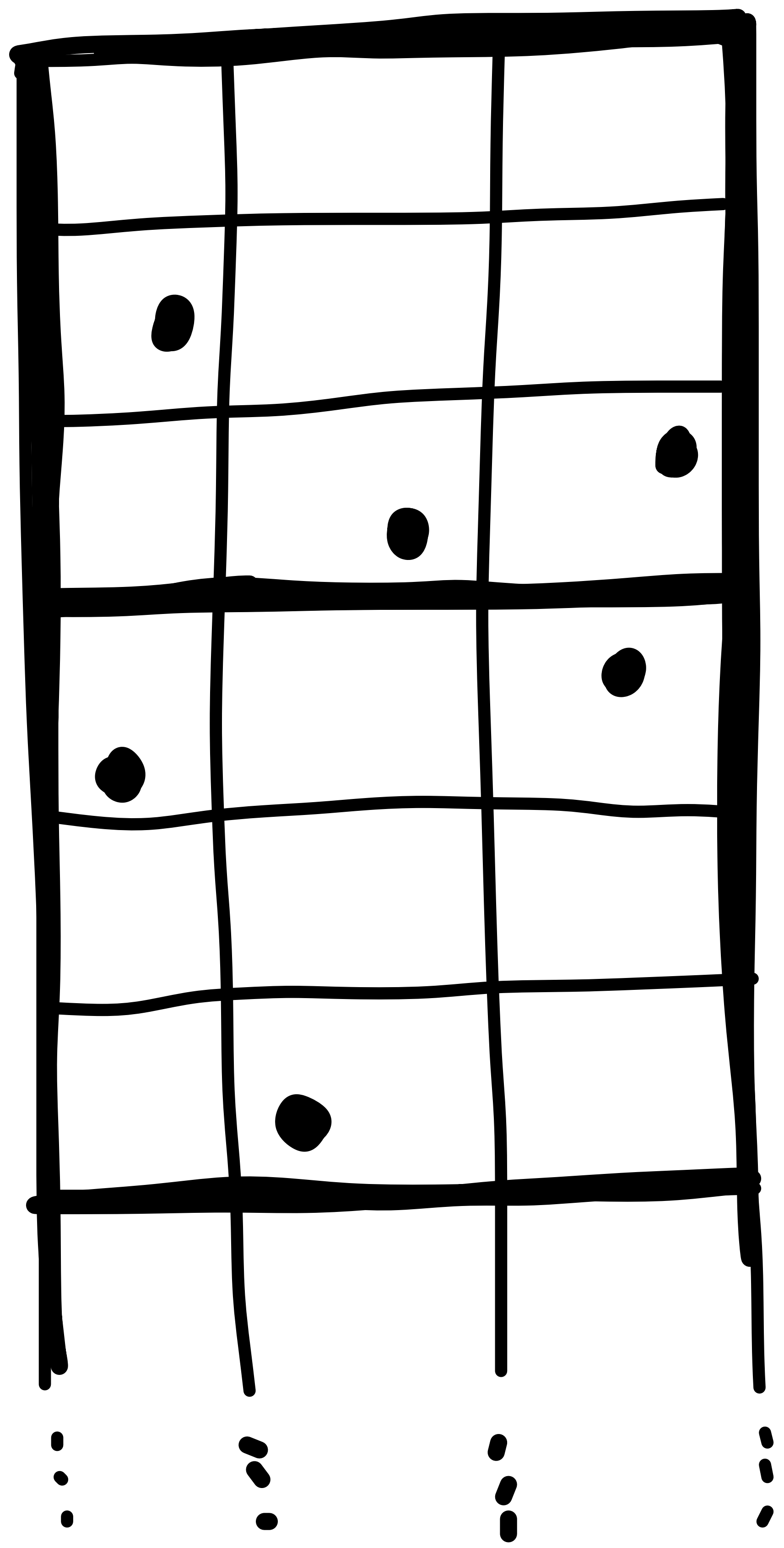

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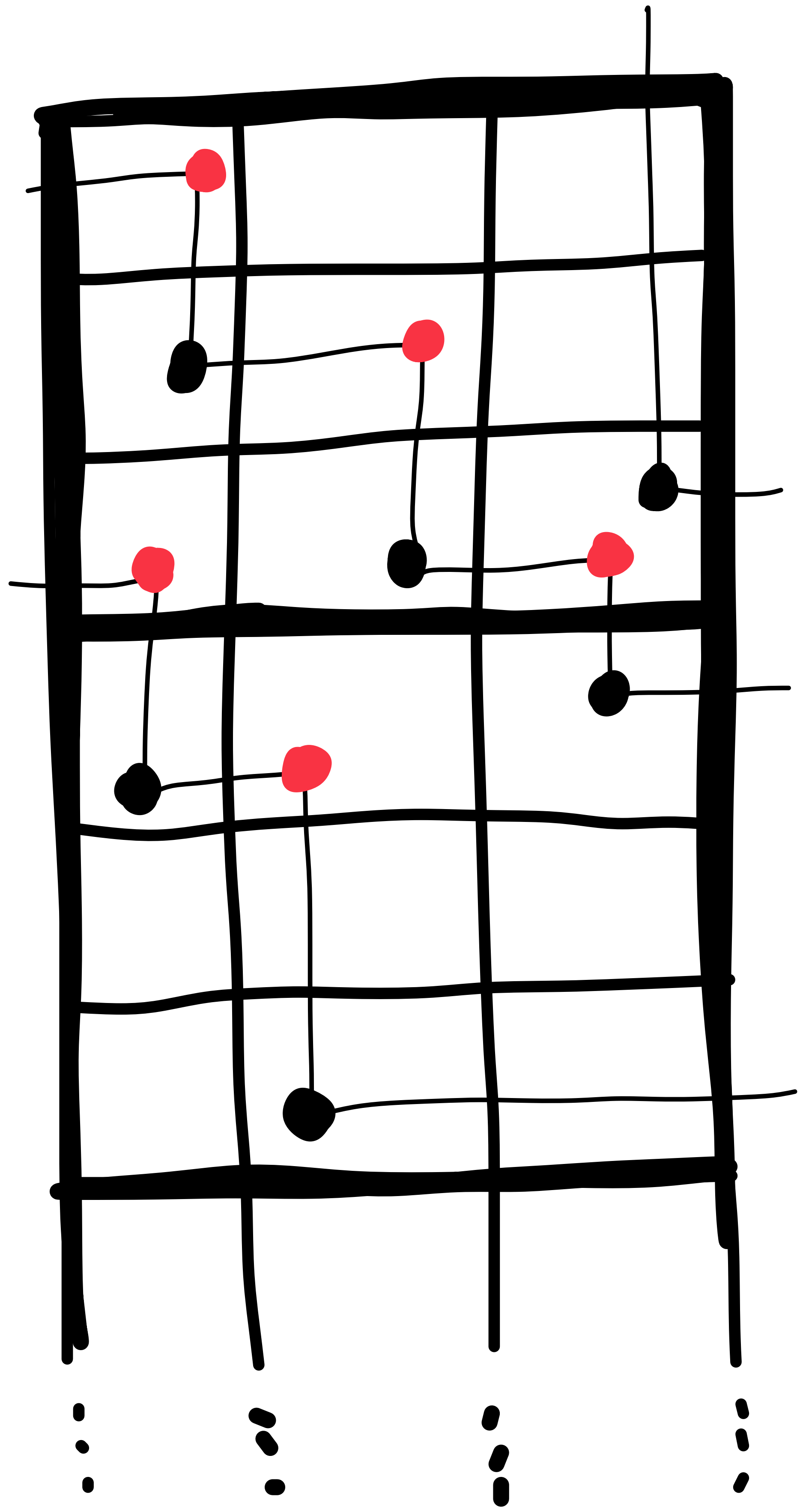

Periodic boundary conditions

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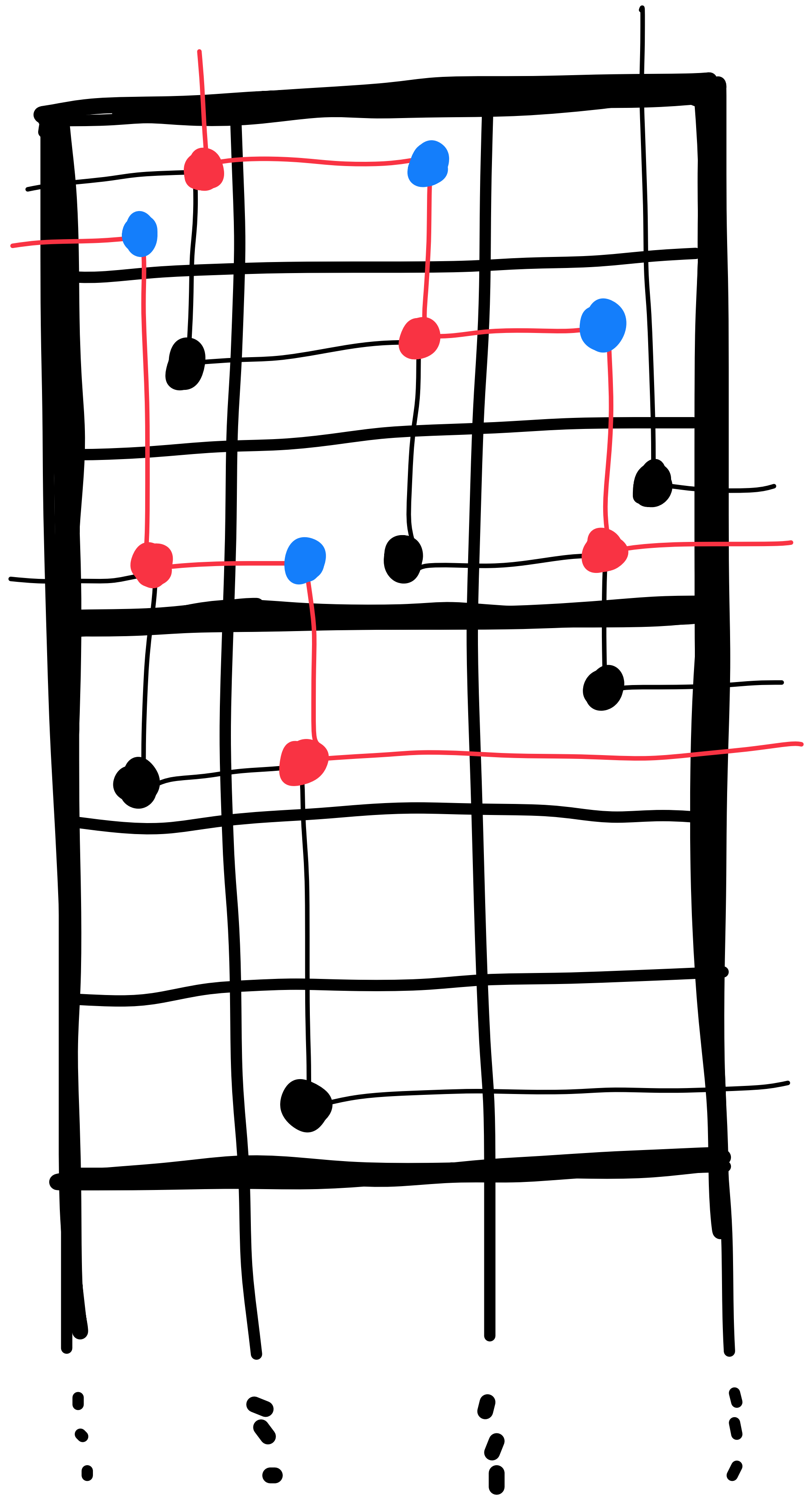
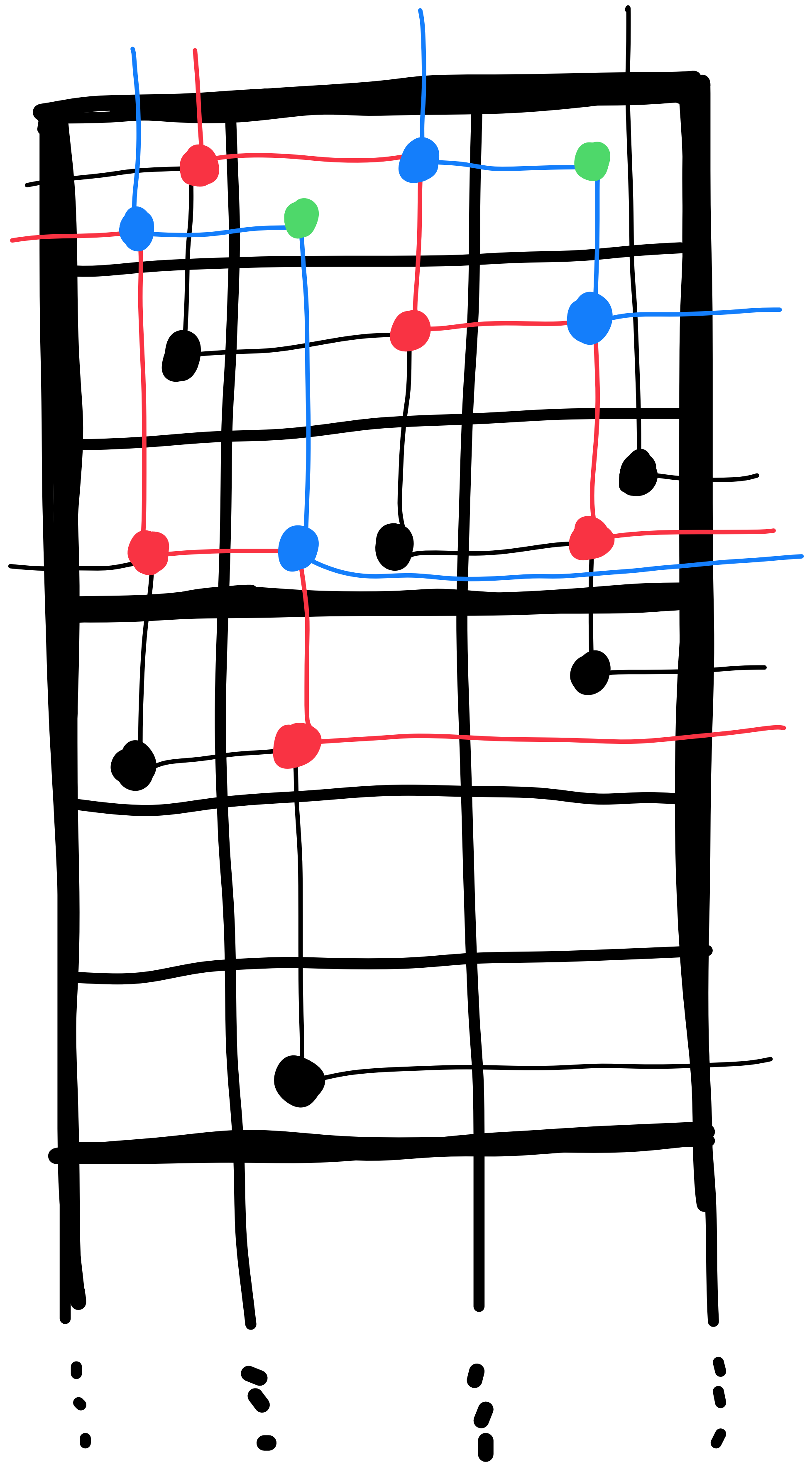


Periodic boundary conditions

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (1,1,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,1,\dots) & (0,0,\dots) & (0,1,\dots) \end{pmatrix} \longleftrightarrow \; ?$$



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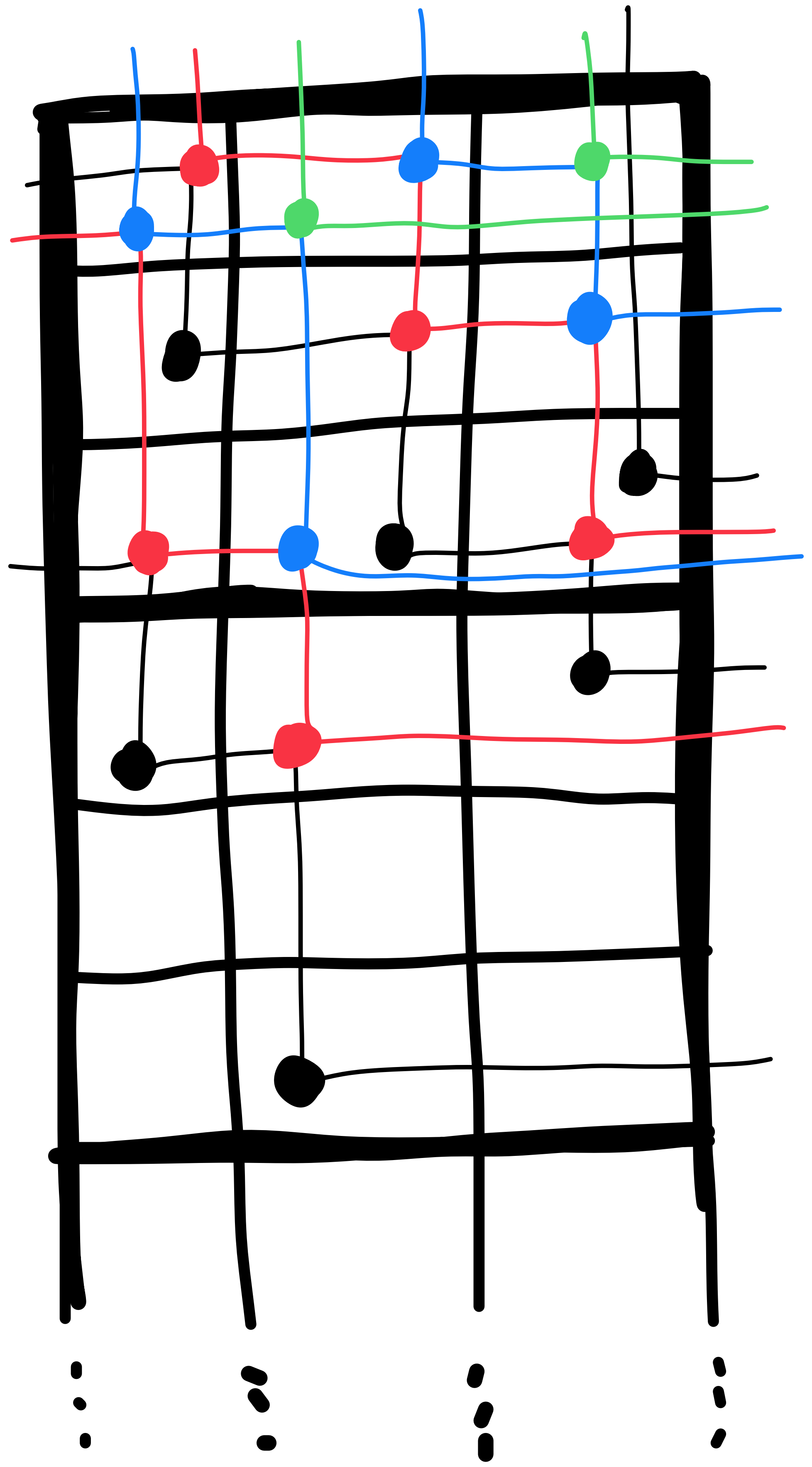

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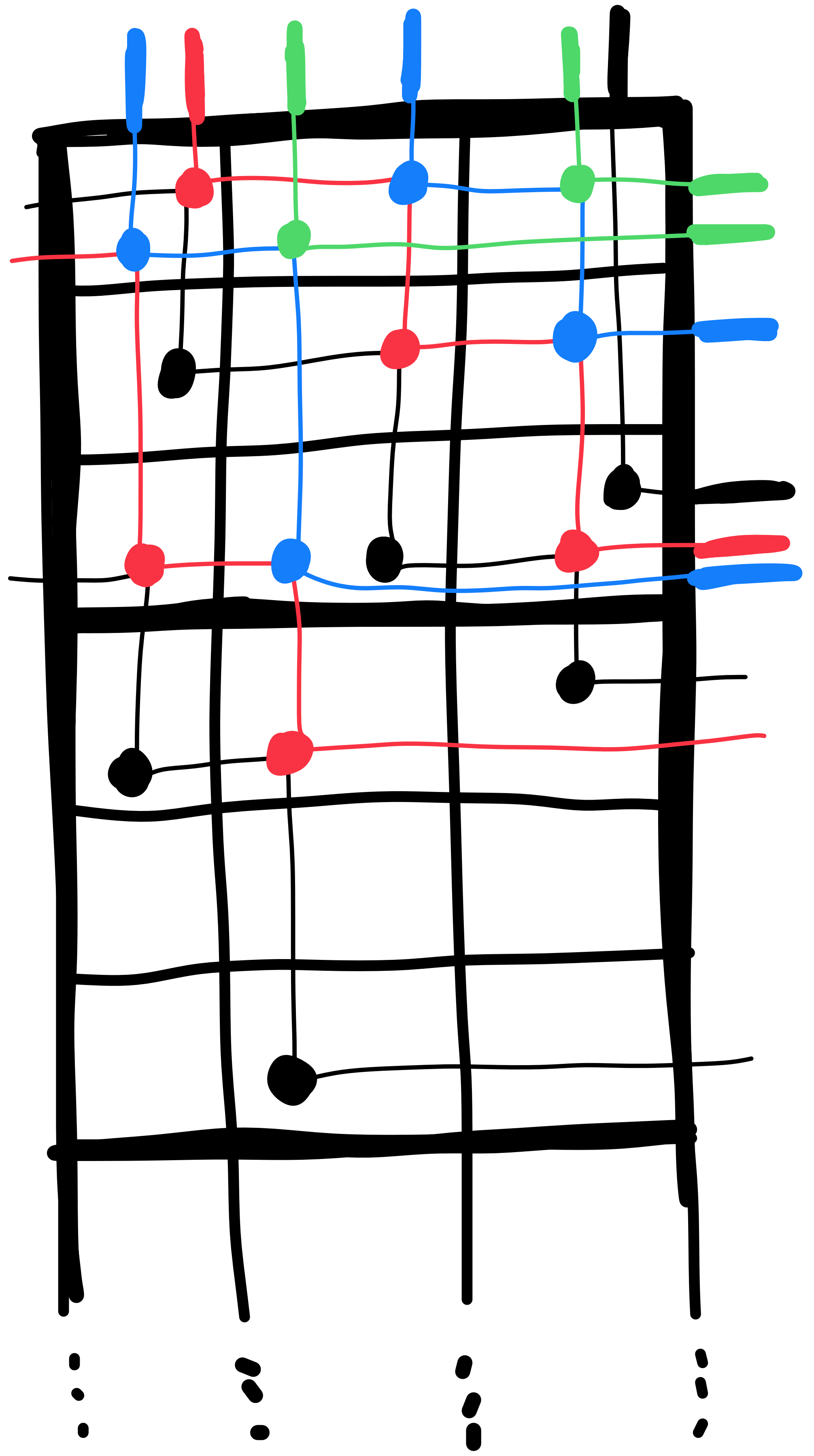

1st row

2nd row

3rd row

4th row

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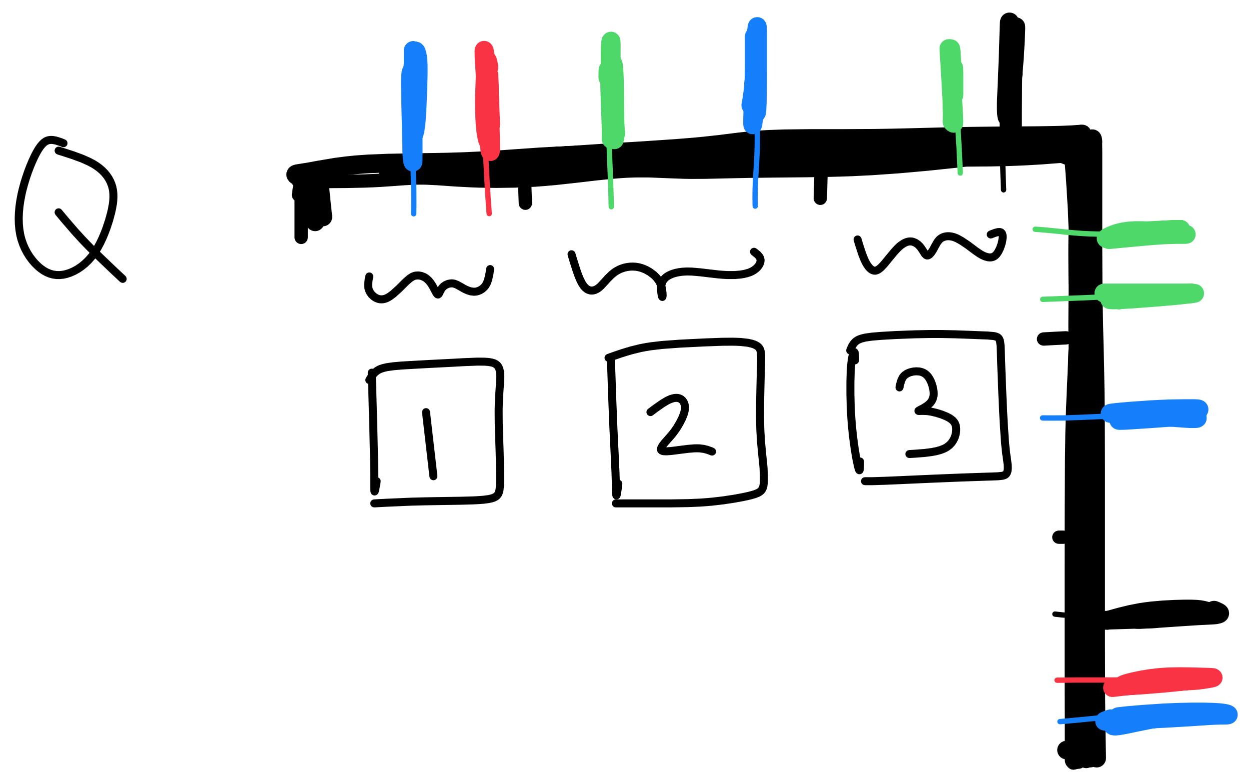

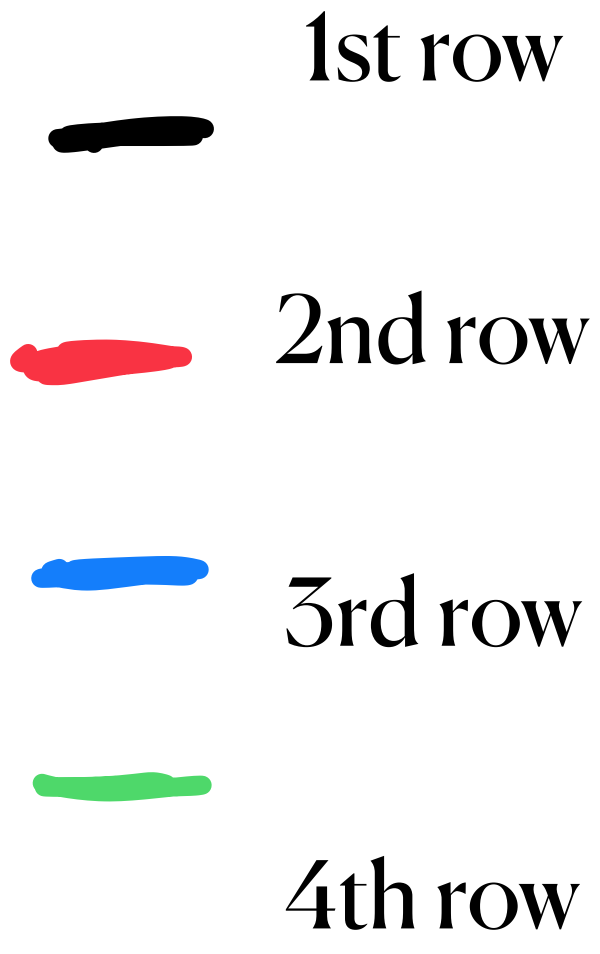

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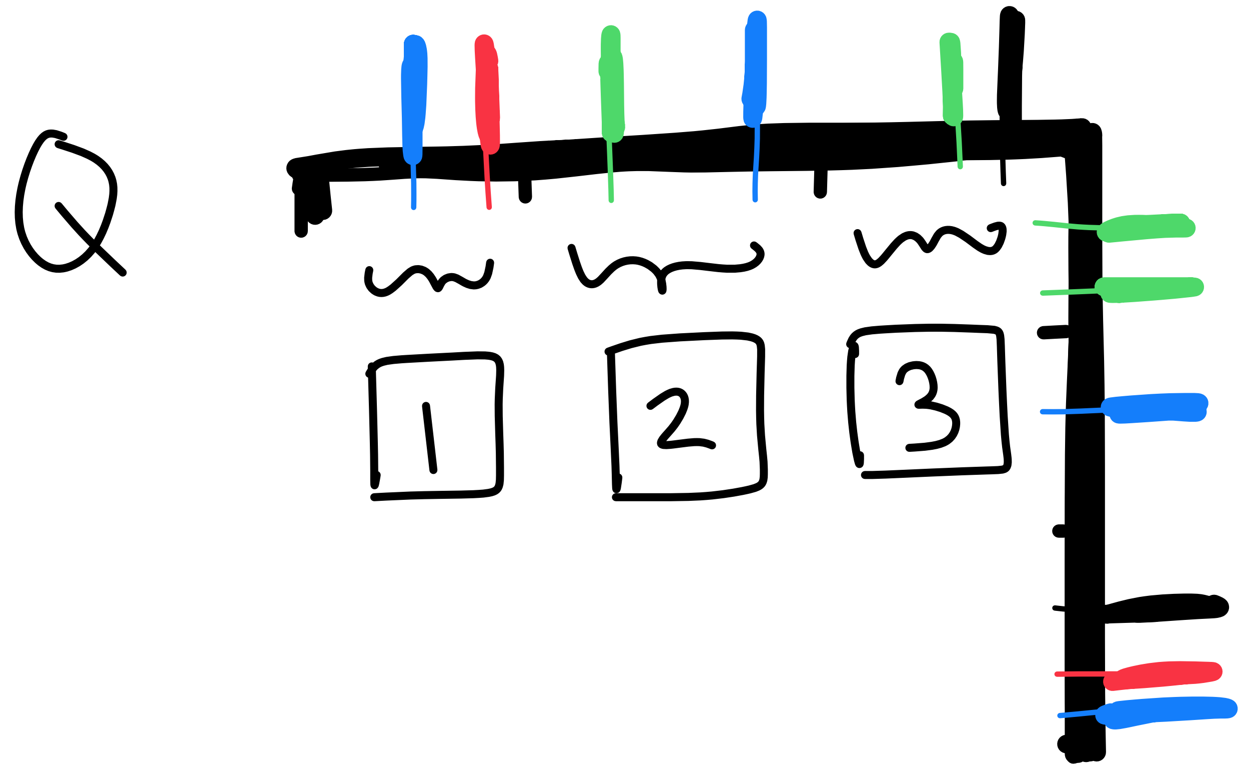

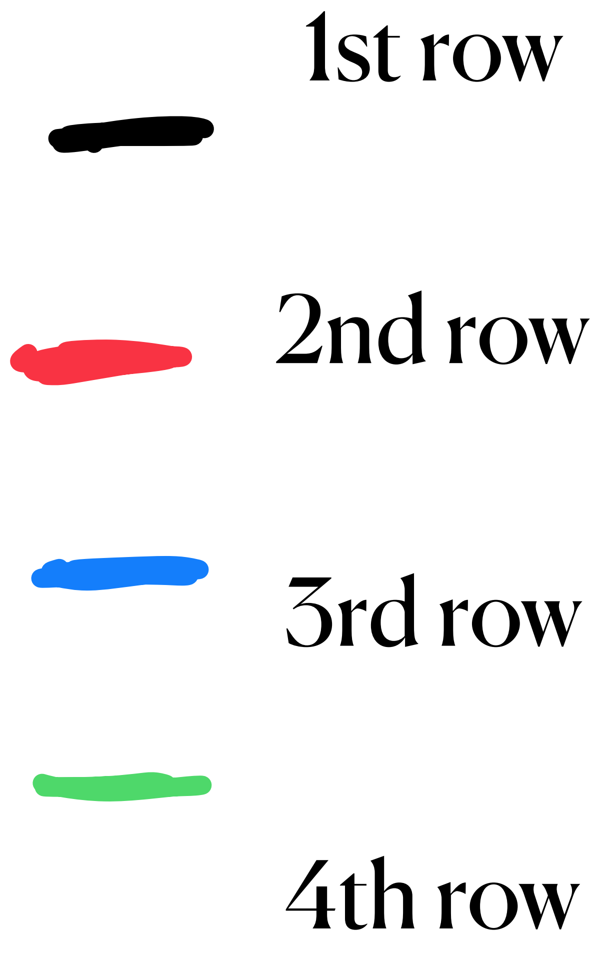

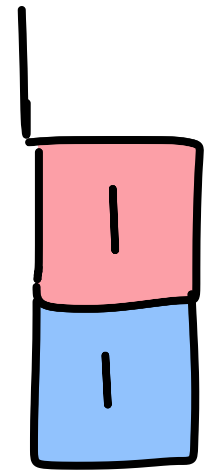

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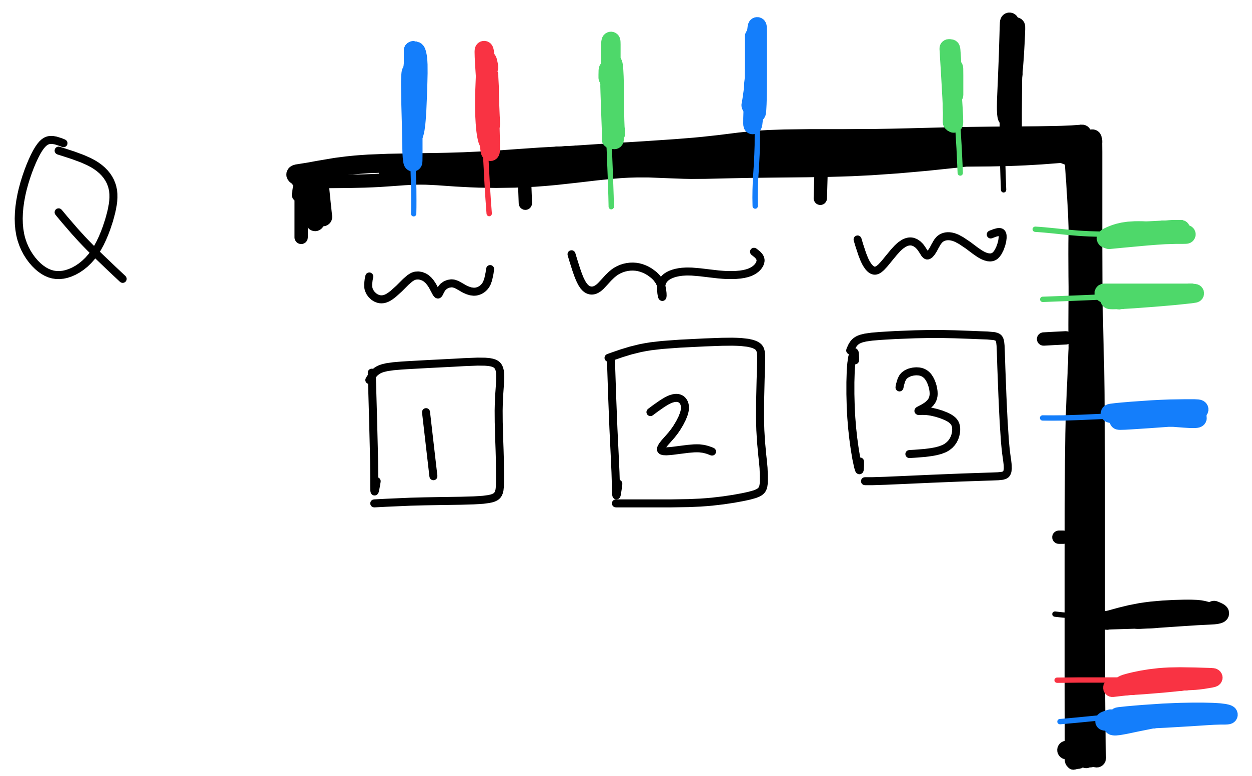
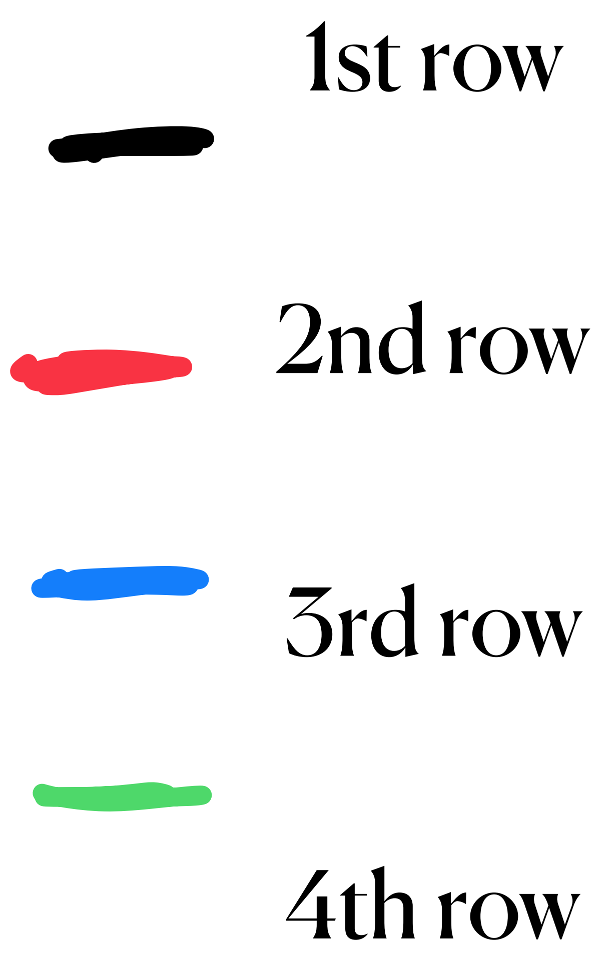
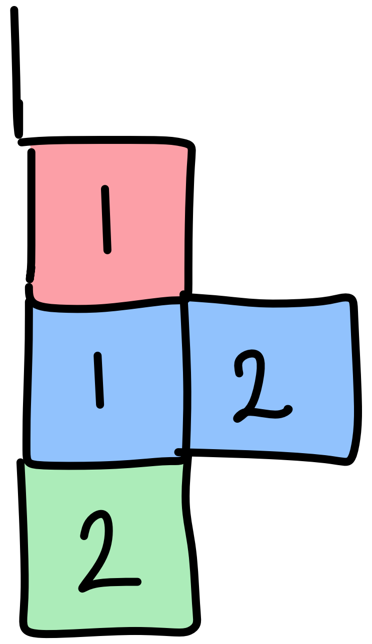


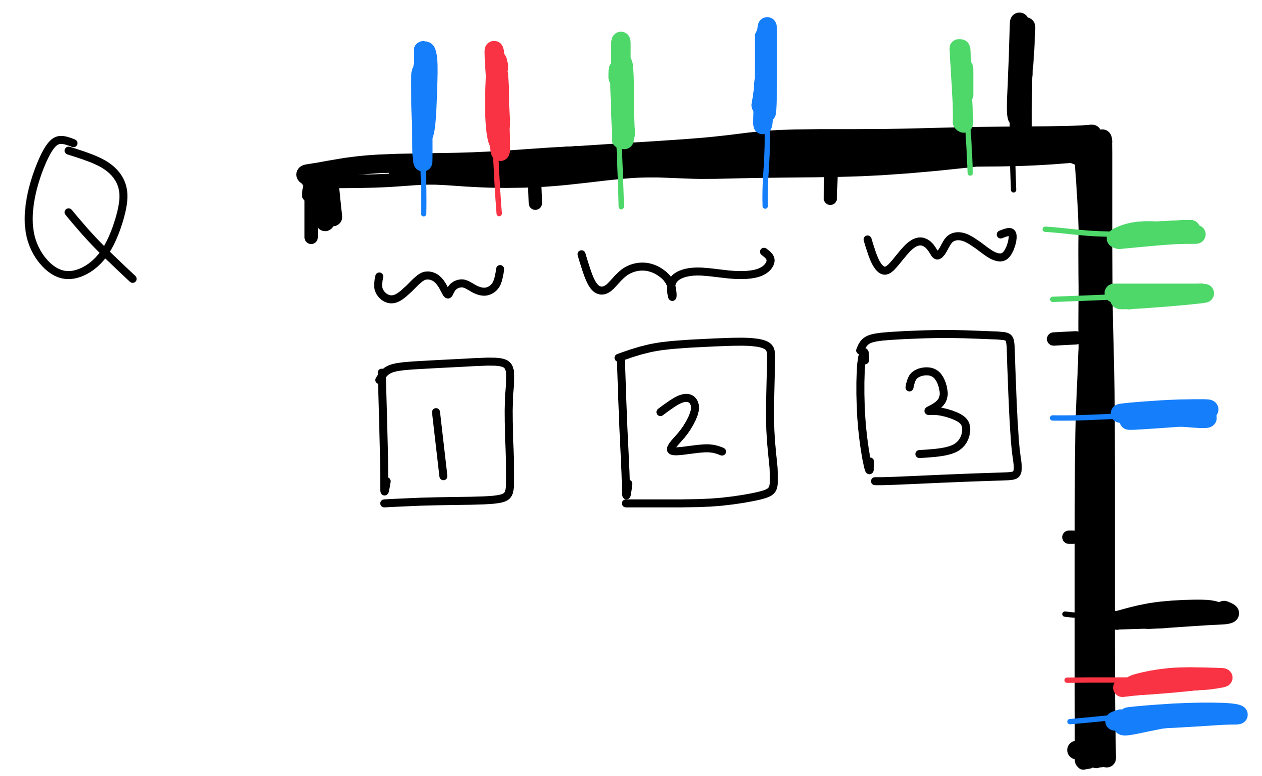




Construction of Υ

$$(M_{i,j}^{k}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

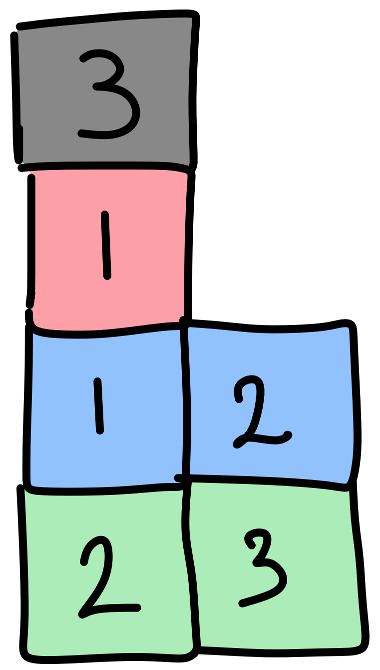


1st row

2nd row

3rd row

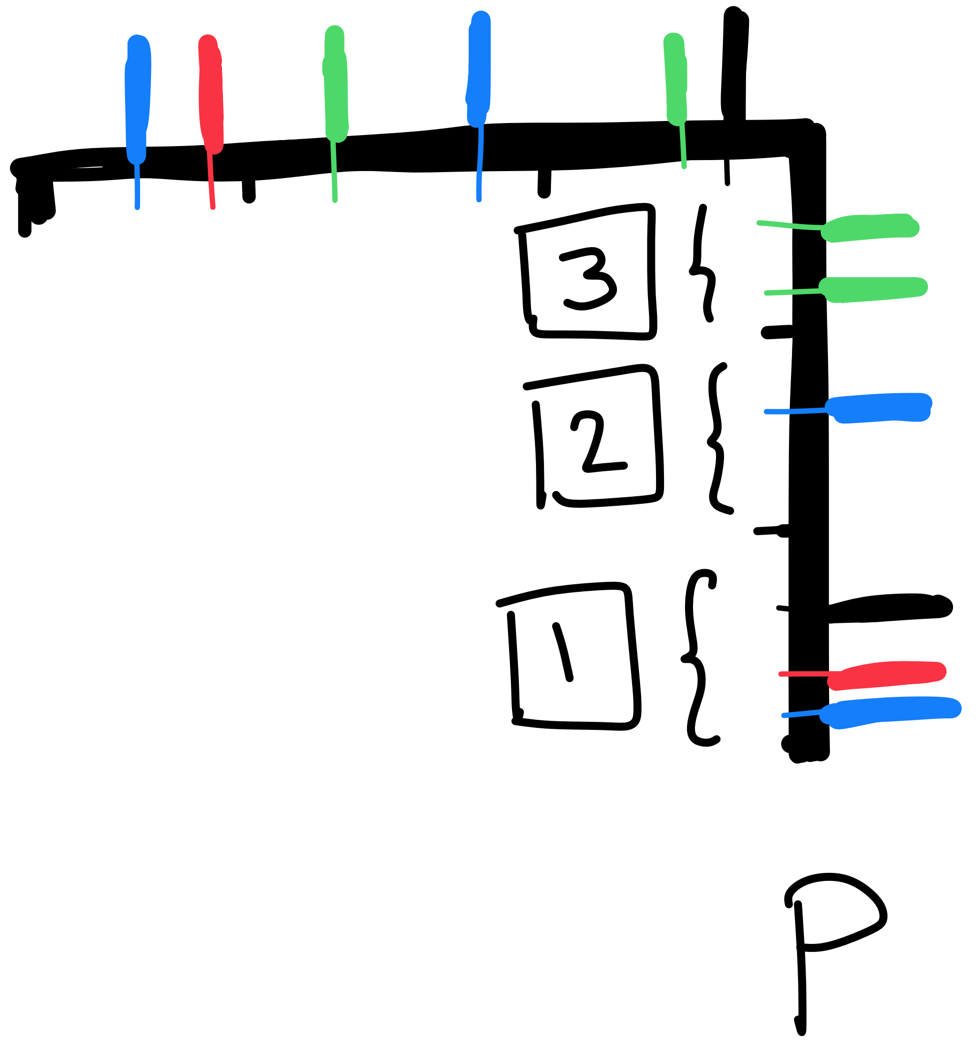
4th row

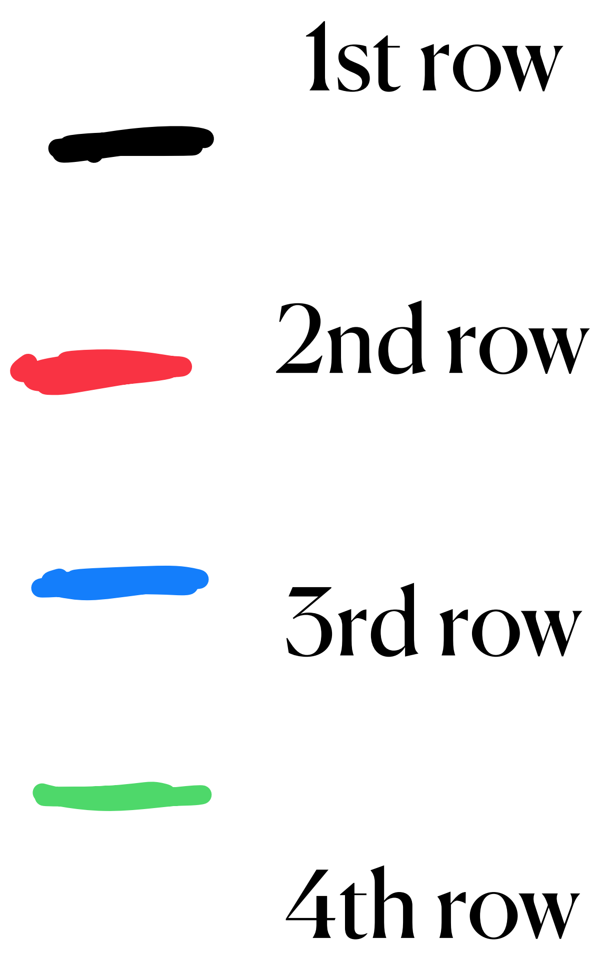


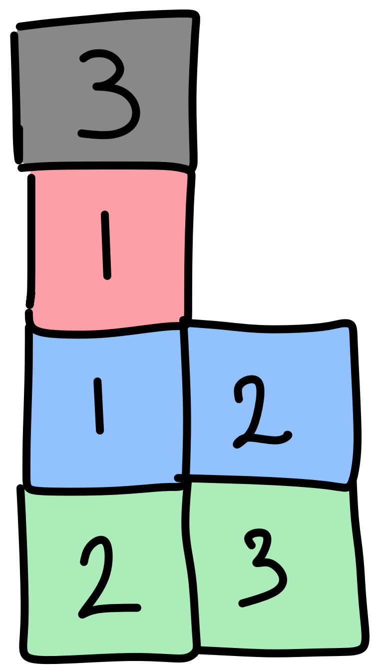
Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$







Construction of Υ

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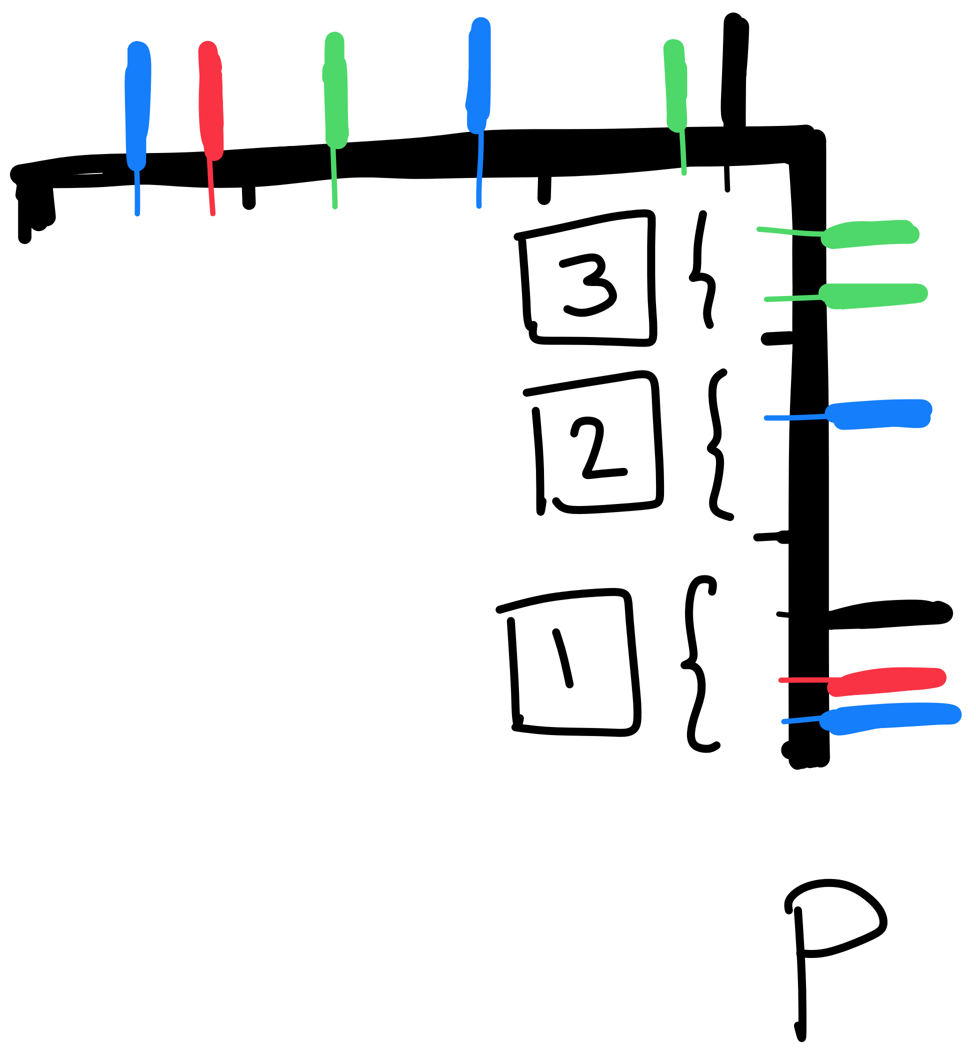

1st row

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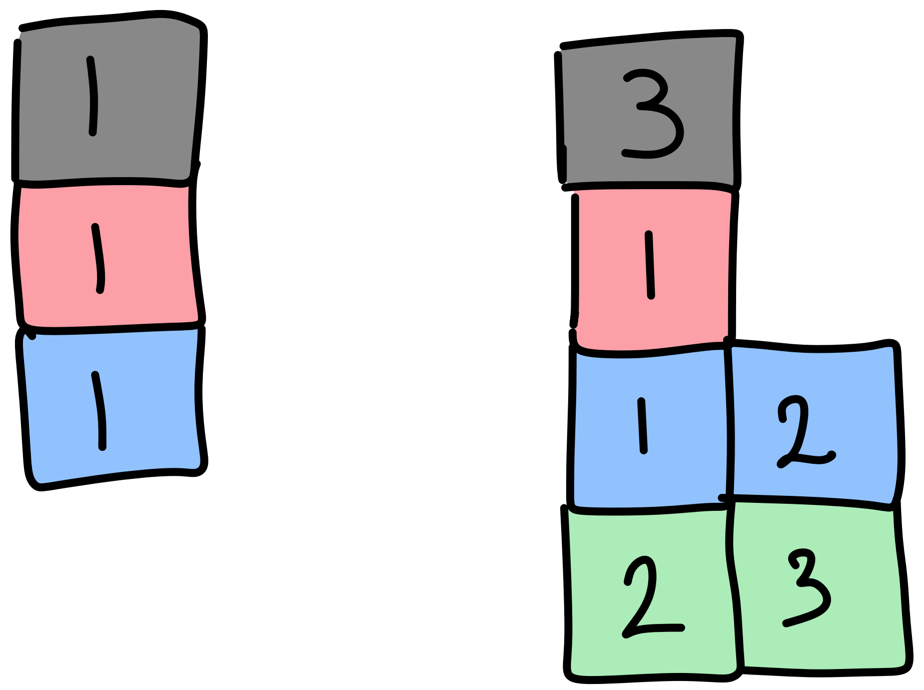

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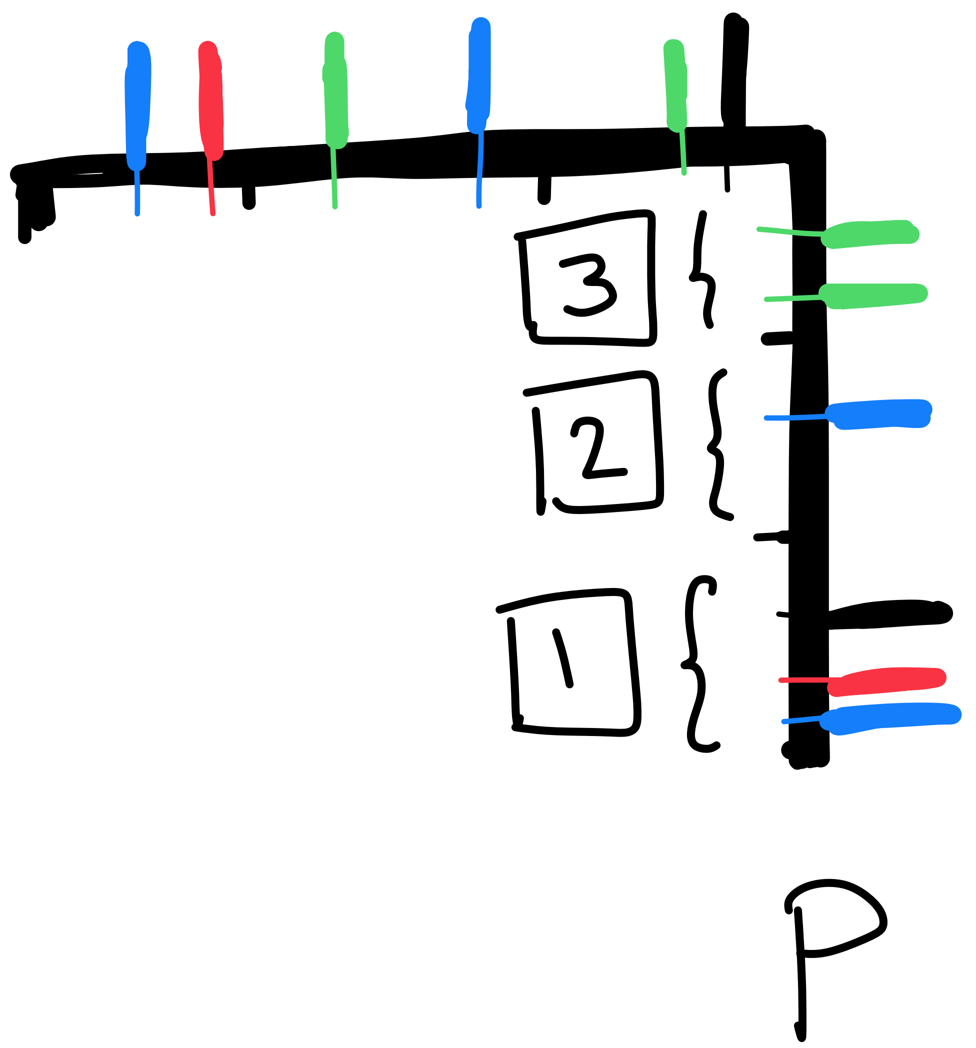
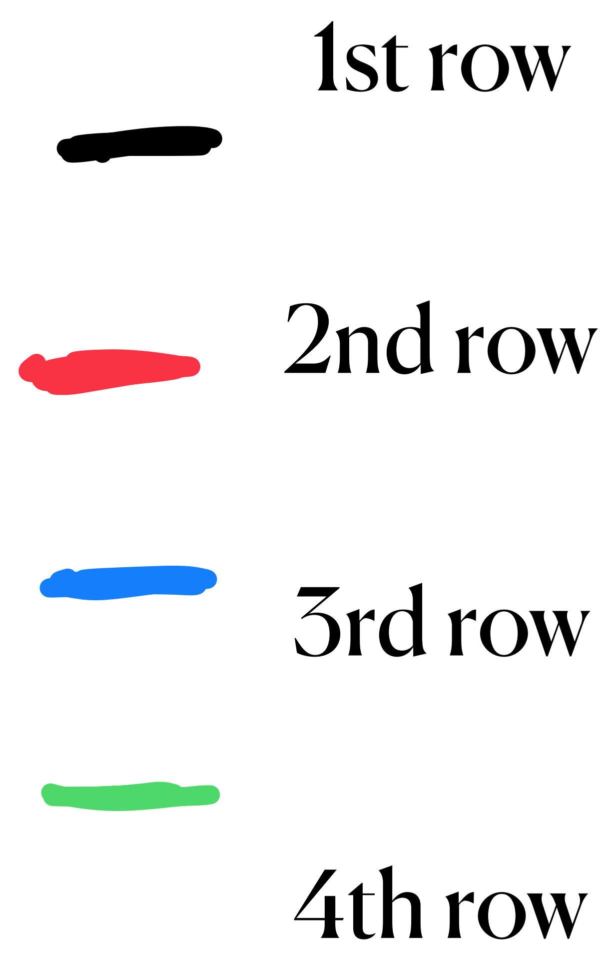
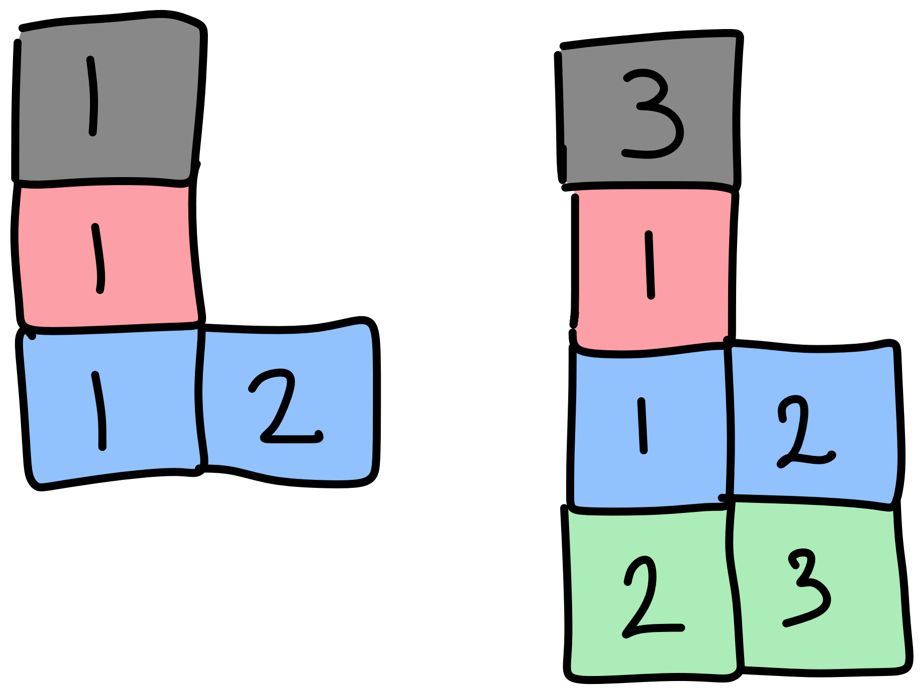
$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (1,1,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,1,\dots) & (0,0,\dots) & (0,1,\dots) \end{pmatrix} \longleftrightarrow \ ?$$



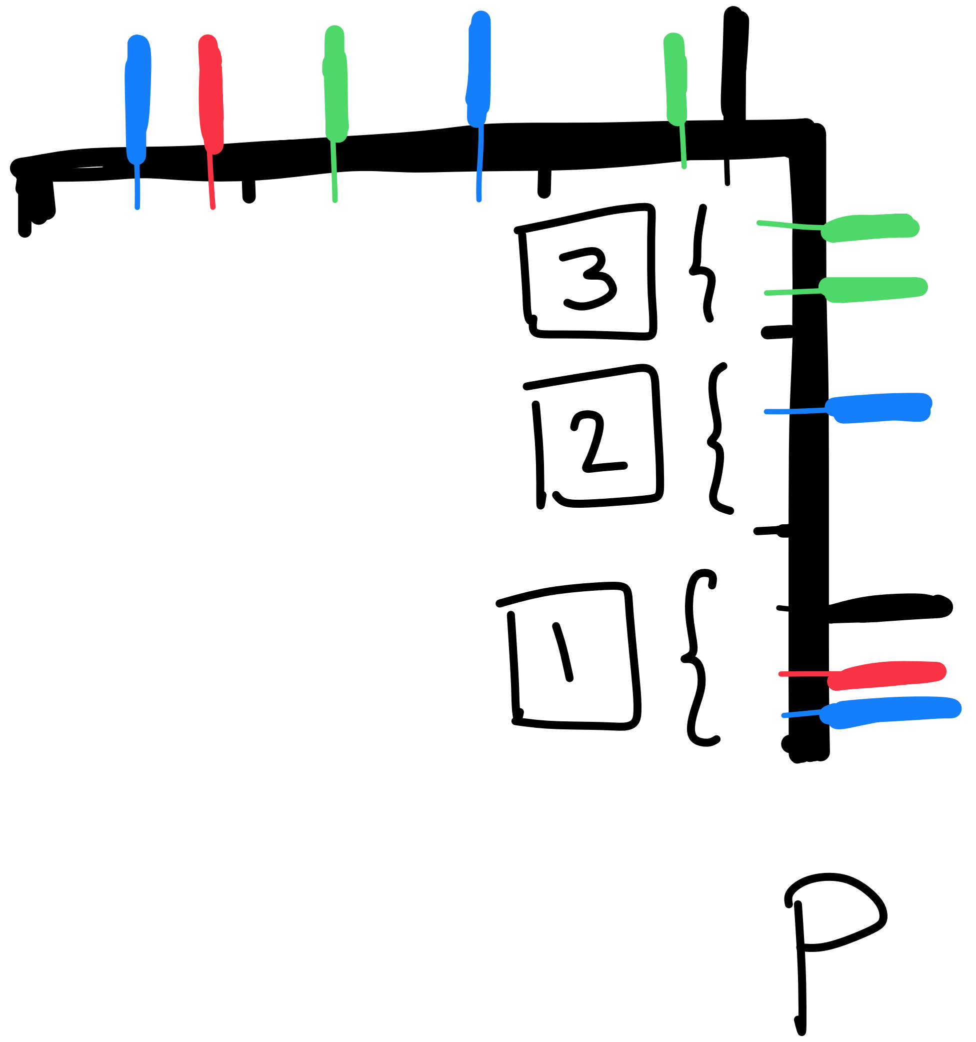


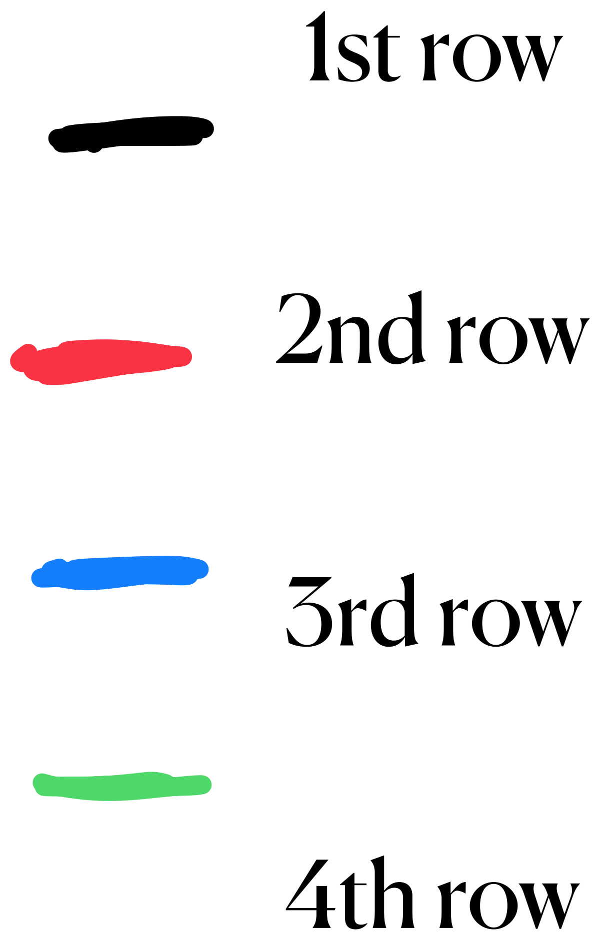


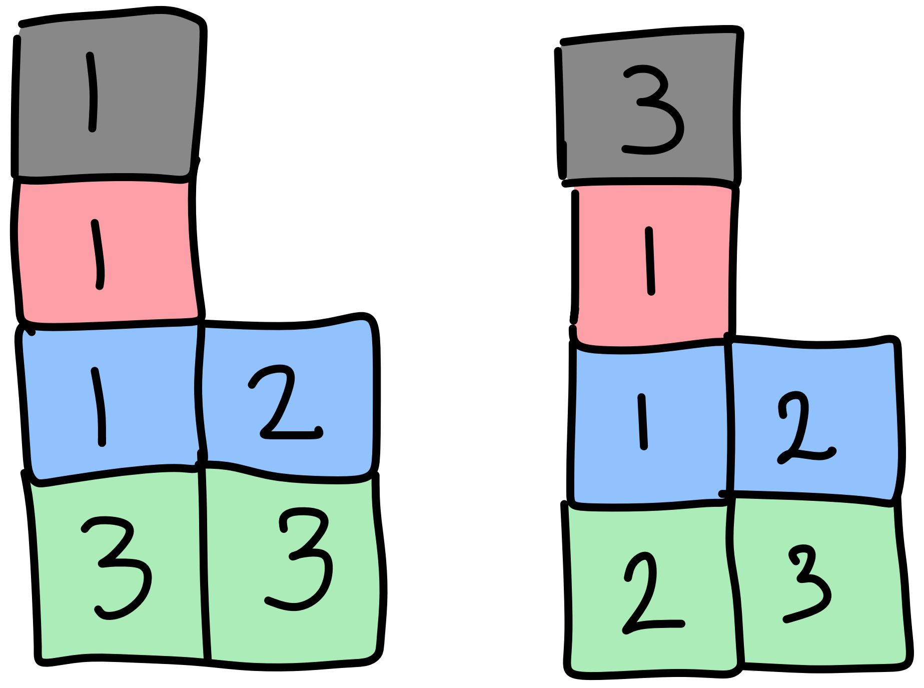
Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$



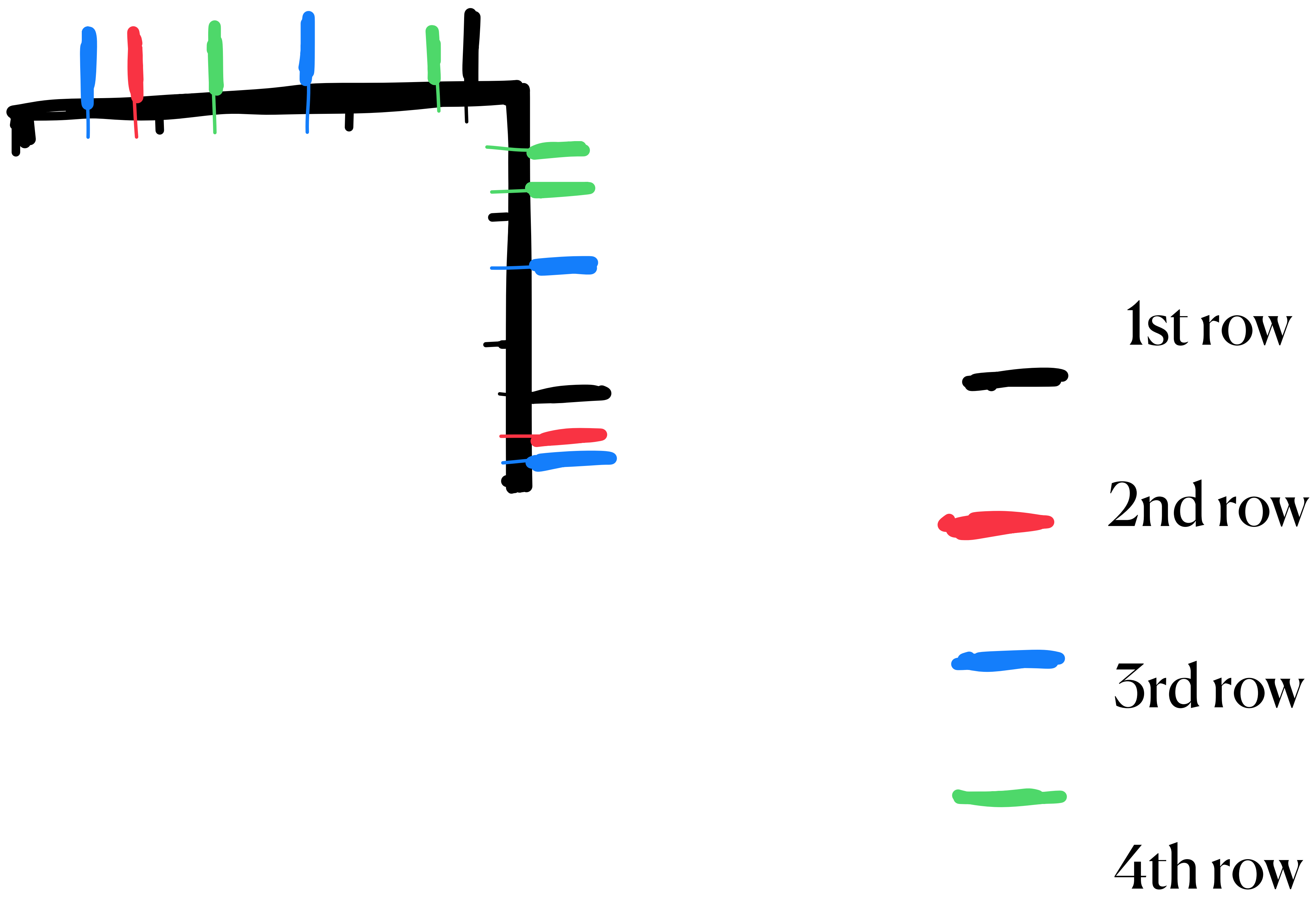


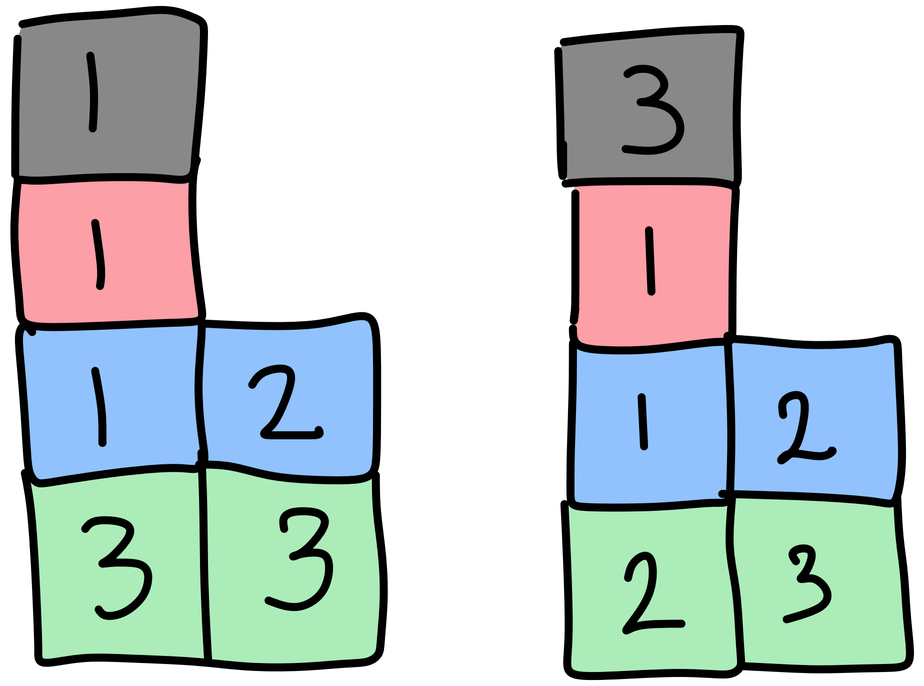


Construction of Υ

$$(M_{i,j}^{k}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$

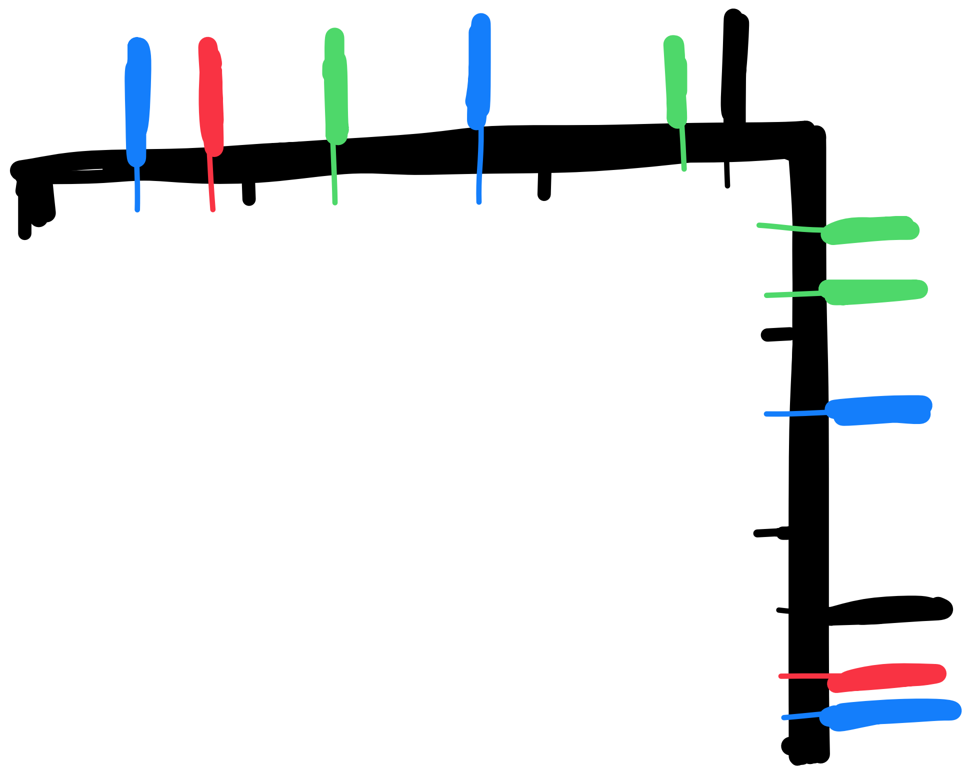




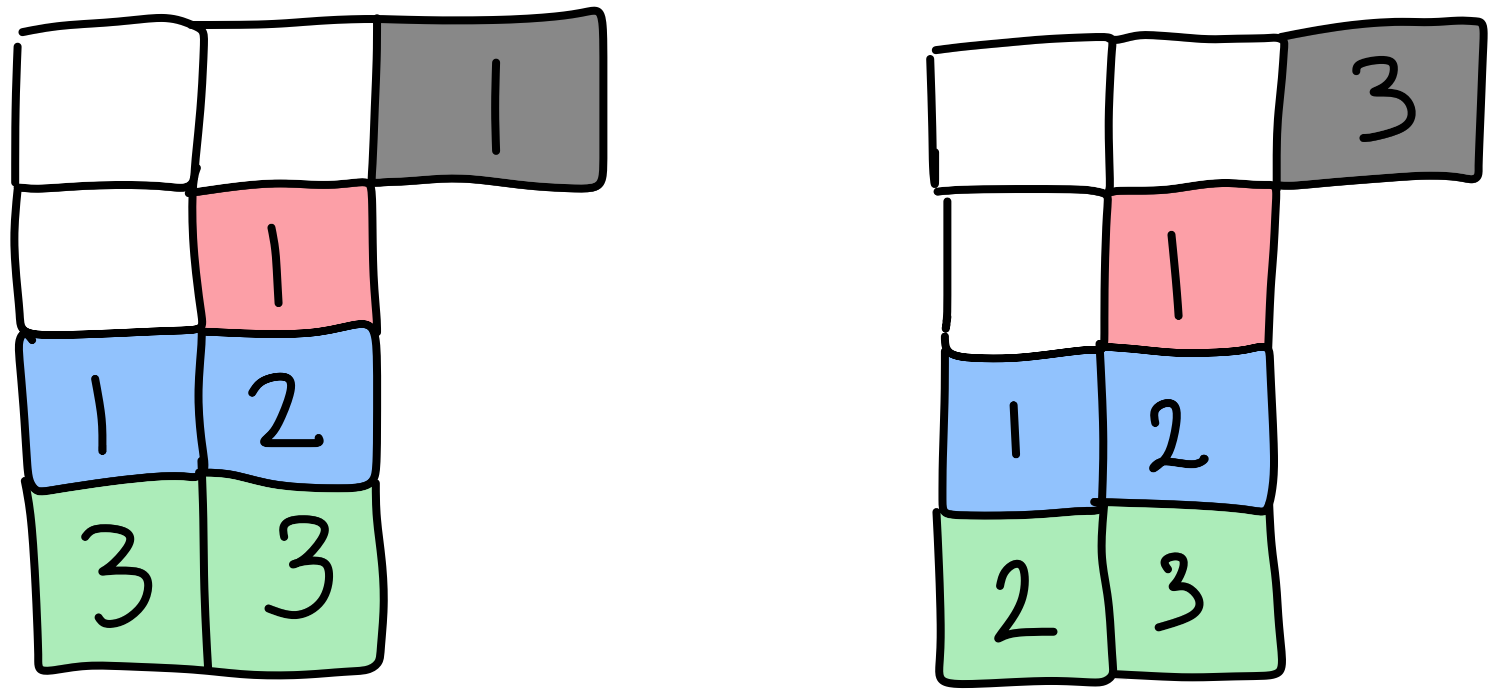
Produce a pair of semi-standard skew tableaux

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$



$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

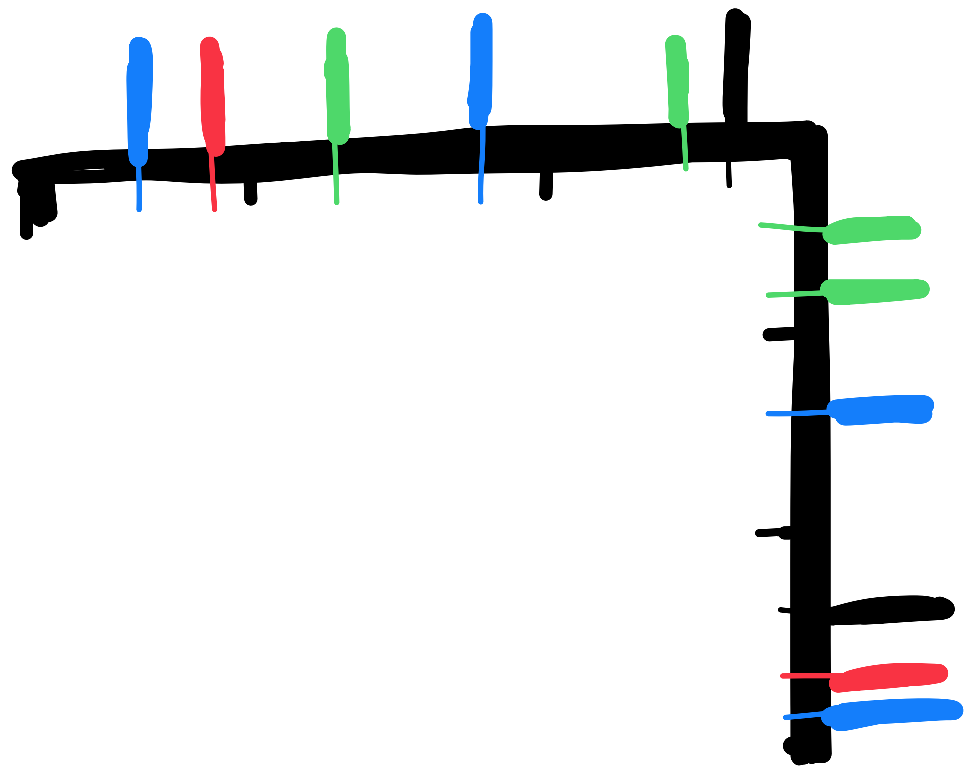


$P \quad , \quad Q$

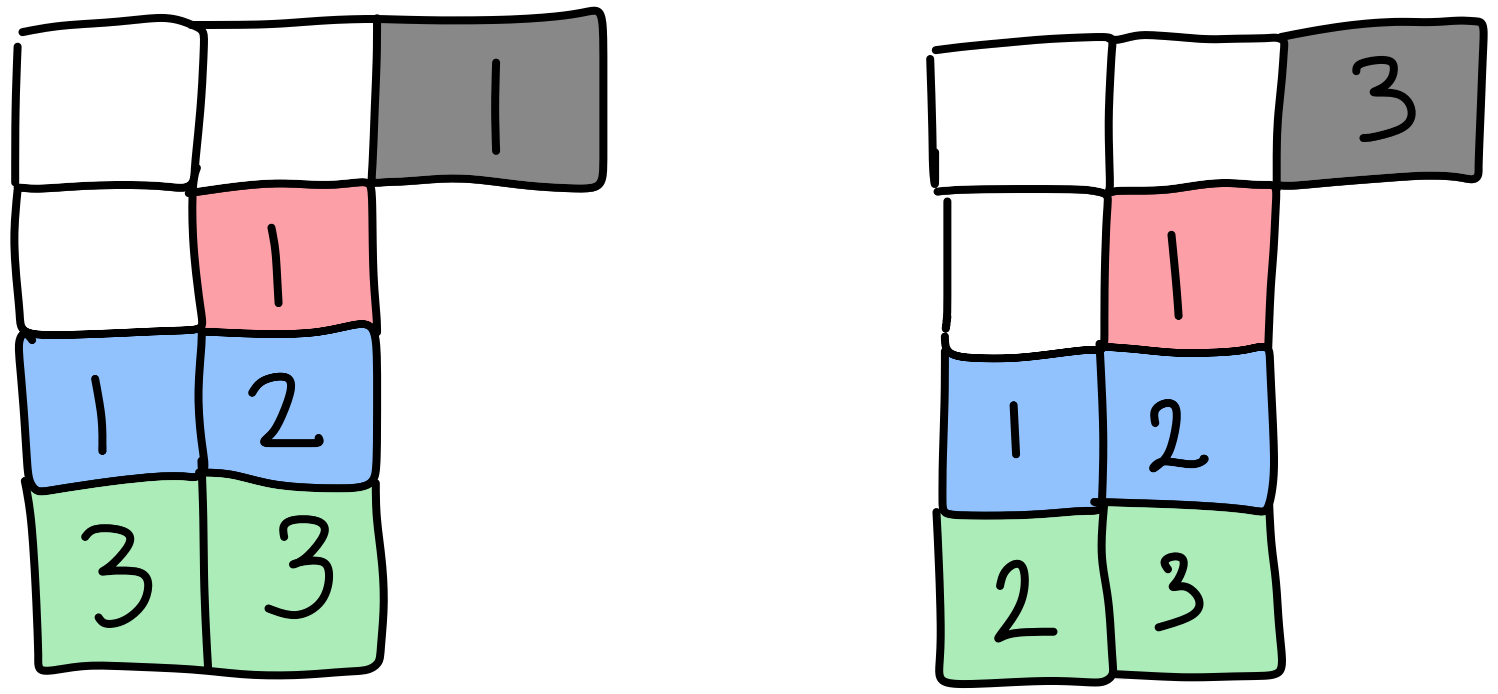
Fact: if ρ = empty shape of P, Q, then $|\rho| = \sum_{k>0} \sum_{i,j=1}^{n} k\, M^k_{i,j}$ [Sagan-Stanley'89]

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$



$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$



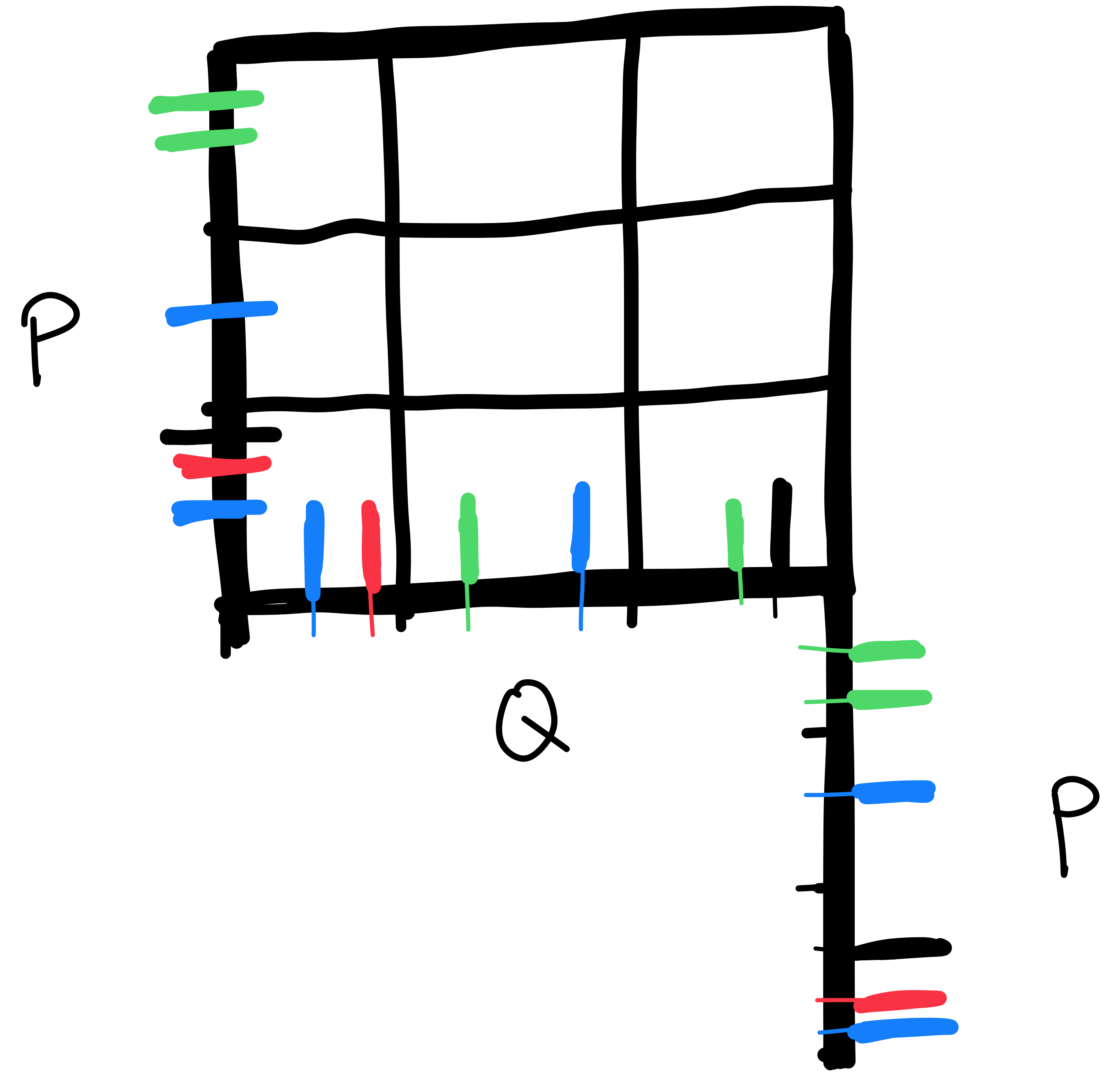
$P \quad , \quad Q$

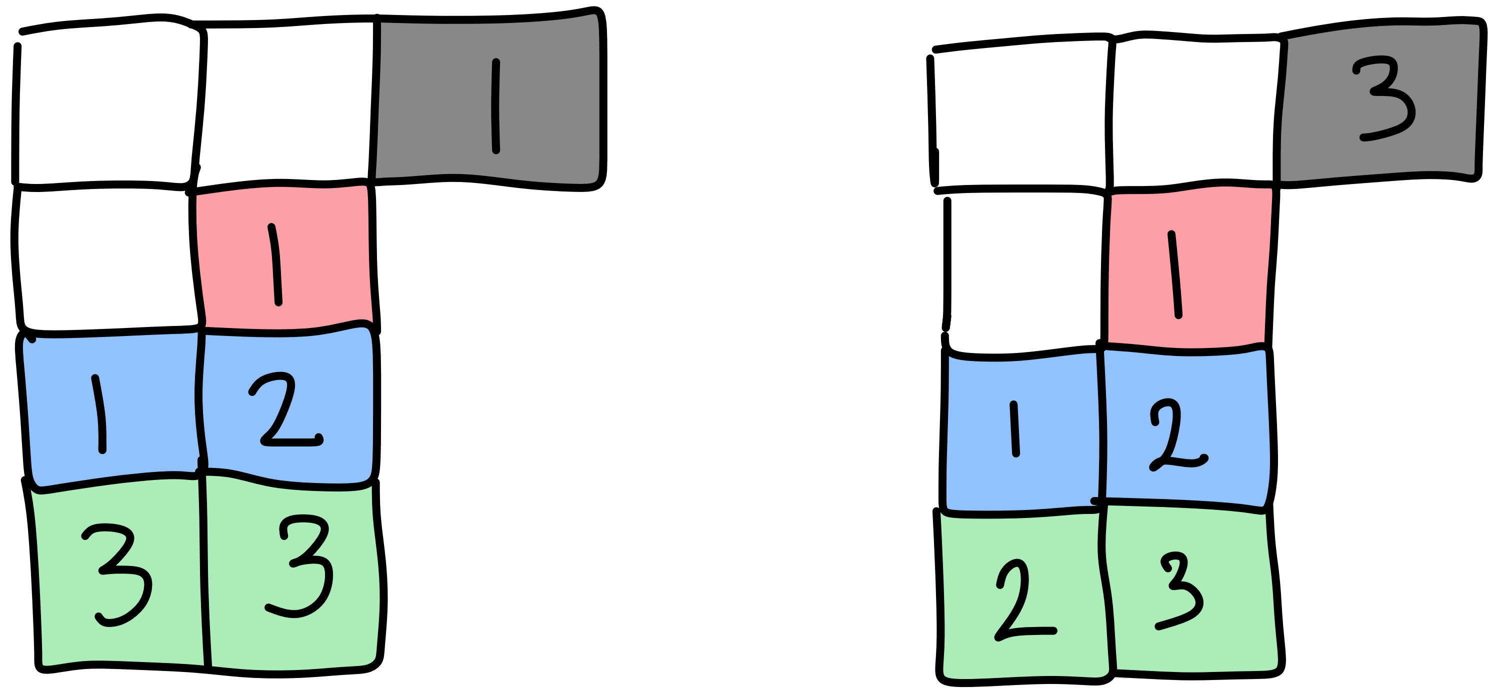
IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$





$P \quad , \quad Q$

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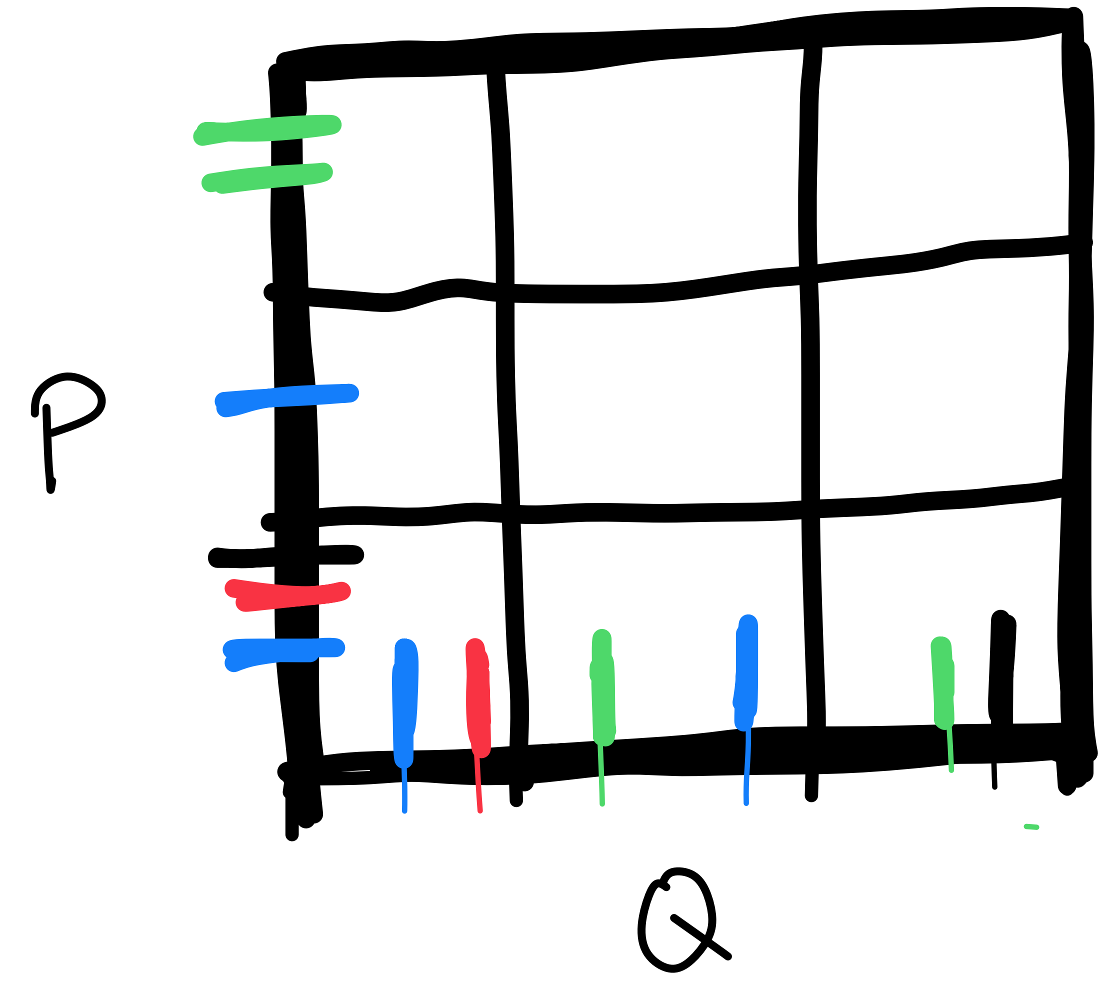
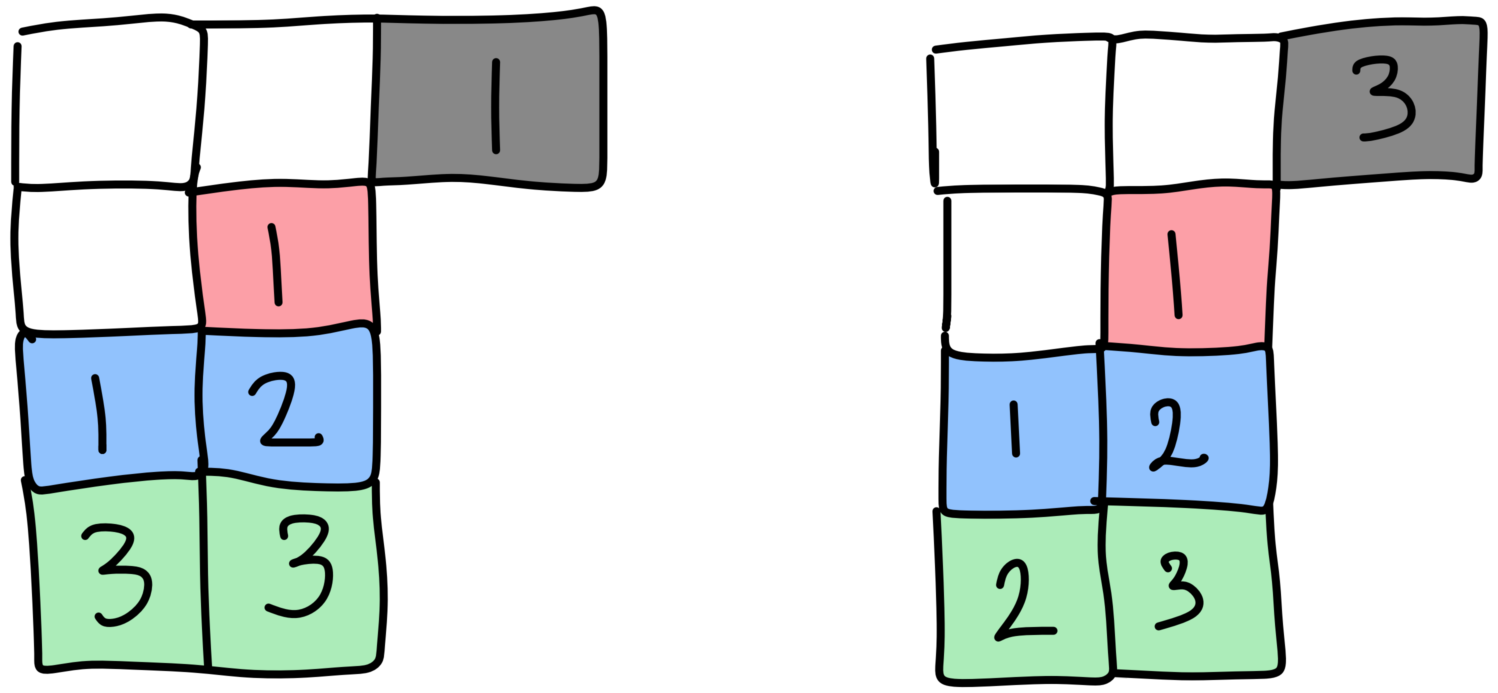
Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$





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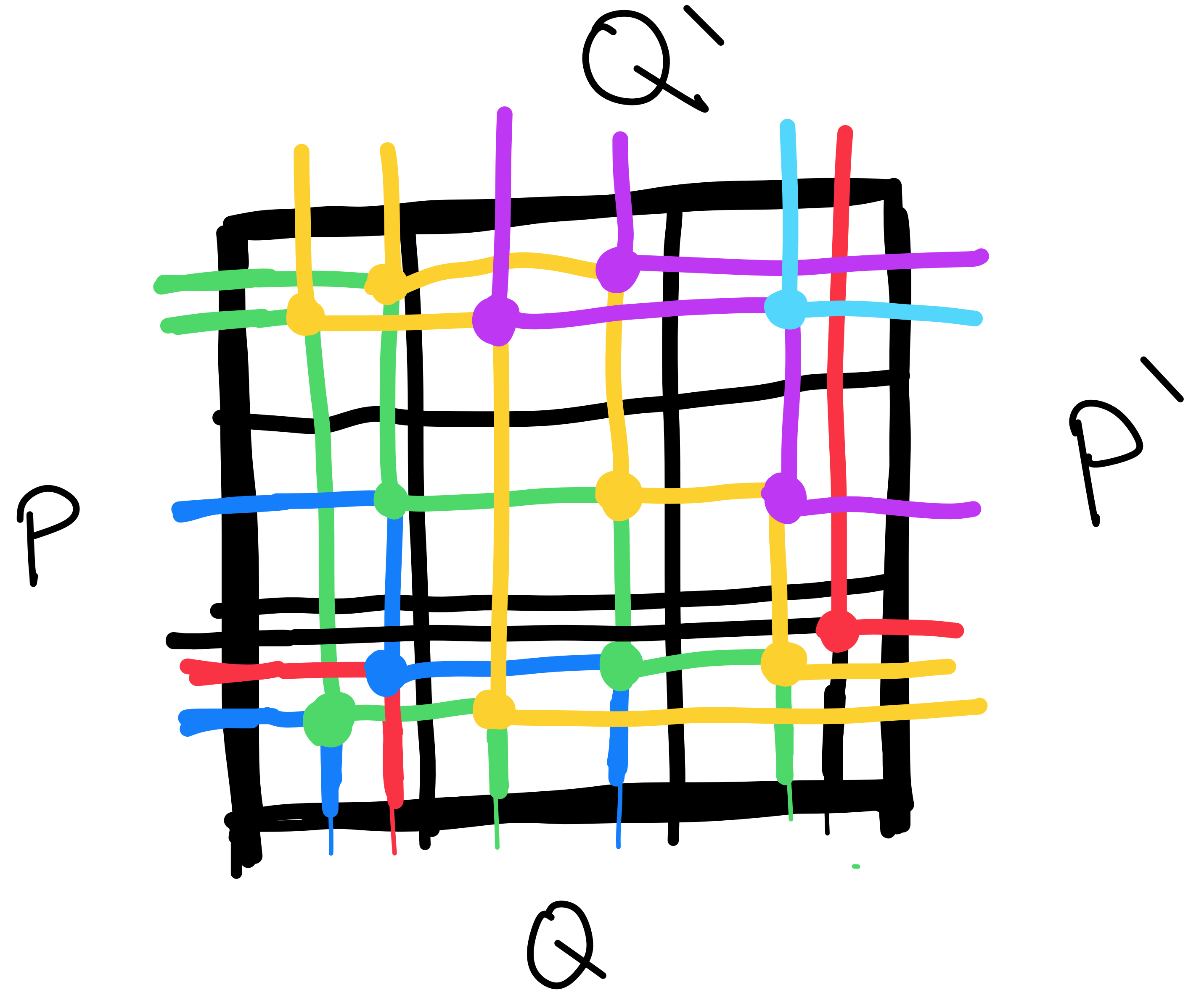
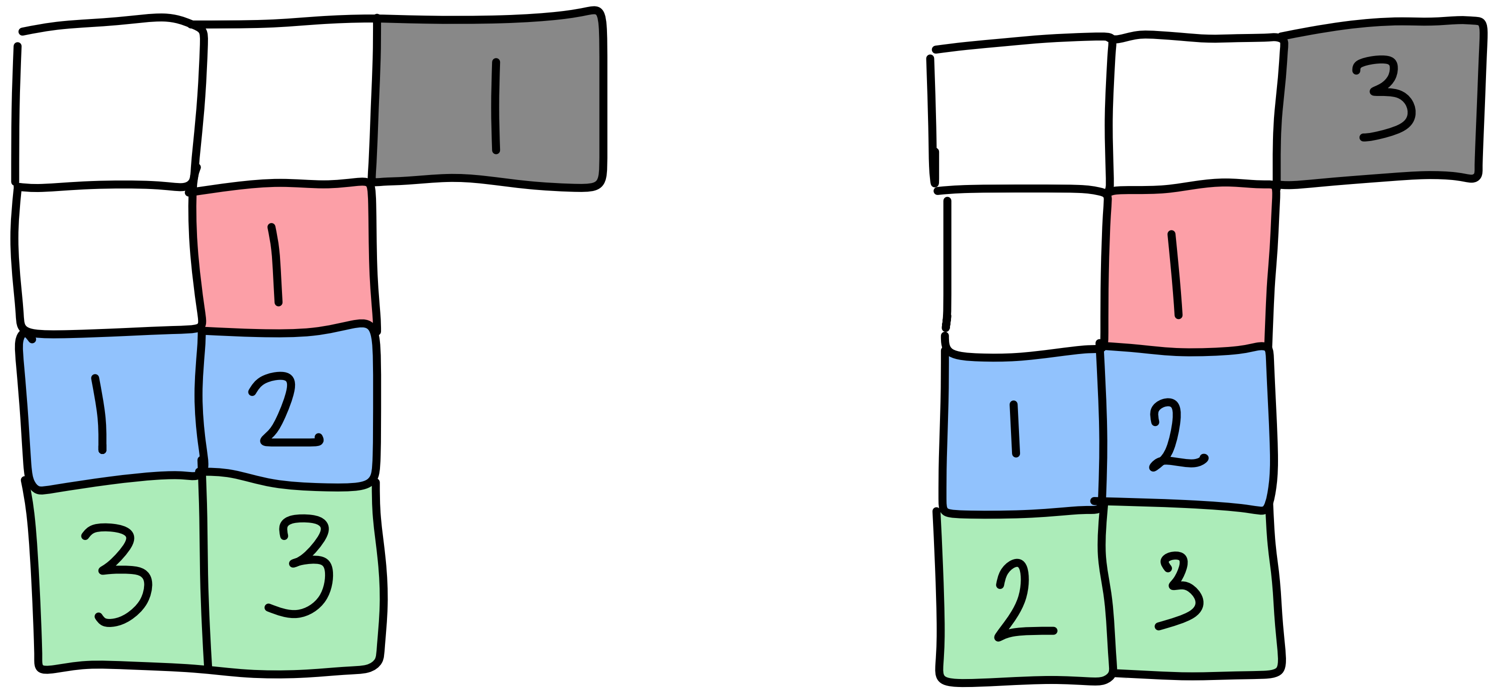
$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$





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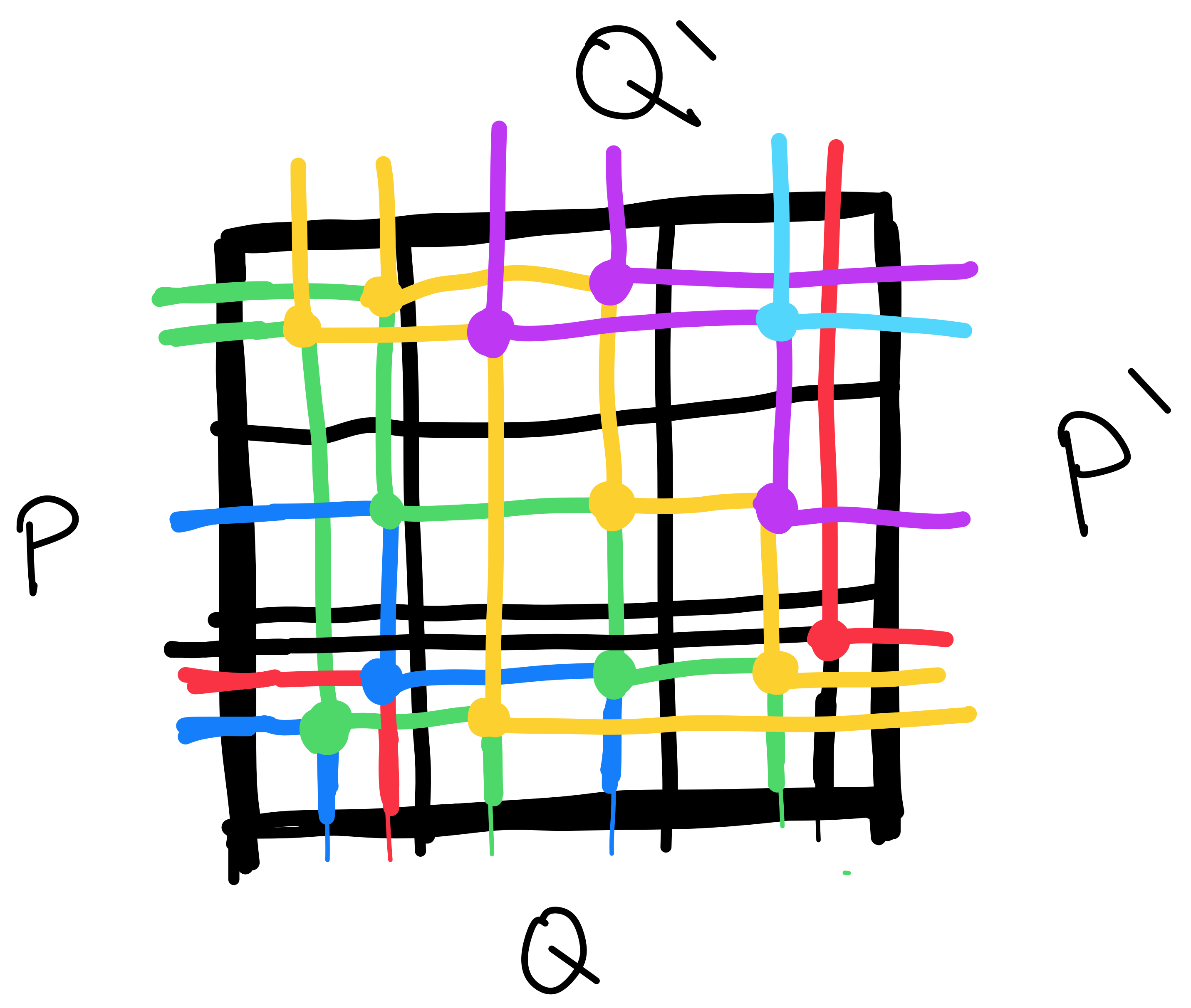
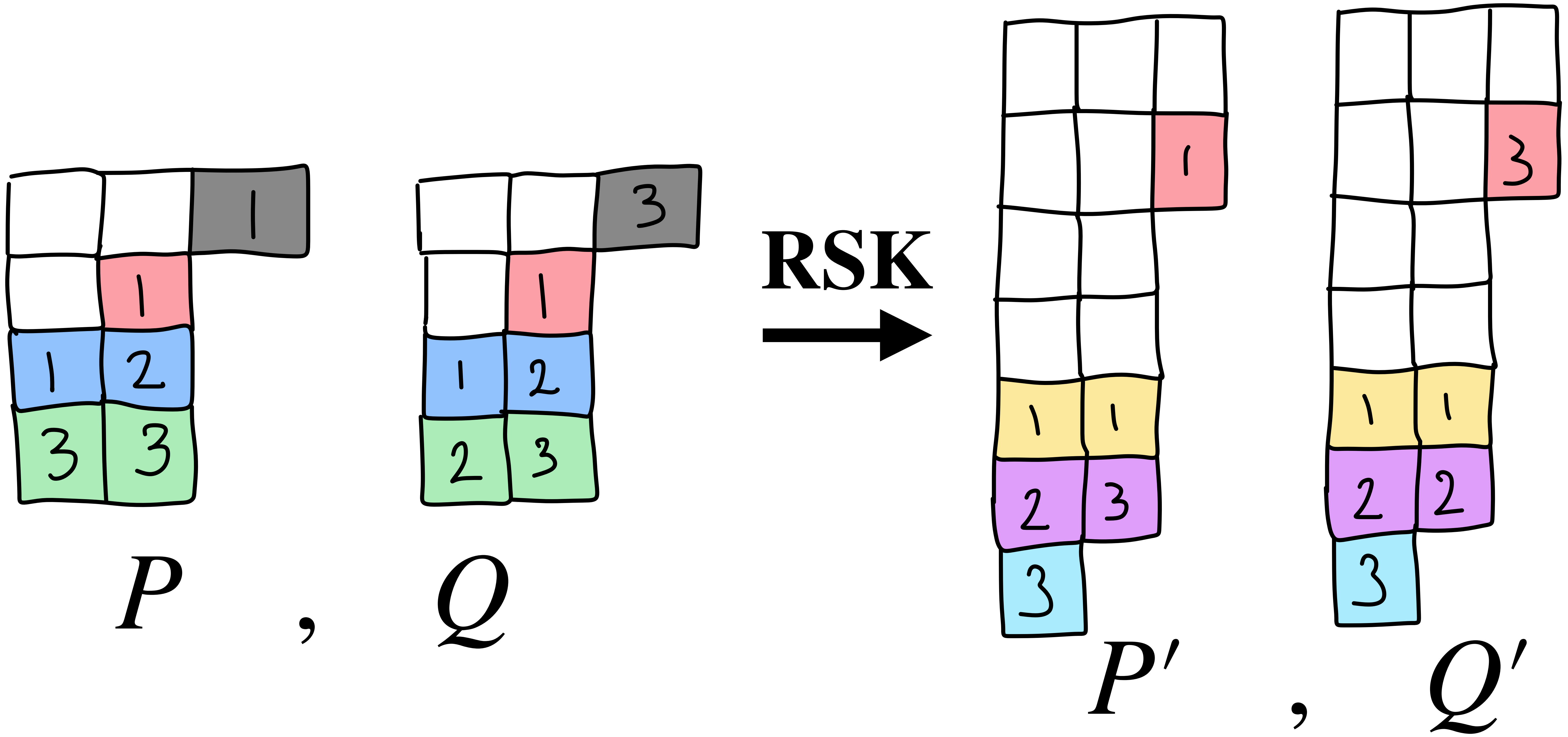
$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$





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$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$

$$(P, Q) \xrightarrow{\textbf{RSK}} (P', Q')$$

IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow ?$$

$$(P, Q) \overset{\textbf{RSK}}{\to} (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

Skew **RSK** dynamics

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (1,1,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,1,\dots) & (0,0,\dots) & (0,1,\dots) \end{pmatrix} \longleftrightarrow \;?$$

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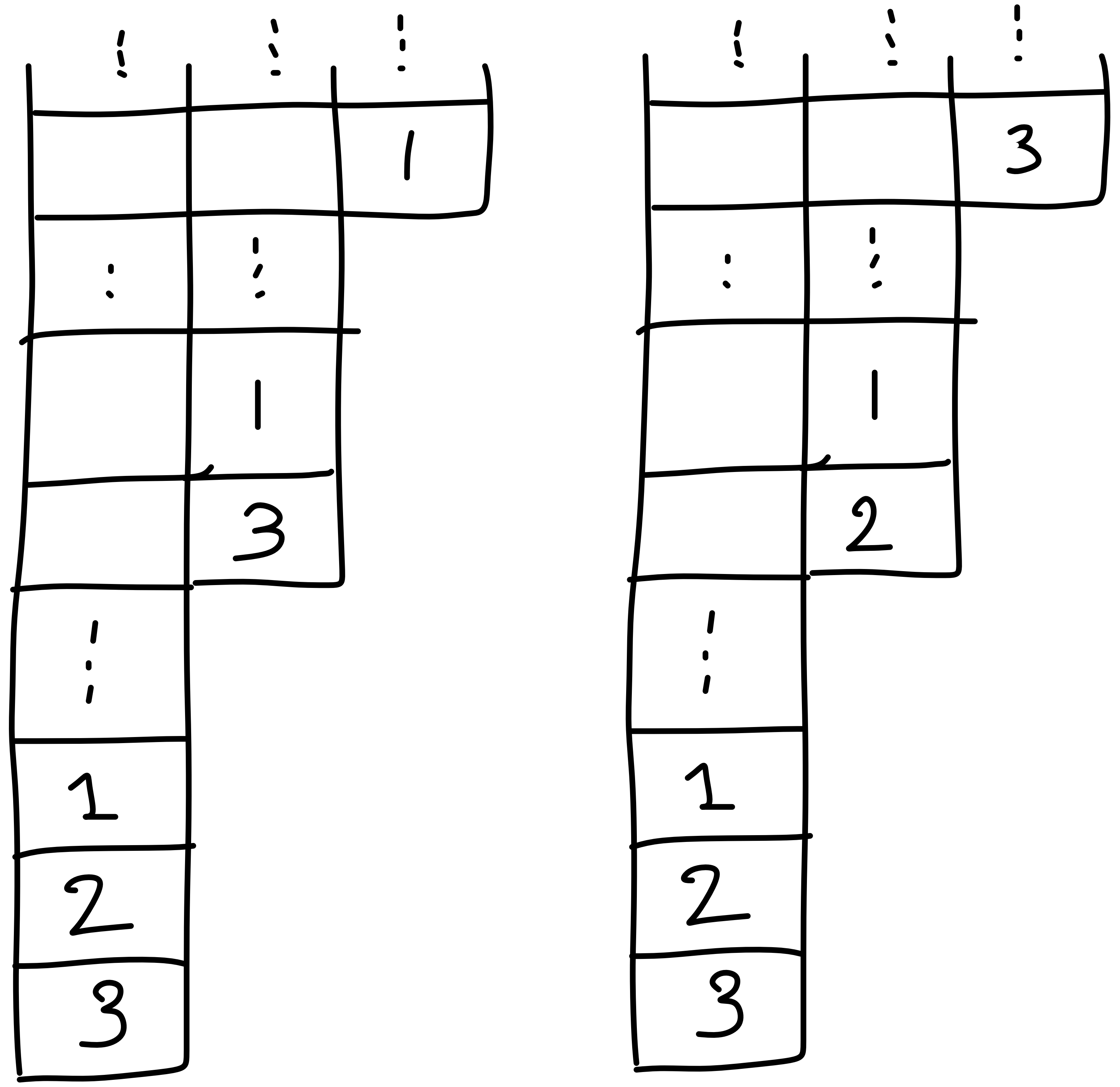
Skew **RSK** dynamics

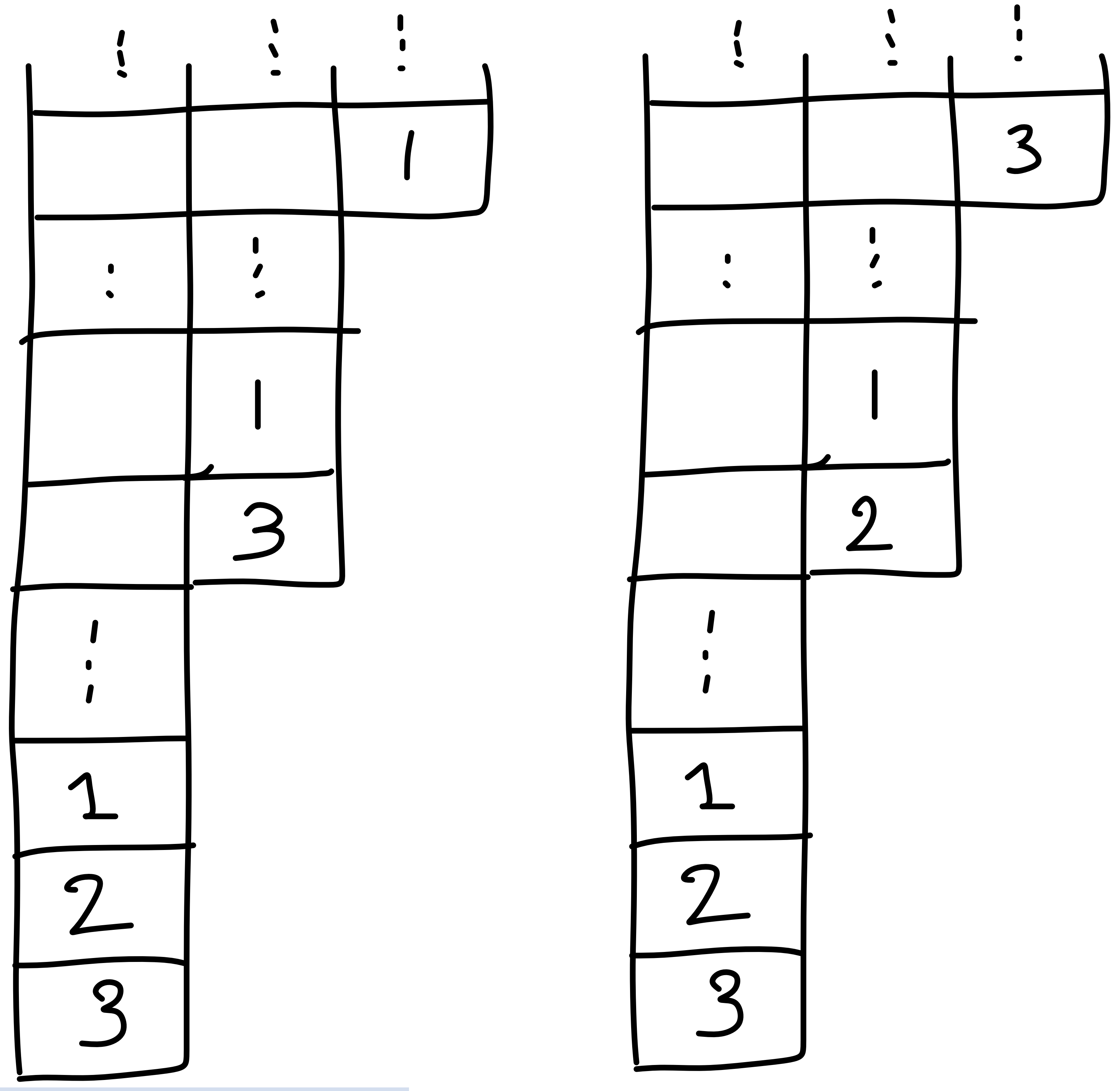


Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become "stable"

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$



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Skew **RSK** dynamics

Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become "stable"

From stable configurations we determine vertically strict tableaux V, W

Construction of Υ

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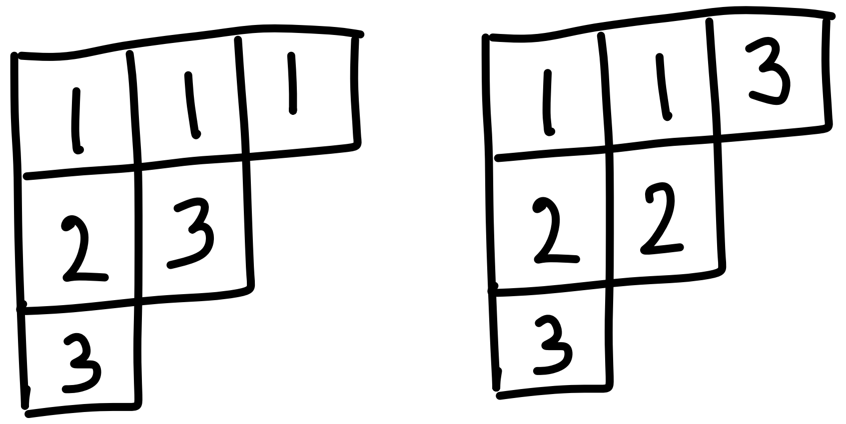
$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \Big($$  $$; \kappa \Big)$$

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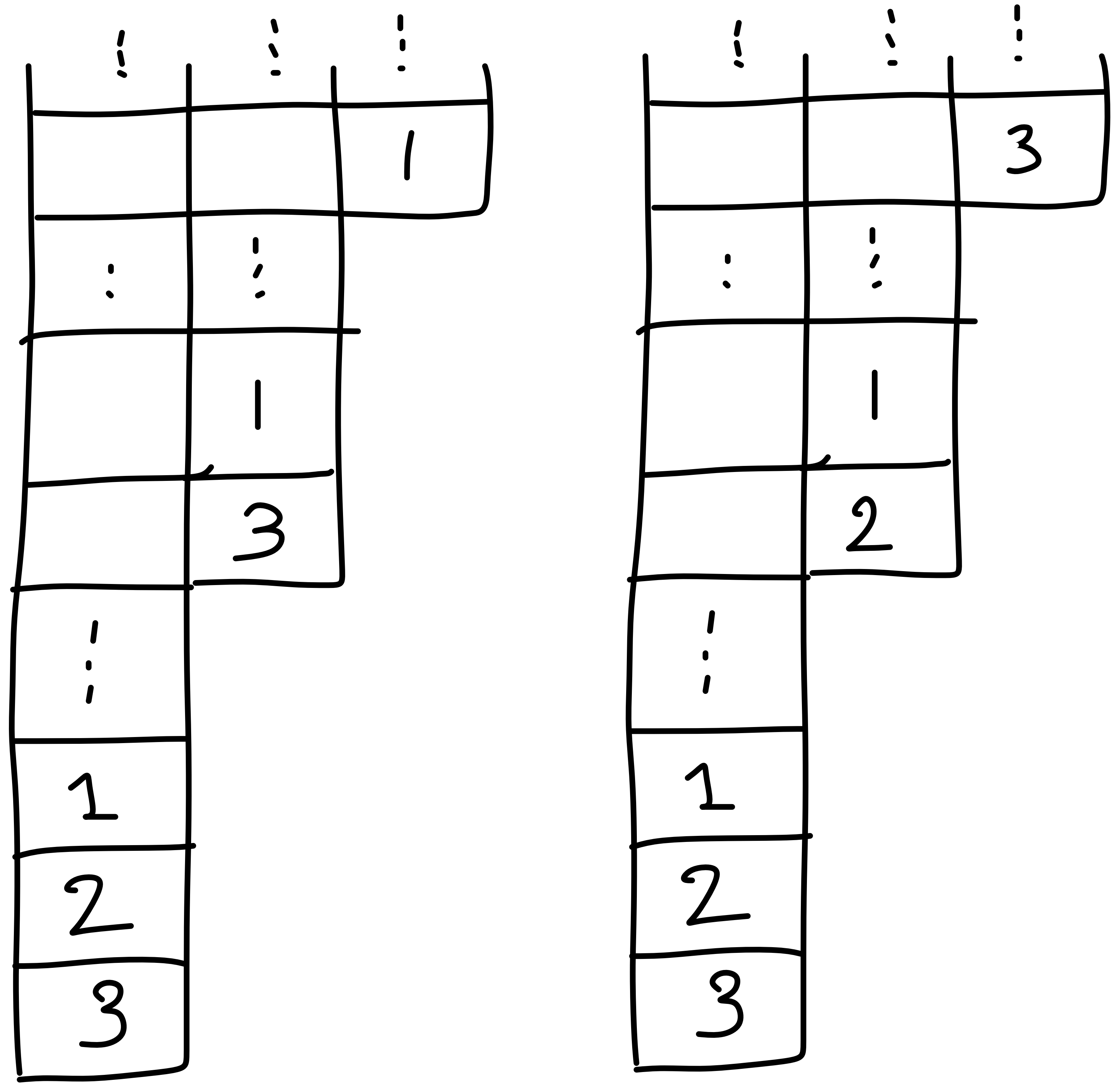
Skew **RSK** dynamics



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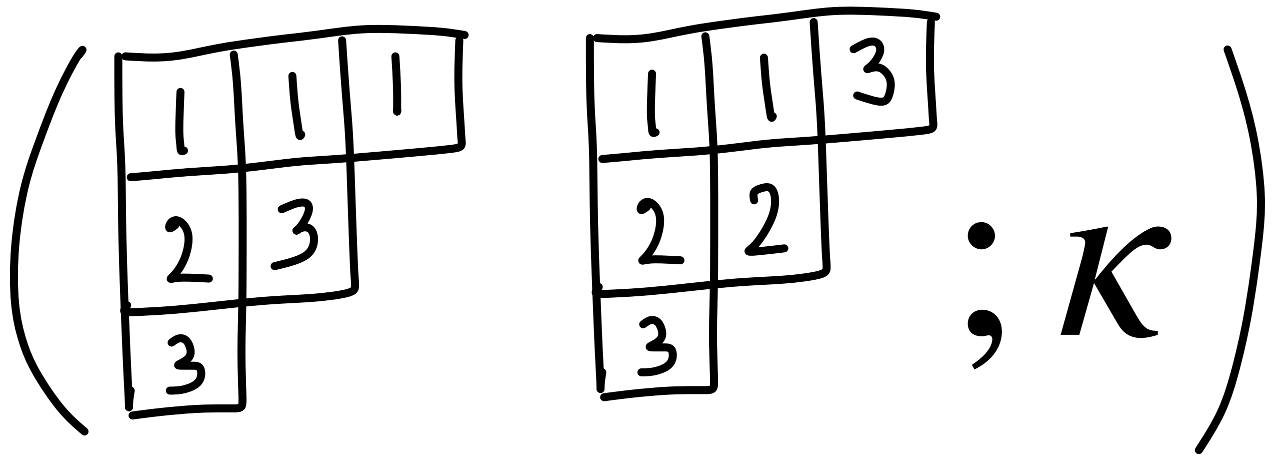
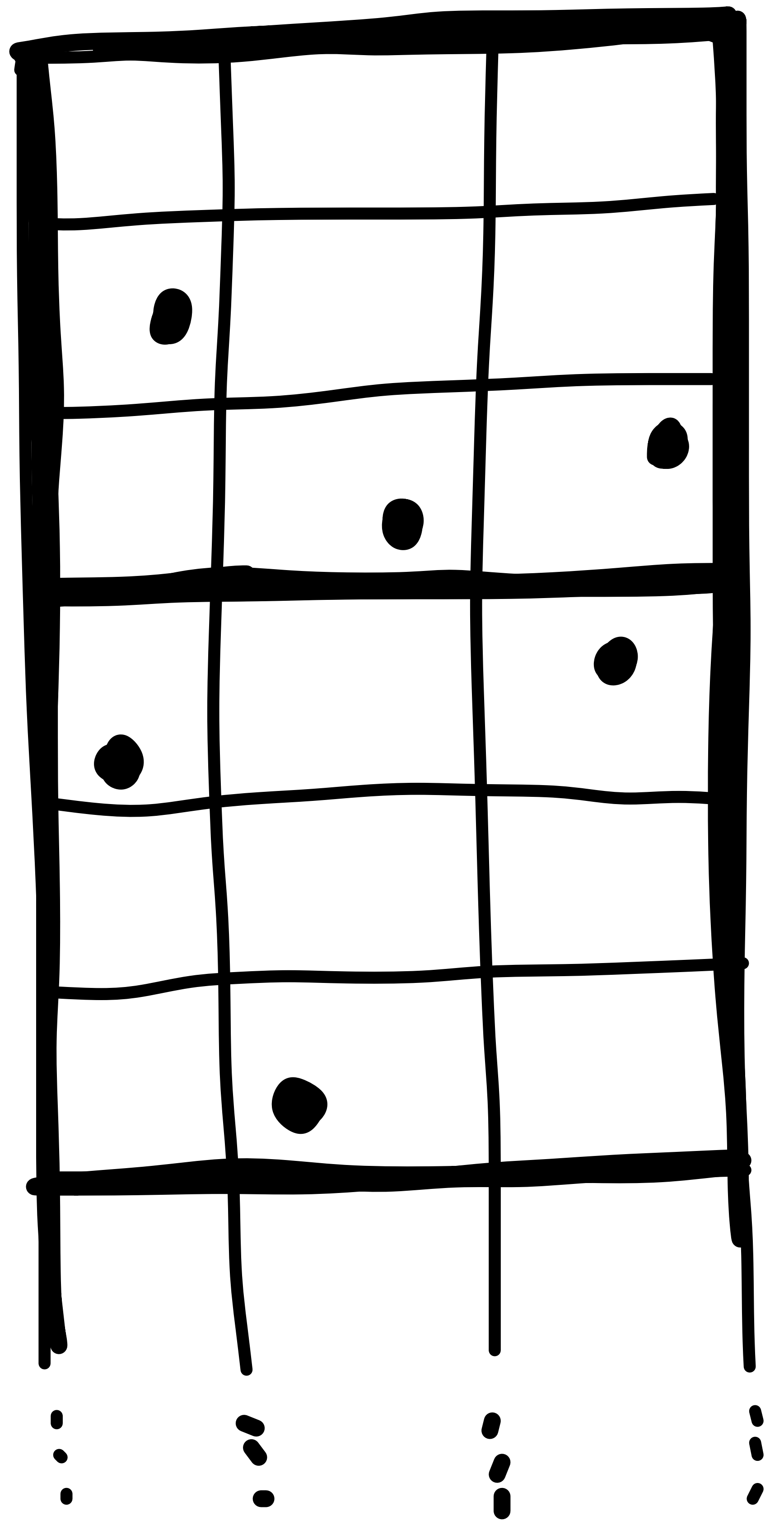


Theorem [IMS'21] (Characterization of asymptotic shapes)

$$\mu_1 = \mathrm{LIS}_1$$

$$\mu_1 + \cdots + \mu_\ell = \mathrm{LIS}_\ell$$



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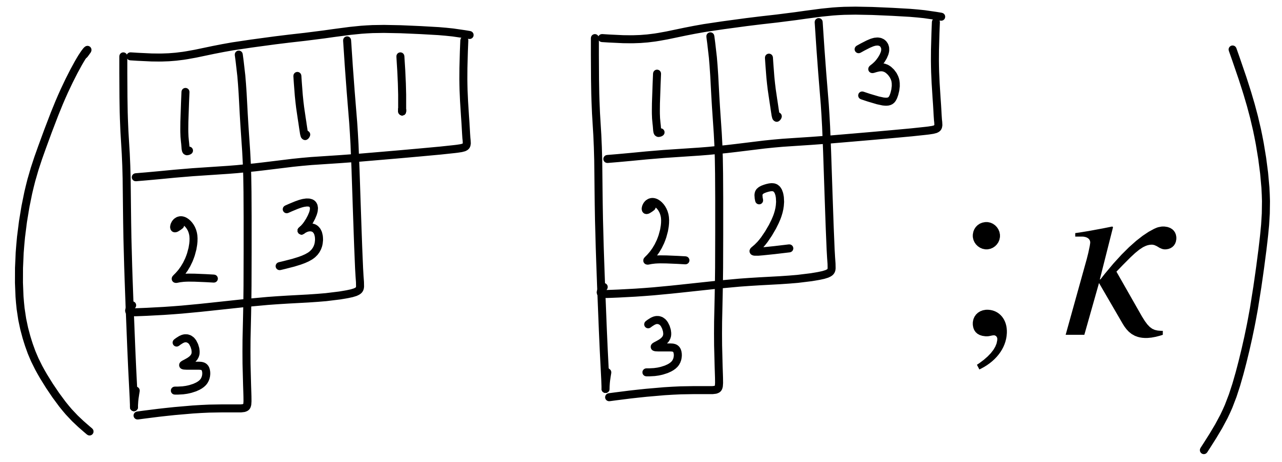

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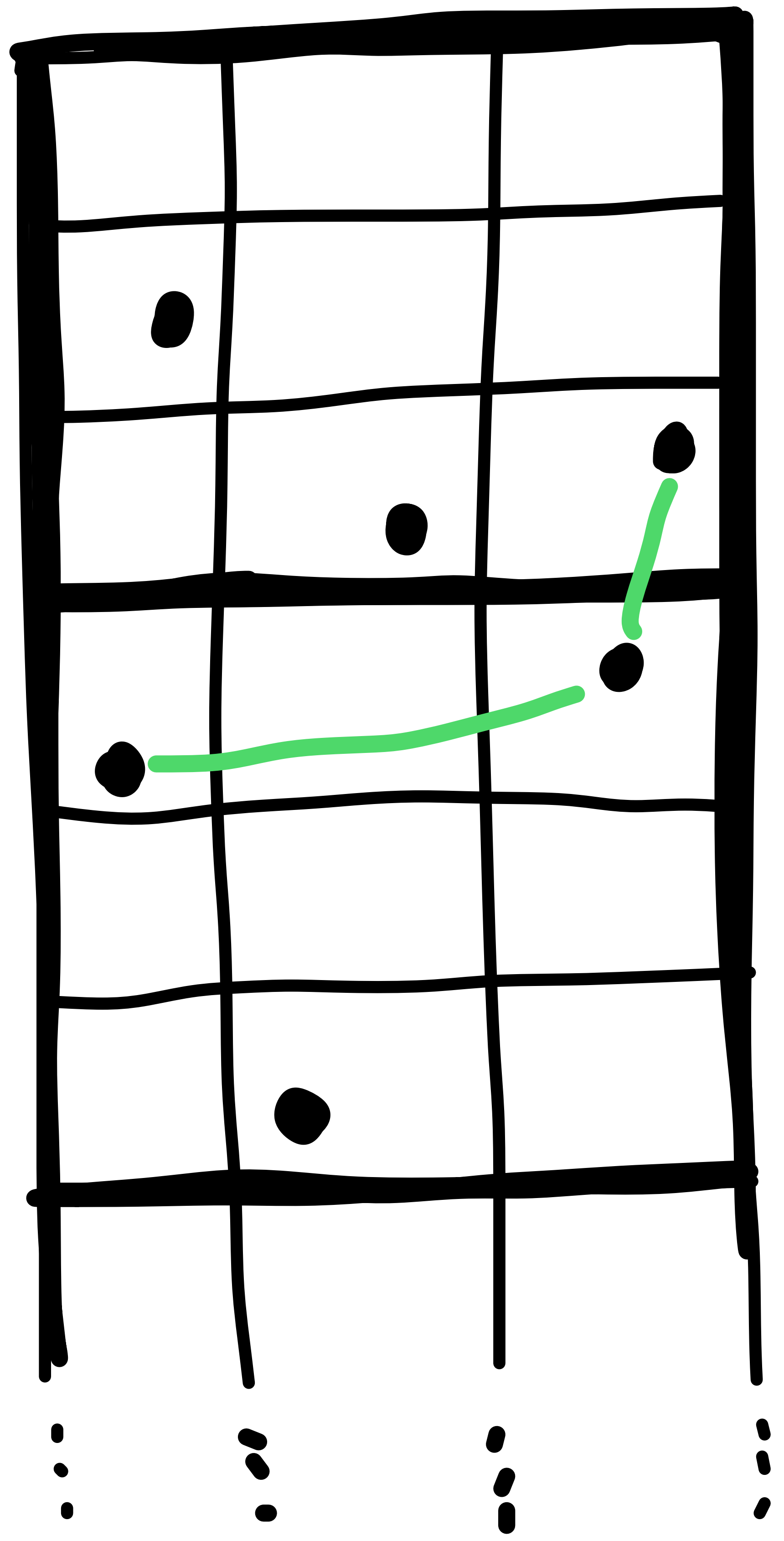

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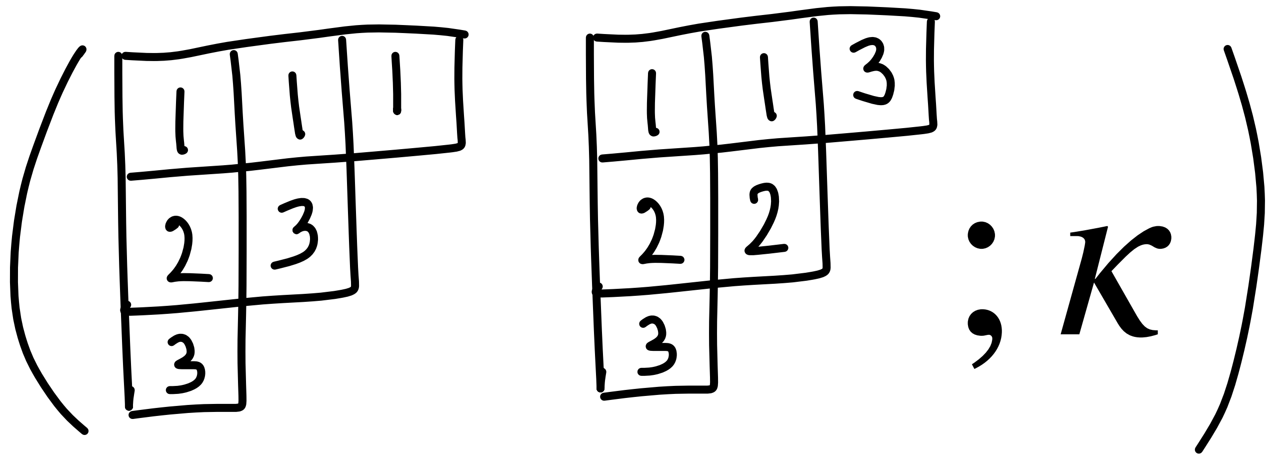
$$\begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow$$



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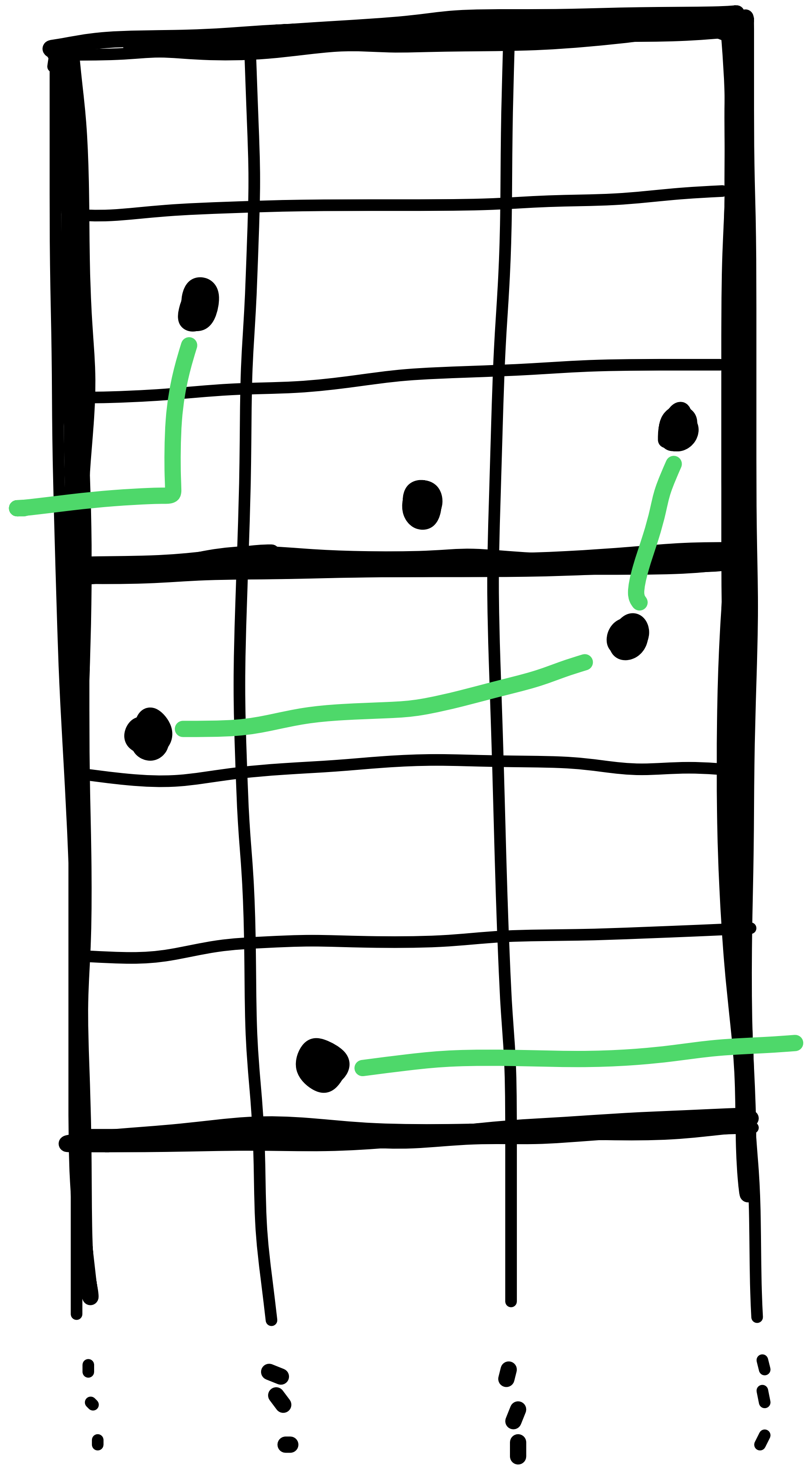

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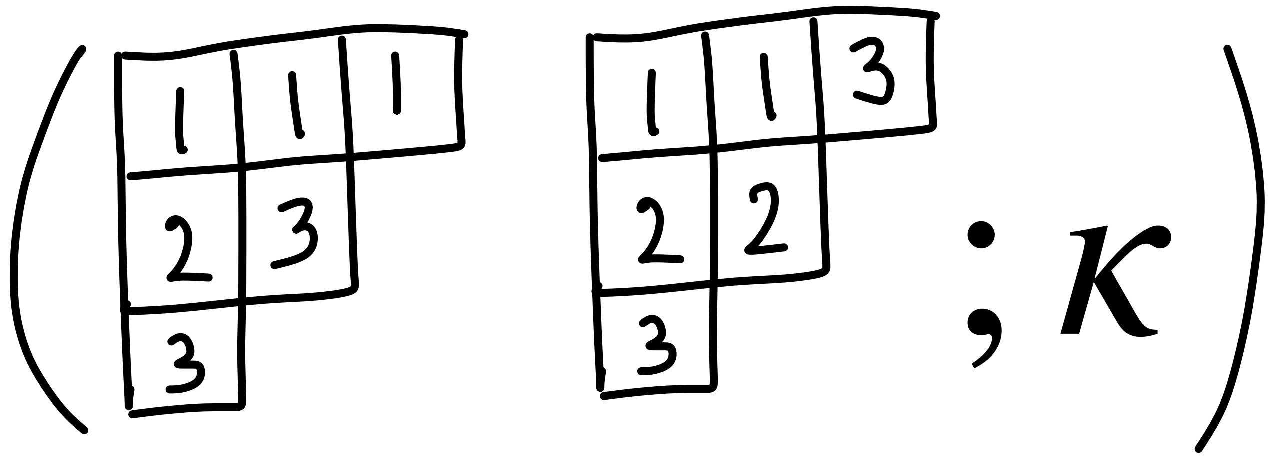

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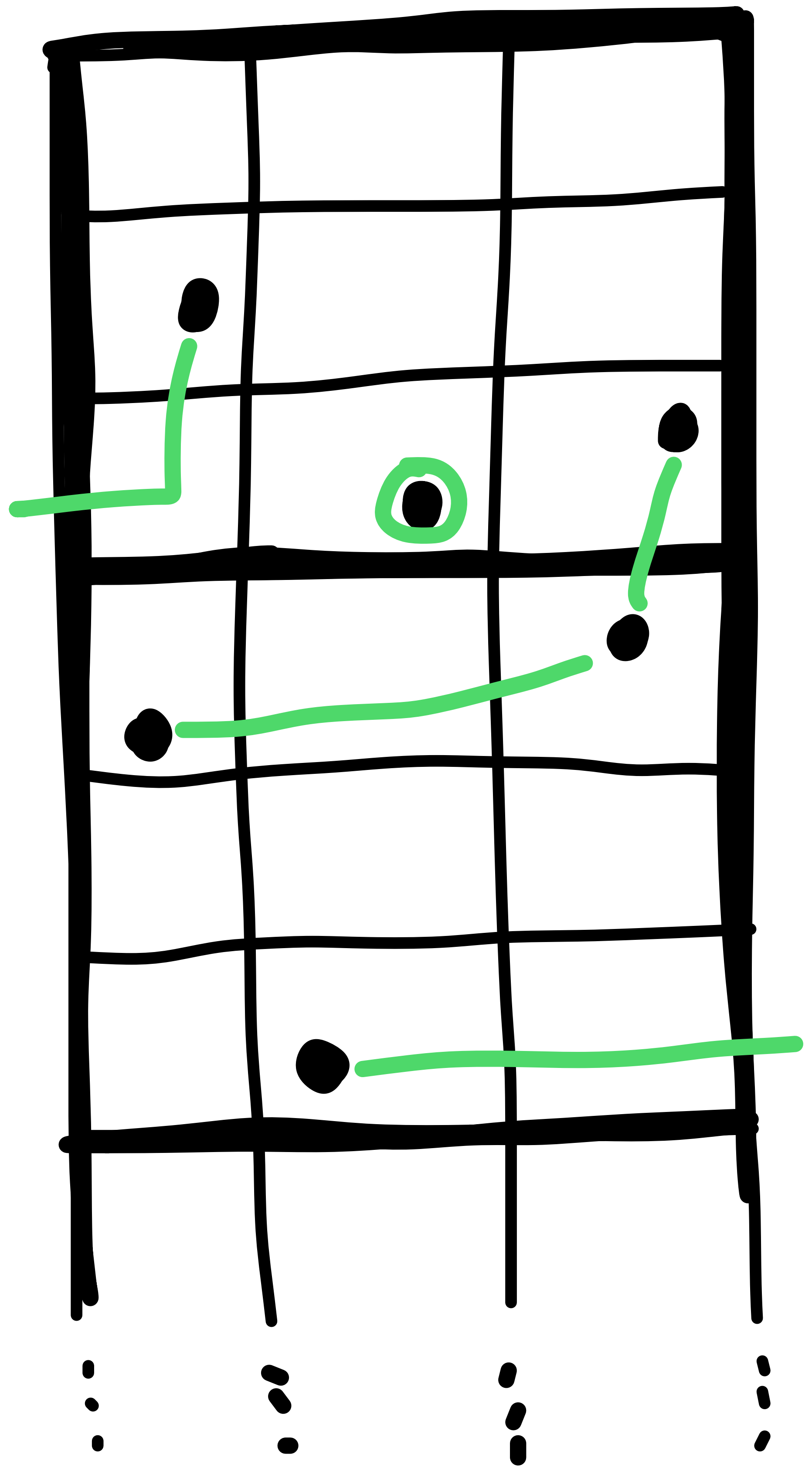
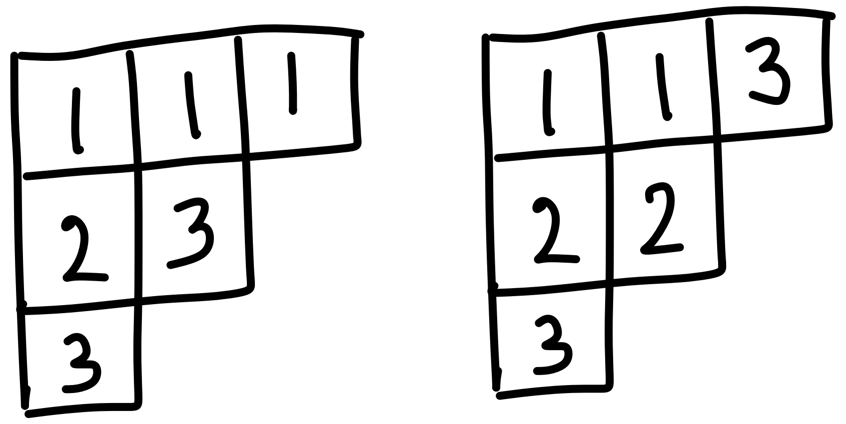


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (1,1,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,1,\dots) & (0,0,\dots) & (0,1,\dots) \end{pmatrix} \longleftrightarrow \Bigg($$



$$; \kappa \Bigg)$$

- It remains to determine κ
- For this we need to consider symmetries of the skew **RSK**

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} ; \kappa \right)$$

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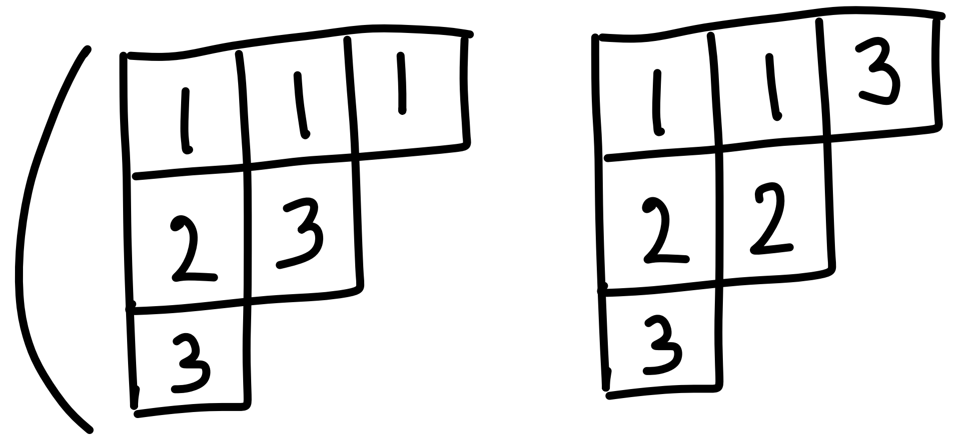

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Construction of Υ

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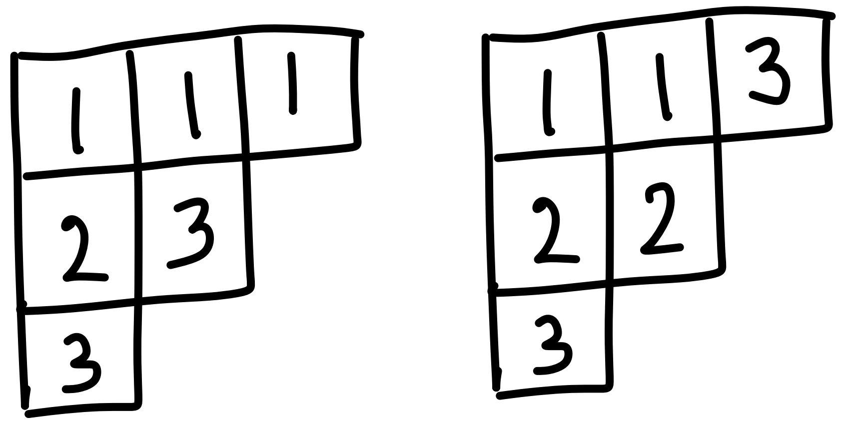

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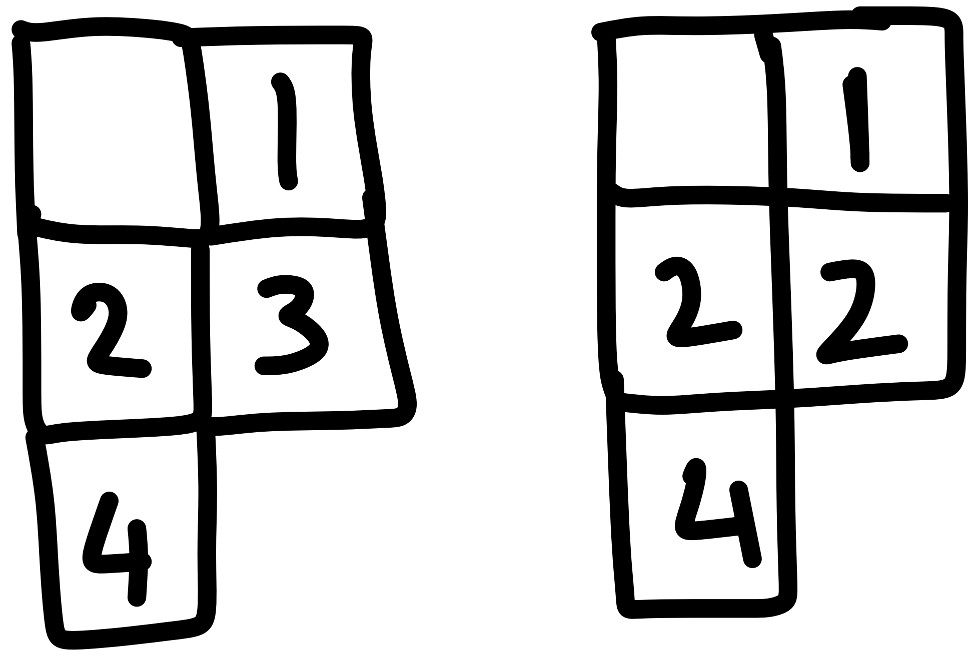
ι_ε = internal insertion with cycling



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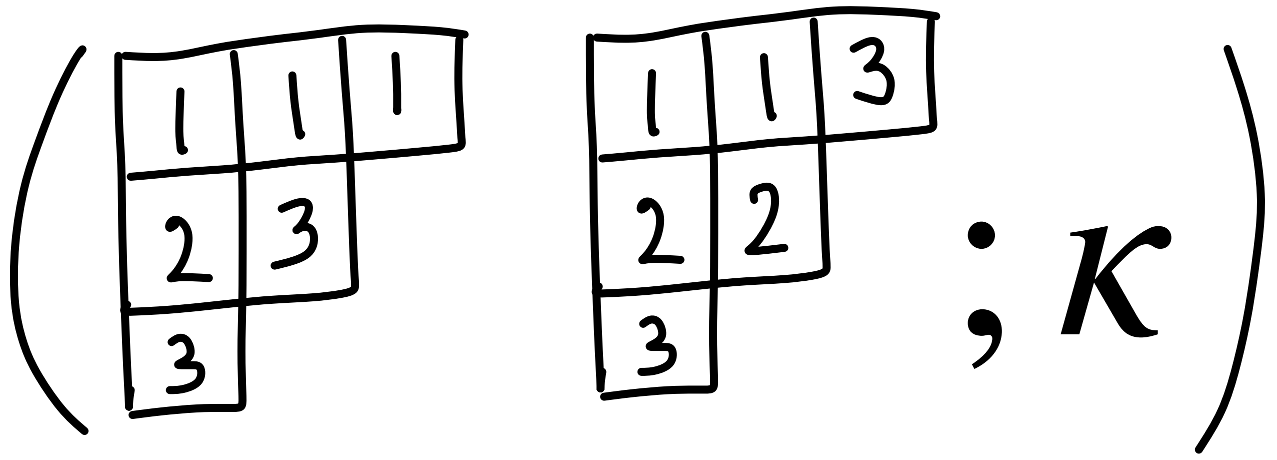

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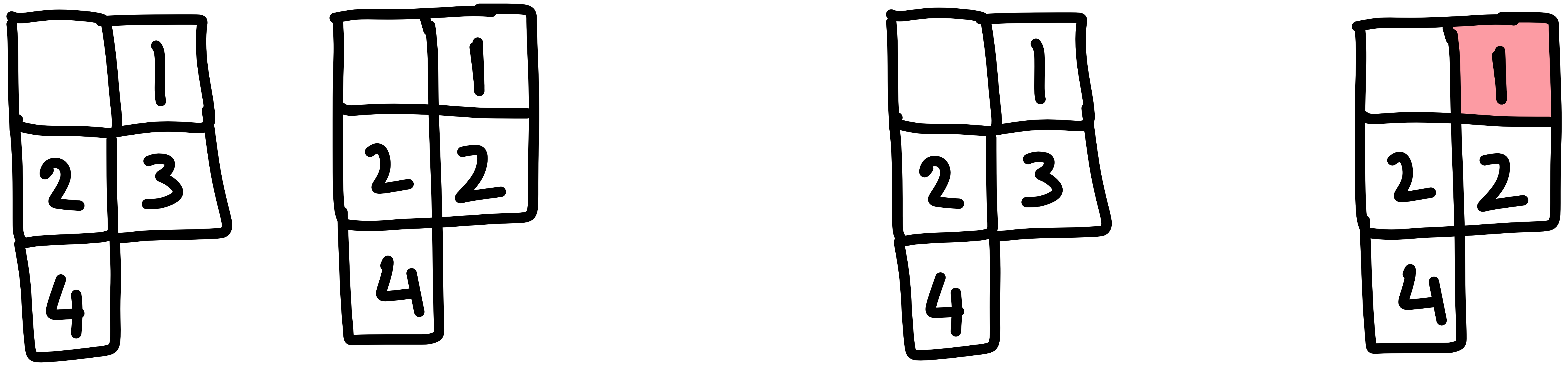

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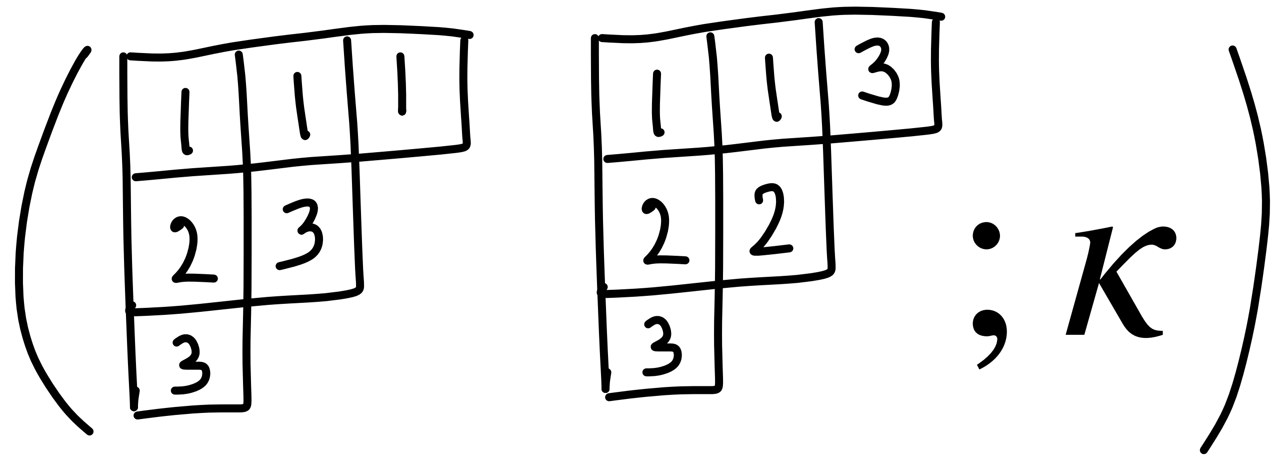

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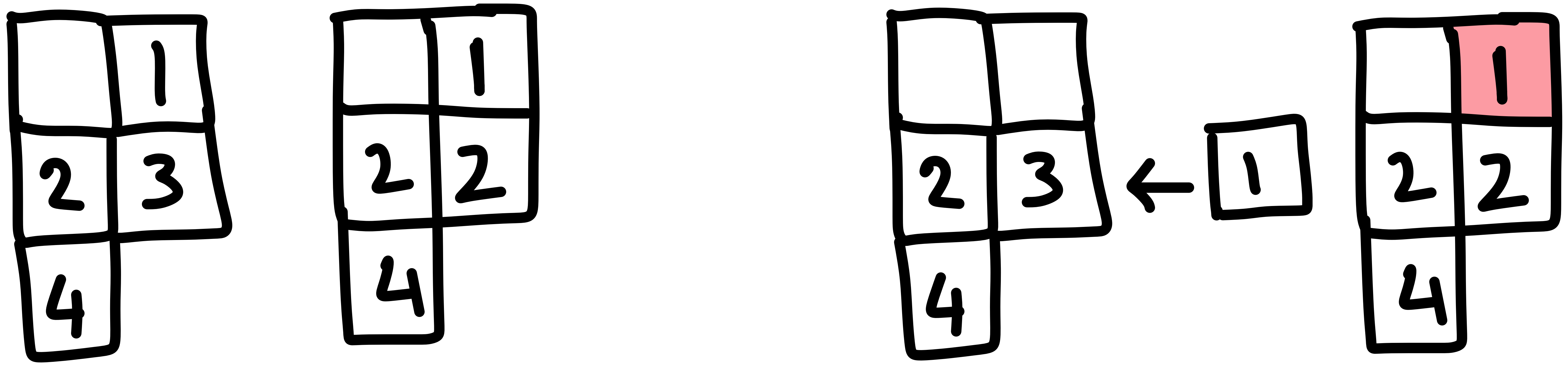

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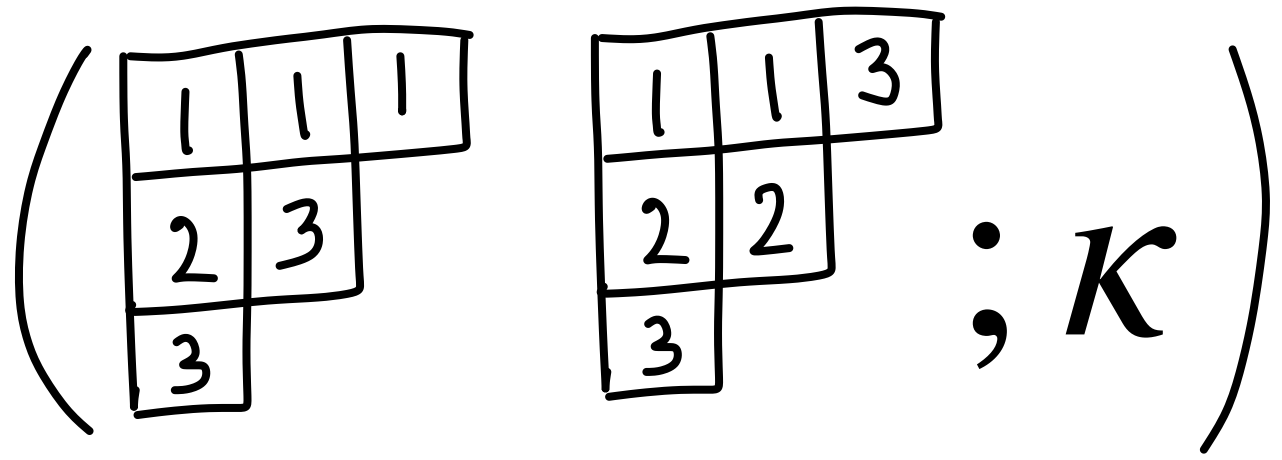

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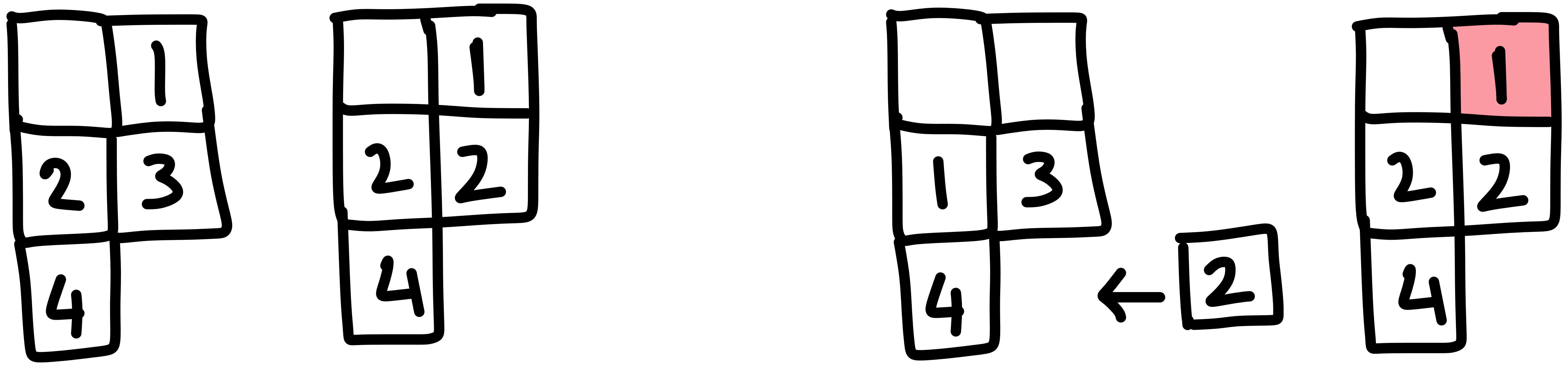

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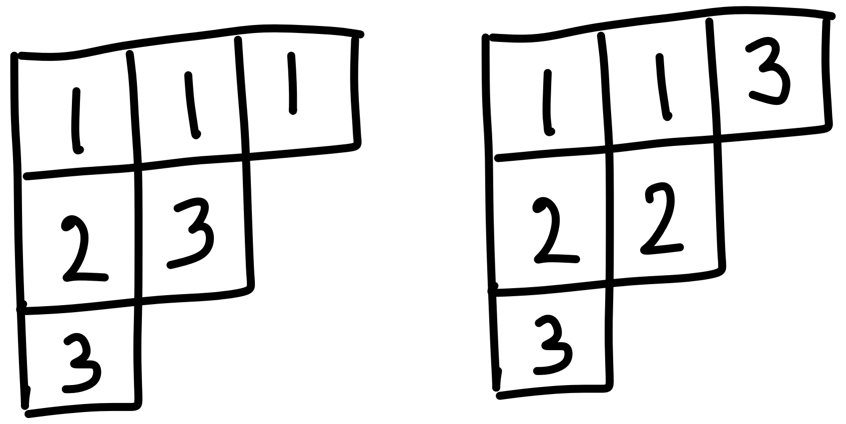

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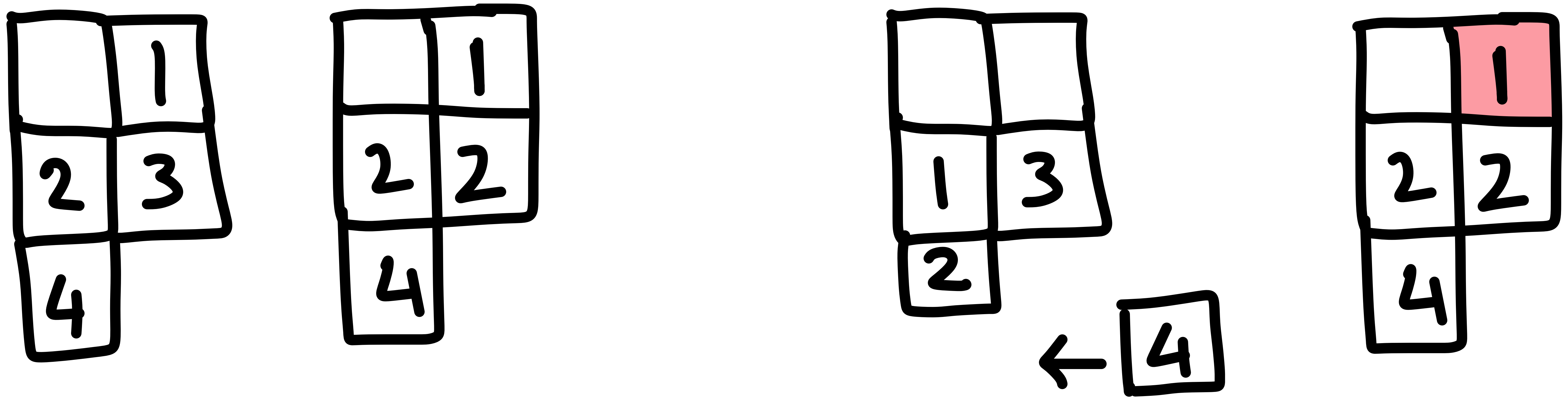

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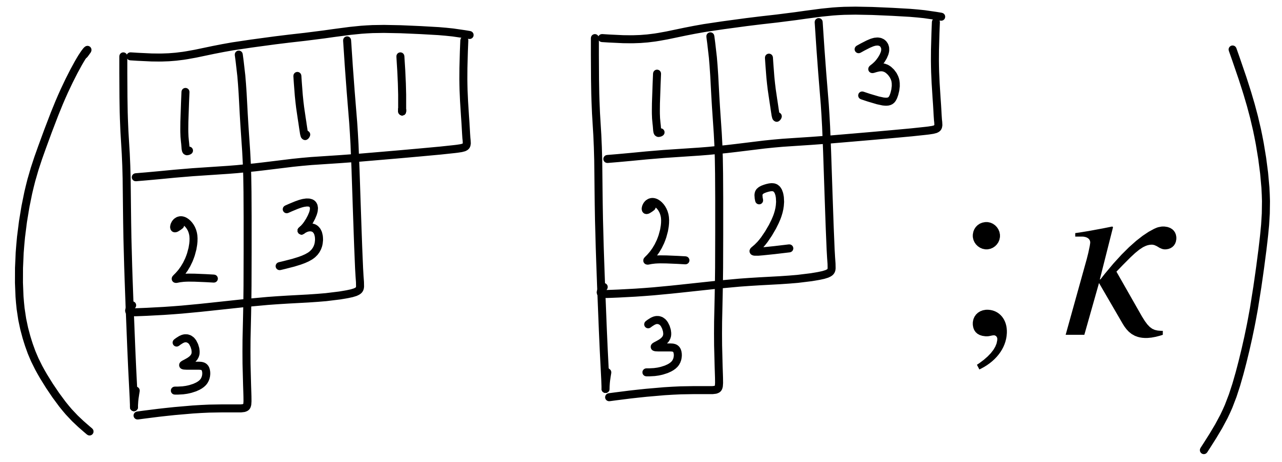

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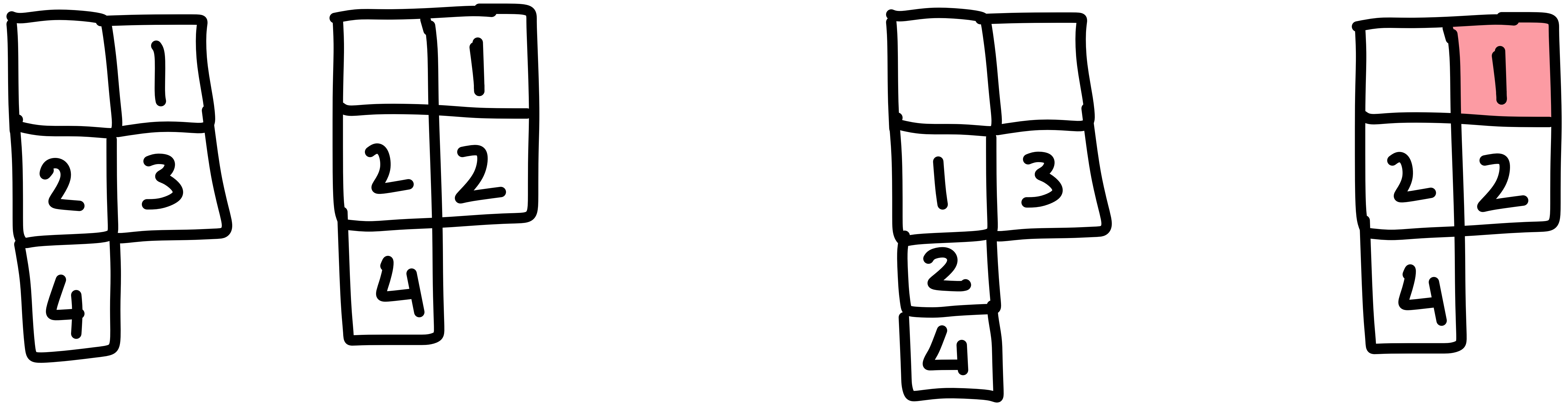

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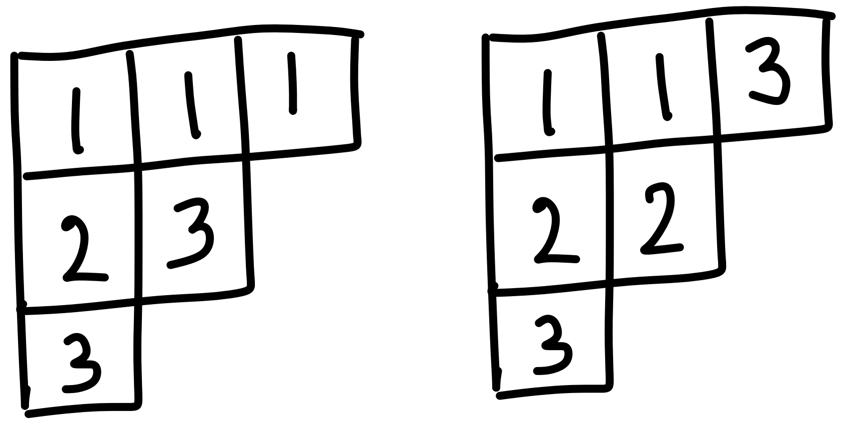

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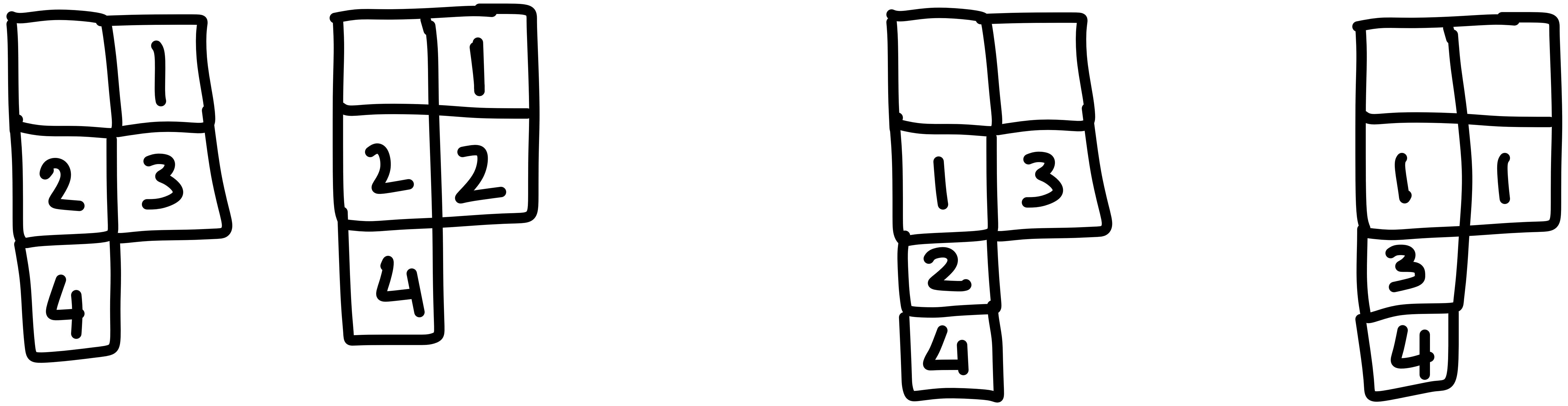

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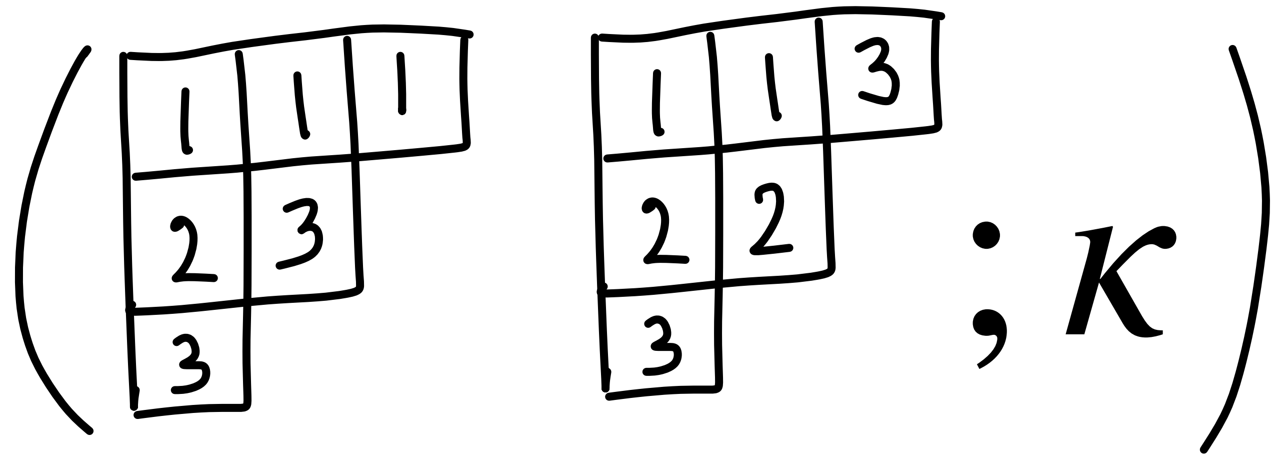

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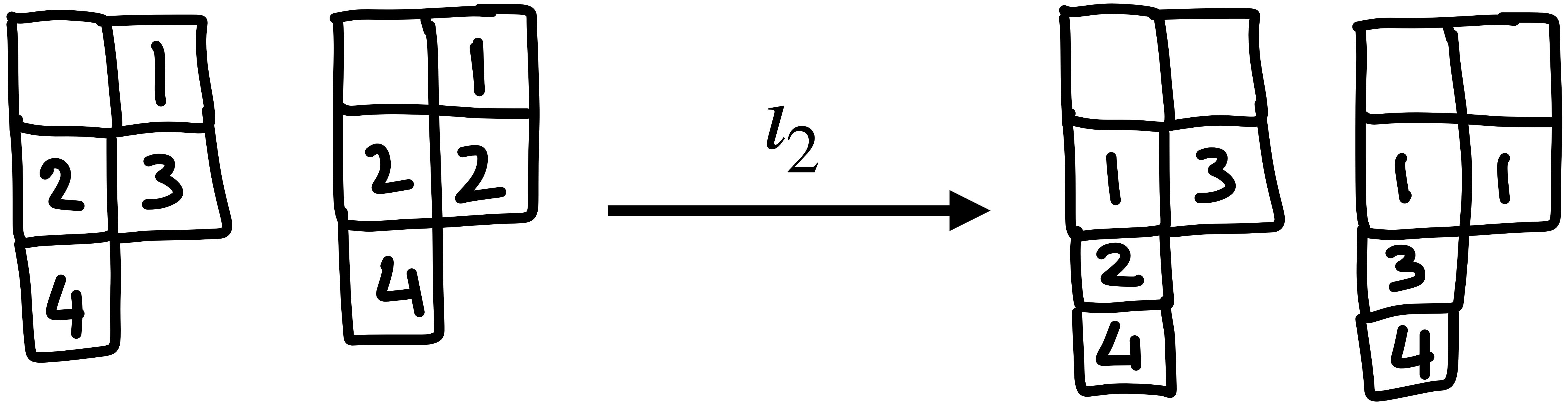

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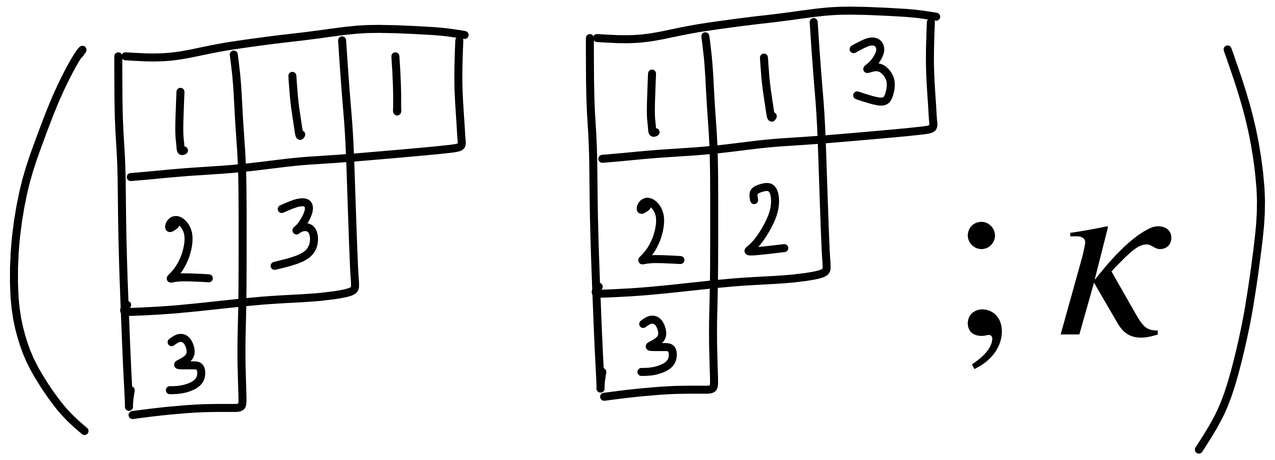
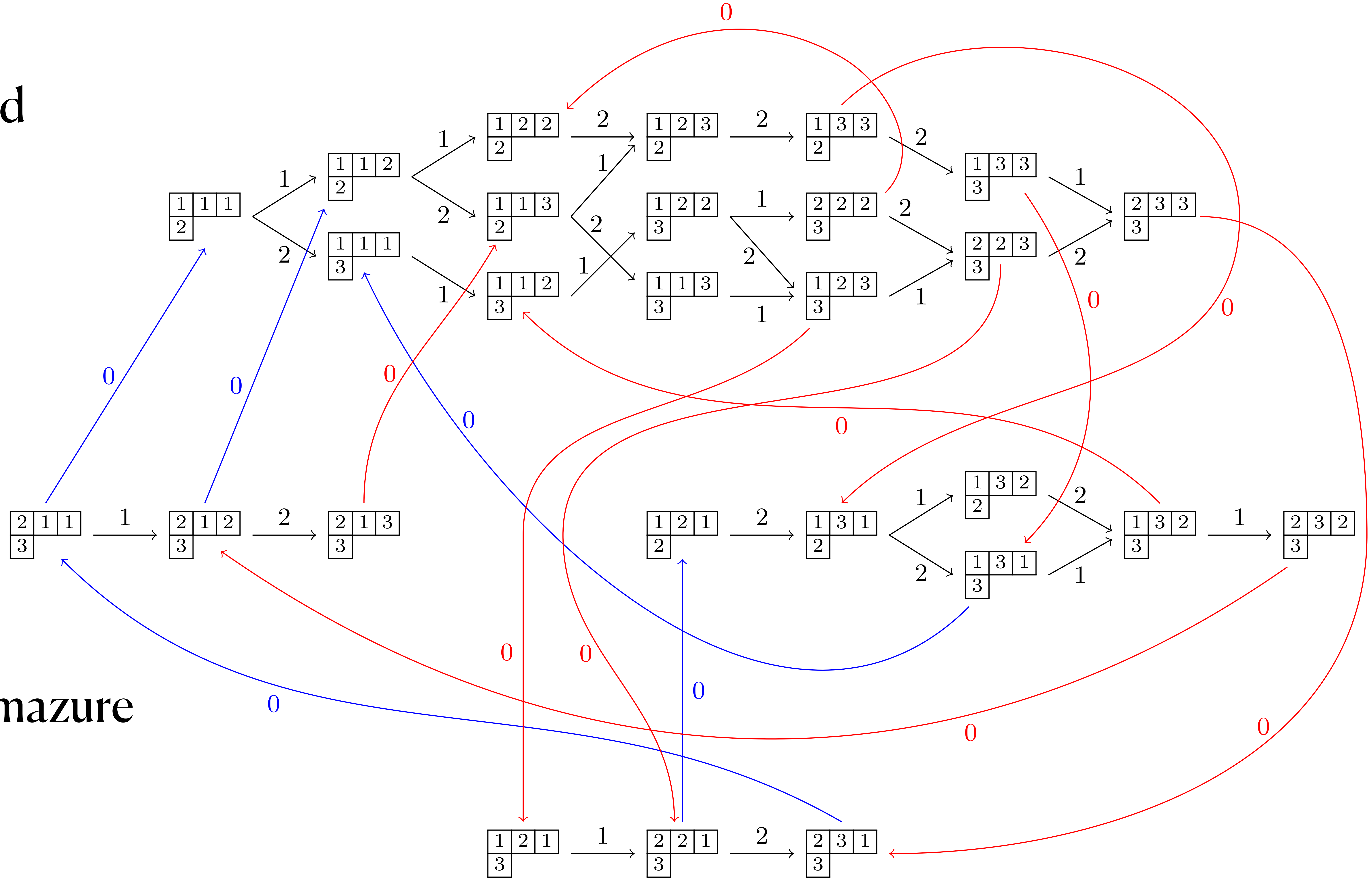


- $\tilde{E}^{(\varepsilon)}_0, \tilde{E}^{(\varepsilon)}_1, \ldots, \tilde{E}^{(\varepsilon)}_{n-1}$, for $i = 1,\ldots,n-1$, $\varepsilon = 1,2$ define an $\widehat{\mathfrak{sl}}_n$ bi-crystal structure on the set of pairs (P, Q) of skew semi-standard tableaux (or equivalently on matrices $(M^k_{i,j})$)
- This makes the projection $(P, Q) \to (V, W)$ a morphism of $\widehat{\mathfrak{sl}}_n$ bi-crystals

Affine crystal graph $VST(\mu)$

- The crystal graph $VST(\mu)$ is connected
  [Kashiwara'90]
- Demazure subgraph: remove "bad" $\tilde{e}_0, \tilde{f}_0$ arrows
- For any V there exist a path on the Demazure subgraph $\mathscr{L}_V : V \mapsto \text{YAM}(\mu)$
- $\mathscr{H}(V) = \#\tilde{f}_0 - \#\tilde{e}_0$ in $\mathscr{L}_V$
  [Schilling-Tingley'12]



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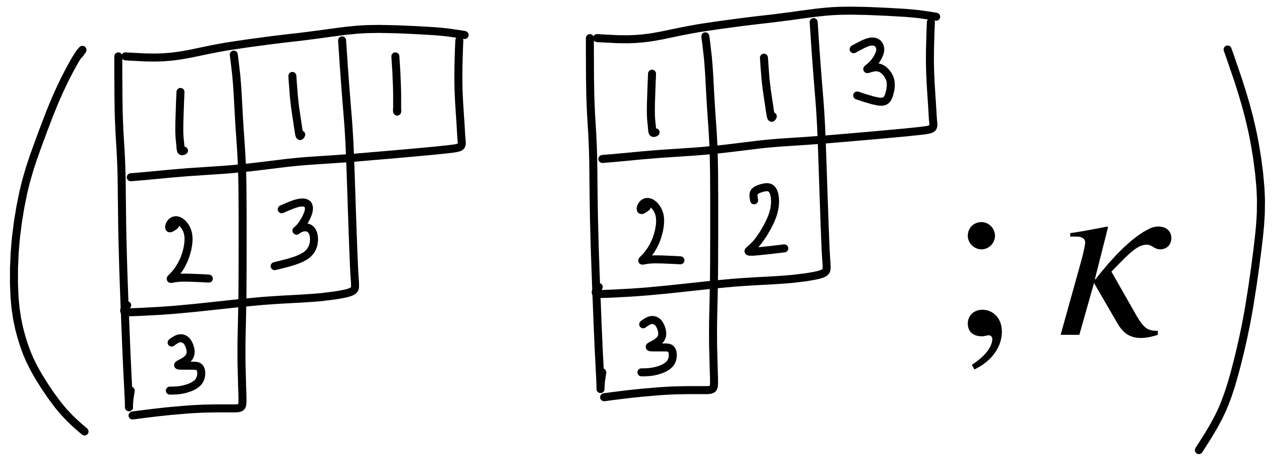
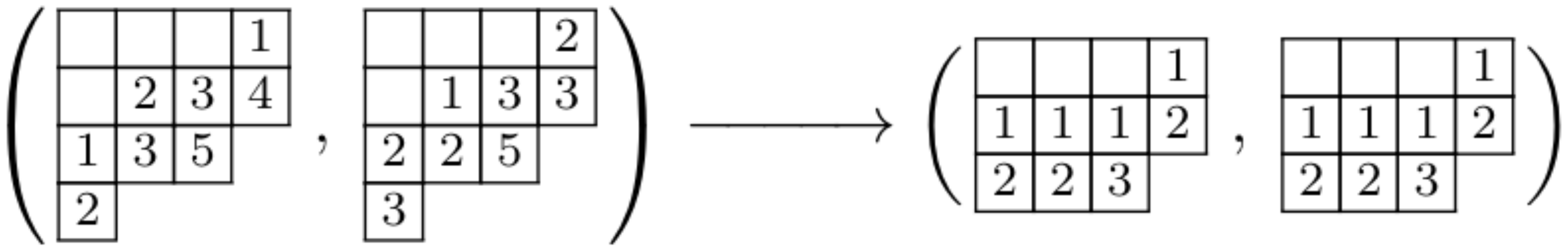


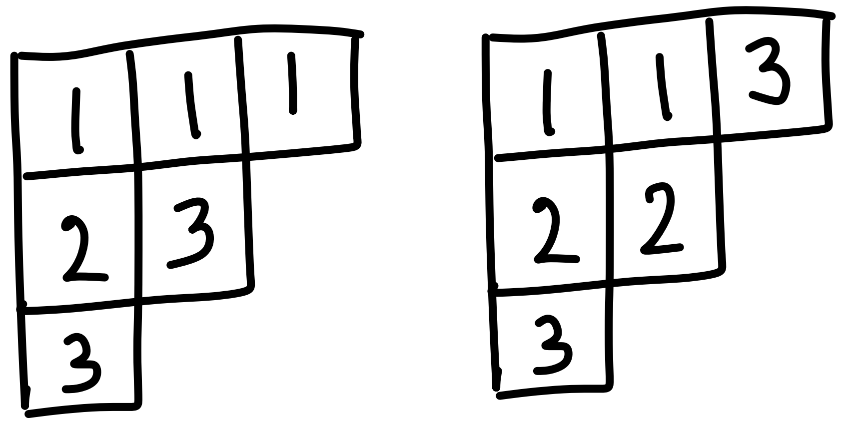
- From the $\widehat{\mathfrak{sl}}_n$ bi-crystal structure on (V, W) we get info about intrinsic energy $\mathscr{H}$
- We can also transport leading maps $\mathscr{L}_V, \mathscr{L}_W \mapsto \mathscr{L}_{P,Q}$ and "trivialize" tableaux P, Q

Example:

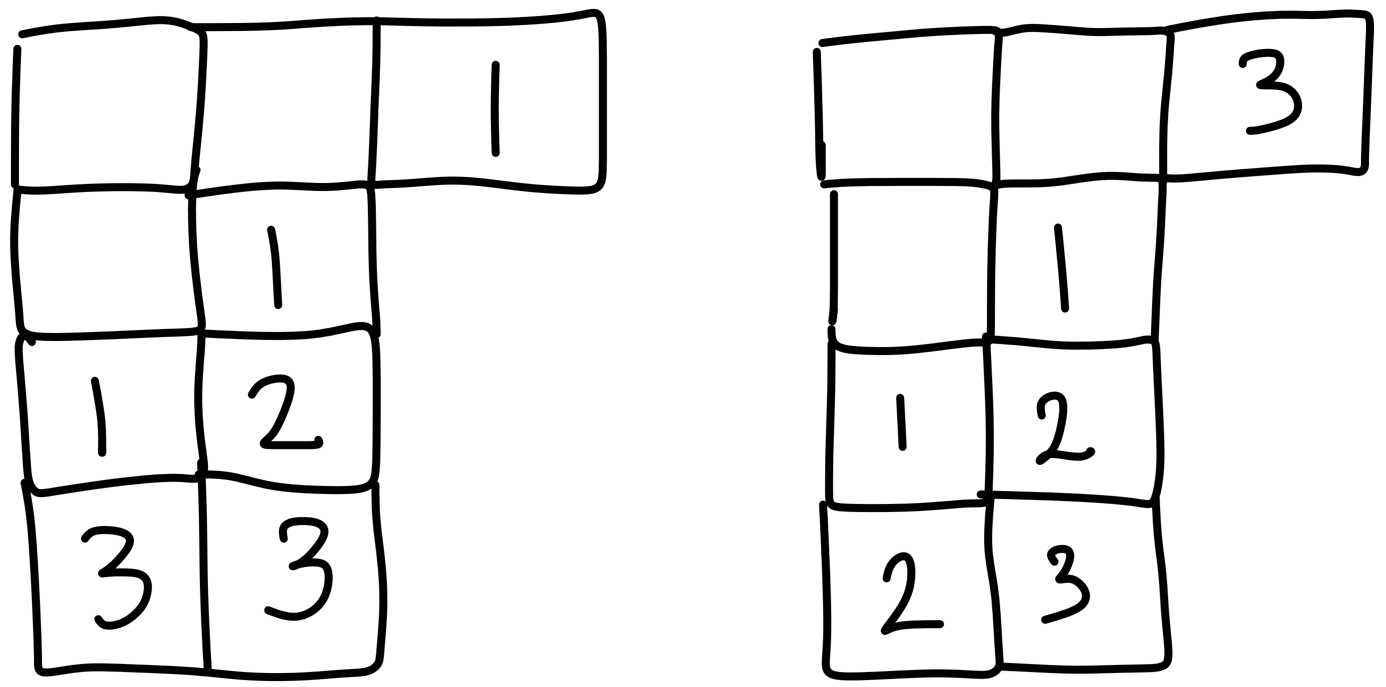


- From the "leading tableau" $(T, T) = \mathscr{L}_{P,Q}$ we read off κ

Construction of Υ

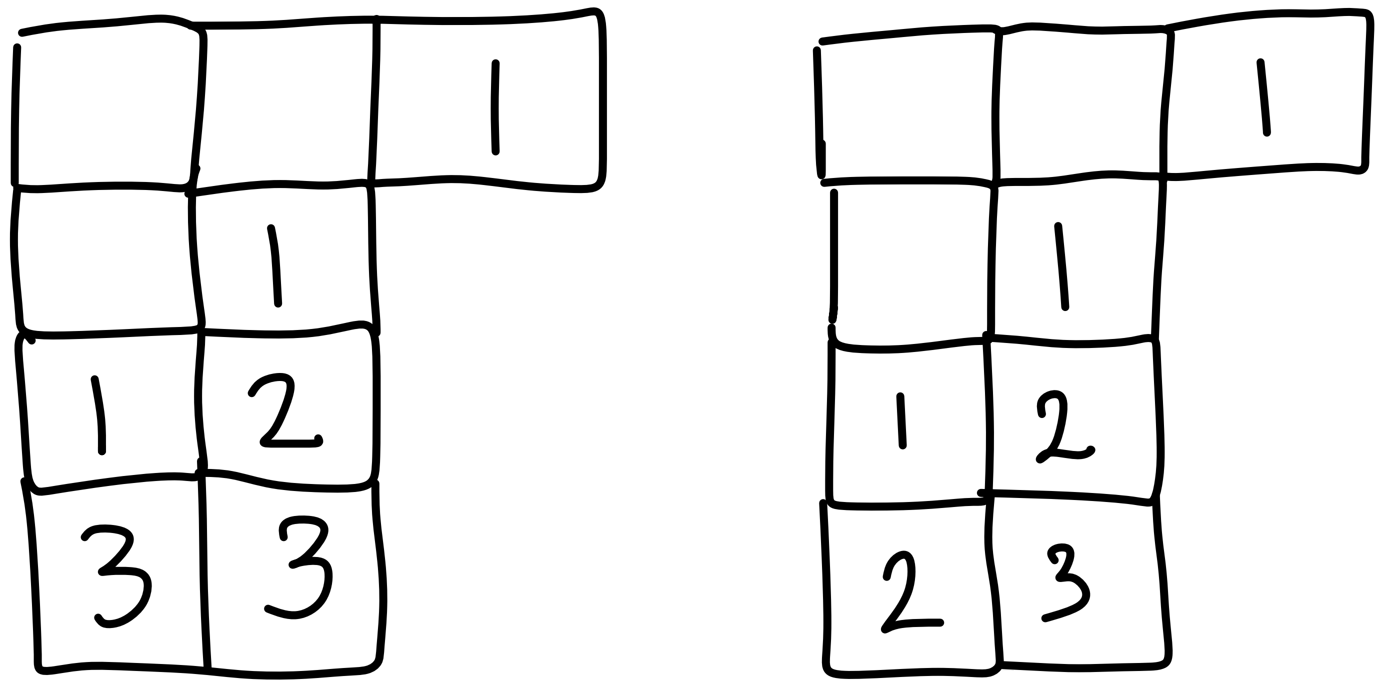
$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa) \quad \begin{pmatrix} (0,0,\ldots) & (1,1,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,1,\ldots) & (0,0,\ldots) & (0,1,\ldots) \end{pmatrix} \longleftrightarrow \Bigg($$  $$; \kappa \Bigg)$$

- In our example



$P \quad , \quad Q$

$$\xrightarrow{\mathscr{L}_{P,Q}}$$



$$\kappa = (1,2,0)$$

To sum up

- $\Upsilon : (M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$ has the following properties:
  - Content preserving i.e. $x^V y^W = \prod_{i,j,k} (x_i y_j)^{M_{i,j}^k}$
  - $\sum_{k>0} \sum_{i,j} k M_{i,j}^k = \mathscr{H}(V) + \mathscr{H}(W) + |\kappa|$
  - (Greene invariants) $\mu_1 + \cdots + \mu_\ell = \mathrm{LIS}_\ell(M)$

Bijective proofs

- Cauchy Identity

$$\sum_{\mu} b_\mu(q) P_\mu(x;q) P_\mu(y;q) = \prod_{k\geq 0} \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j q^k}$$

- Kawanaka-Littlewood Identity

$$\sum_{\mu} b_\mu(q) P_\mu(x;q^2) = \prod_{k\geq 0} \prod_{i=1}^{n} \frac{1}{1 - x_i q^k} \prod_{1\leq i<j\leq n} \frac{1}{1 - x_i x_j q^{2k}}$$

Conclusion

- We construct a bijective *q*-extension of the RSK correspondence Υ
- With Υ we can prove bijectively the Cauchy identities (CI) for *q*-Whittaker polynomials
- It is the first time a bijective proof is given for CI of Macdonald polynomials outside of the Schur case
- With Υ we can prove bijectively refinements of the CI relating q-Whittaker and skew-Schur polynomials (Sasamoto's seminar)
- Our bijective theory gives direct connection between solvable non-free fermionic models and positive temperature free fermionic models (Sasamoto's seminar)