Joint moments of multi–species q–Boson

Jeffrey Kuan

Texas A&M

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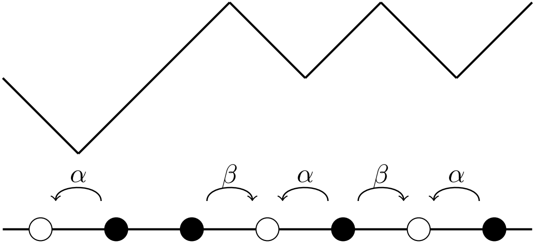
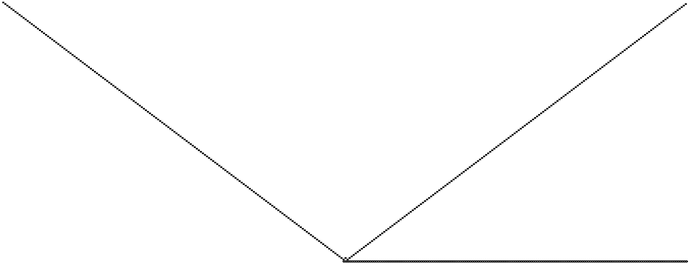
Consider the asymmetric simple exclusion process on a one–dimensional lattice, introduced by MacDonald–Gibbs–Pipkin ('68), Spitzer ('70):



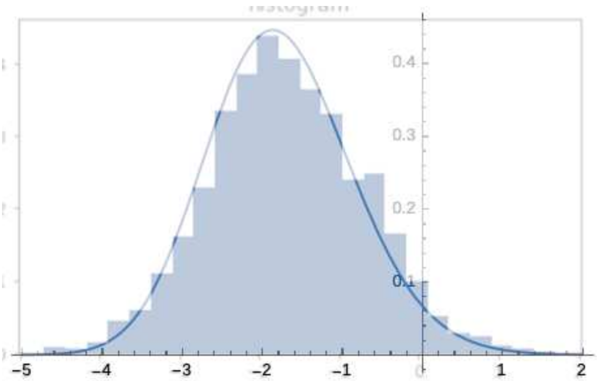
Consider *step* initial conditions, when $x_m(0) = m$ for $m \geq 1$, where x_m denotes the location of the m–th left–most particle:



Time: 2.2596858750679063

Lattice Size: 168
q= 0.25

Computer simulations for 2139 samples of $x_{100}(2000)$ when $\alpha = 1 - \beta = 0.75$, with the help of Texas A&M University High Performance Research Computing :



Known results for single–point fluctuations with *step initial conditions*:

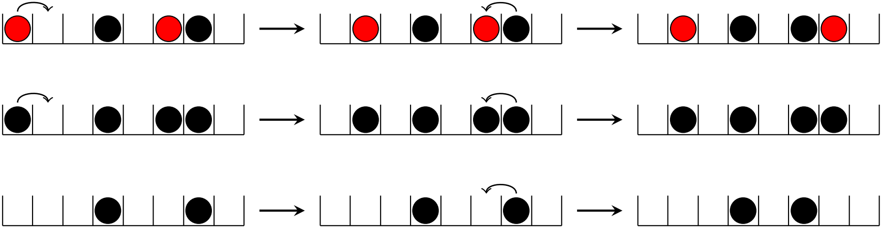
- Strong law of large numbers (LLN) for the totally asymmetric simple exclusion process (TASEP) when $\alpha = 1, \beta = 0$ (Rost 81): $t^{-1}x_{mt}(t) \to c_1$ a.s.
- Strong LLN of ASEP (Liggett '85).
- The central limit theorem (CLT) for TASEP, using determinantal formulas
  $$t^{-1/3}(x_{mt}(t) - c_1 t) \to F_2$$
  where F_2 is the Tracy–Widom distribution (Johansson '99), which is named after, and first discovered by, Tracy and Widom and 1993 in the context of random matrix theory.
- Convergence to F_2 holds for ASEP with step initial conditions (Tracy–Widom '08).

This is a *universal* distribution in the KPZ class, analogous to the Gaussian distribution, which is named after (Gauss 1809) but first discovered by (Adrain 1808) 🇺🇸

Known results for *multi*–point fluctuations with *step initial conditions*:

- For (discrete–time) TASEP, the fluctuations are the Airy$_2$ process (Johansson '02), discovered by (Prähofer–Spohn '01); the spatial fluctuations have a 2/3 exponent.
- Generalized to ASEP (Quastel–Sarkar '20); see also (Dimitrov '20) for two–point fluctuations of the stochastic six vertex model.

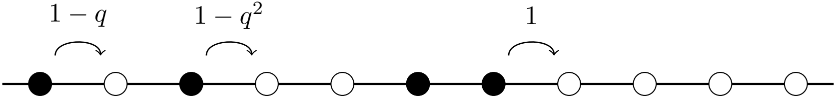
Introduced by Liggett '76, the two–species ASEP can be described as two coupled ASEPs.



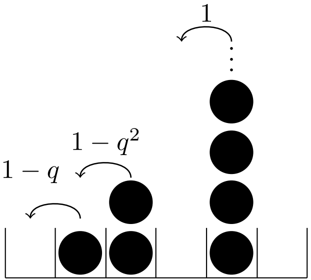
Not much is known, or even conjectured, about asymptotics of multi–species model. It is not immediately clear what the “right” question is.

- In Chen–de Gier–Hiki–Sasamoto–Usui '21, it is shown that the crossing probability of the AHR model is a F_2 times a Gaussian.

Consider q–TASEP, which was introduced by (Borodin–Corwin '11):



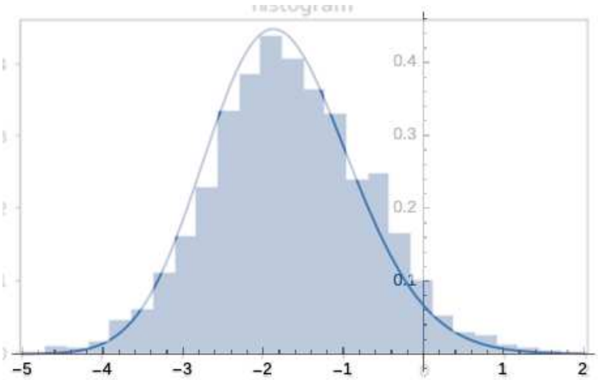
Its gaps evolve as the q–Boson introduced in (Sasamoto–Wadati '98):



Time: 1.6063027272771961

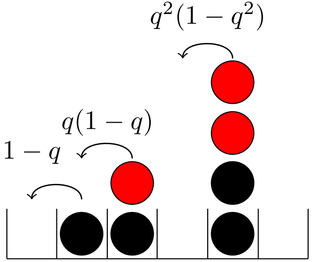
Lattice Size: 400
q= 0.6

Computer simulations of 3010 samples of $x_{200}(2000)$ when $q = 0.6$, again with the help of the Texas A&M University High Performance Research Computing :



The fluctuations are again the Tracy–Widom distribution F_2 [Ferrari–Vető '13], [Barraquand '14].

The multi–species q–Boson was introduced by Takeyama '15:



Given a particle configuration η, let $N_x^{(j)}(\eta)$ denote the number of particles of species $1,\ldots,j$ that are to the right of x.

Theorem (K. 202?)

Suppose that $\eta(t)$ *is a multi–species* q*–Boson with initial conditions with infinitely many* i*th species particles at* $-M_i$ *for some integers* $0 < M_n < \ldots < M_1$. *Fix non–negative integers* $k_1,\ldots,k_n$. *Set* $N = k_1 + \ldots + k_n$ *and define* M_j $(1 \leq j \leq N)$ *by* $M_{k_1+\ldots+k_m+1} = M_{k_1+\ldots+k_m+2} = \cdots = M_{k_1+\ldots+k_{m+1}}$. *Then*

$$\mathbb{E}\left[\prod_{j=1}^{n} q^{k_j N_0^{(n+1-j)}(\eta(t))}\right] = \mathbb{E}\left[\prod_{j=1}^{n} q^{k_j N_{M_j}^{(1)}(\xi(t))}\right]$$

where $\xi(t)$ *is a single–species* q*–Boson starting with infinitely many particles at* 0.

In words: the single–point fluctuations of the multi–species q–Boson are the same as the multi–point fluctuations of the single–species q–Boson, which should be the Airy$_2$ process.

Conjecture: The single–point fluctuations of the multi–species q–Boson (and other models in the KPZ class with step initial conditions) are the Airy_2 process.

Similar statements were proved in greater generality in Borodin–Gorin–Wheeler '19, Galashin '20, Bufetov–Korotkikh '20 for stochastic vertex models in the quadrant. The result here does not (immediately) follow from those statements [?].

Outline of proof:

- Duality reduces calculations of the n–th q–moments to the n–particle system.
- Write the Green's function (transition probabilities) of the n–particle system.
- Apply a symmetrization identity to the Green's function.

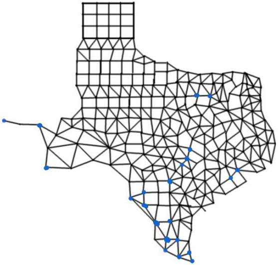
Suppose η_t and ζ_t are Markov processes with state spaces X and Y respectively, and let $D(\eta,\zeta)$ be a bounded measurable function on $X \times Y$. The processes η_t and ζ_t are said to be *dual* to one another with respect to D if

$$\mathbb{E}_\eta D(\eta_t,\zeta) = \mathbb{E}_\zeta D(\eta,\zeta_t) \text{ for all } t \geq 0.$$

An equivalent definition of duality (on discrete state spaces): If the generators L_X and L_Y are viewed as $X \times X$ and $Y \times Y$ matrices respectively, and D is viewed as a $X \times Y$ matrix, then

$$L_X D = D L_Y^T.$$

Consider the symmetric exclusion process on an arbitrary graph $\mathcal{G}$ with jump rates given by a symmetric stochastic $\mathcal{G} \times \mathcal{G}$ matrix $p(x, y)$:



Each site has an exponential clock of rate 1, with all clocks independent. When the clock at site x rings, the particle there (if there is one there) chooses a site y according to the probabilities $p(x, y)$. If y is unoccupied, the particle jumps to y; else the jump is blocked.

Let $\eta_t(x)$ denote the number of particles (zero or one) at lattice site $x \in \mathcal{G}$ at time $t \geq 0$.

Let $A_t \subseteq \mathcal{G}$ denote the lattice sites with a particle at time $t \geq 0$. Then the duality result of Spitzer ('70), Schütz–Sandow ('94)

$$\mathbb{P}(\eta_t(x) = 1 \text{ for all } x \in A) = \mathbb{P}(\eta(x) = 1 \text{ for all } x \in A_t)$$

The process $\eta_t(x)$ can have infinitely particles, whereas A_t has finitely many particles. So duality reduces an infinite–particle system to a finite–particle system.

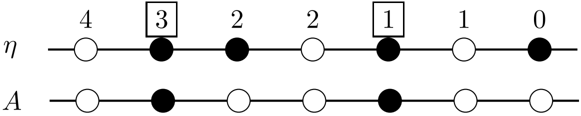
Using algebraic methods, Schütz '95 proves the following: (where $q = \sqrt{\beta/\alpha} \in (0,\infty)$ is the asymmetry parameter)

Theorem (Schütz '95)

Assuming closed boundary conditions ($S = \{a, a+1, \ldots, b\}$) or the infinite line ($S = \mathbb{Z}$), ASEP is dual to another ASEP with respect to the function

$$\tilde{D}(\eta, A) = \begin{cases} \prod_{x \in A} q^{2N_x(\eta)+2x}, & \text{if } \eta(x) = 1 \text{ for all } x \in A \\ 0, & \text{else.} \end{cases}$$

where $N_x(\eta) = \#$ of particles to the right of lattice site x.



Taking $q = 1$ recovers the simple case of Spitzer's result.

The duality for ASEP can be found *algebraically*.
Associate to each particle a basis vector of $\mathbb{C}^2$.
Associate to each particle configuration a basis vector of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L \text{ copies}}$, where L is the lattice size.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathbb{C}^2$ is a representation of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices, (also the Lie algebra of the Lie group $SU(2)$), which has basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{creation operator}$$

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{annihilation operator}$$

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{number operator}$$

The action is given by explicit multiplication: for example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Lie algebra $\mathfrak{sl}_2$ also acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ through the co–product:

$$\Delta(a) = \sum_{j=1}^{L} 1^{\otimes j-1} \otimes a \otimes 1^{\otimes L-j}, \quad a = e, f, h.$$

In words, $a \in \mathfrak{sl}_2$ acts on the jth lattice site and fixes all other lattice sites. Note that the co–product is symmetric in the left–right directions.
The generator of the symmetric exclusion process can be written as $\Delta(C)$, where C is the Casimir element of $\mathfrak{sl}_2$, which commutes with all other elements ($CX = XC$). Thus, taking $L = \Delta(C)$ and $D = \Delta(F)$ for an appropriate annihilation operator F shows the duality $LD = DL$ defining intertwining.

The asymmetry occurs through quantization. The Drinfeld–Jimbo ('86) quantum group has q–deformed generators, relations and co–product, where we have the formal power series $q = e^{\hbar} = \sum_k \frac{\hbar^k}{k!}$, now with co–product

$$\begin{array}{rr} \Delta(e) = 1 \otimes e + e \otimes 1 & \Delta(e) = q^h \otimes e + e \otimes 1 \\ \Delta(f) = 1 \otimes f + f \otimes 1 & \Delta(f) = 1 \otimes f + f \otimes q^{-h} \\ \Delta(h) = 1 \otimes h + h \otimes 1 & \Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1} \end{array}$$

We can again check directly that the generator of ASEP commutes with the quantum group:

$$L_{\text{ASEP}} \cdot \Delta(u) = \Delta(u) \cdot L_{\text{ASEP}} \text{ for all } u \in \mathcal{U}_q(\mathfrak{sl}_2).$$

Using time reversibility of ASEP, $L_{\text{ASEP}} = V L_{\text{ASEP}}^T V^{-1}$ for a diagonal matrix V, we get

$$L_{\text{ASEP}} \cdot \underbrace{\Delta(u)V}_{D} = \cdot \underbrace{\Delta(u)V}_{D} \cdot L_{\text{ASEP}}^T,$$

In Carinci–Giardina–Redig–Sasamoto ('14), they consider the case when $\mathfrak{g} = \mathfrak{sl}_2$ and V is the irreducible representation of dimension $m + 1 \in \mathbb{Z}_+$, and C is the Casimir element of $\mathcal{U}_q(\mathfrak{sl}_2)$. Then m particles may occupy a site and the asymmetry parameter is q^m. This process is called ASEP(q, j) (for $j = m/2$), and it satisfies a duality which generalizes the Schütz ASEP duality.

Let A_i denote the location of the ith species particles in A. Let $\eta(x)$ denote the species of the particle at x.

Theorem (K. (16))

The n–species ASEP is dual to another n–species ASEP with respect to the function

$$\tilde{D}(\eta, A) = \begin{cases} \prod_{i=1}^{n} \prod_{x_i \in A_i} q^{2N^{(i)}_{x_i}(\eta) + 2x_i}, & \text{if } 1 \leq \eta(x_i) \leq i \text{ for all } x_i \in A_i \\ 0, & \text{else.} \end{cases}$$

where $N^{(i)}_x(\eta) = \#$ *of particles to the right of* x *of species* $\{1, \ldots, i\}$.

Concurrent with Belitsky–Schütz (16), and generalizing $n = 2$ Belitsky–Schütz (15), K. (15).

Applying a charge–parity symmetry to the ASEP(q, j), one finds:

Theorem (K. (16))

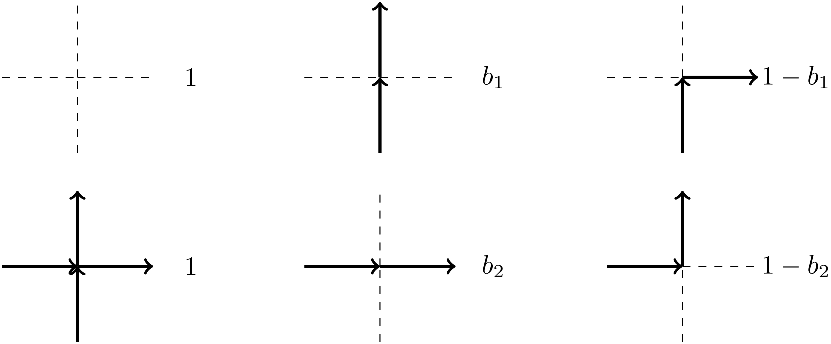
The multi–species q–Boson satisfies duality with respect to

$$D(\eta, A) = \prod_{i=1}^{n} \prod_{x_i \in A_i} q^{2N_{x_i}^{(i)}(\eta)}.$$

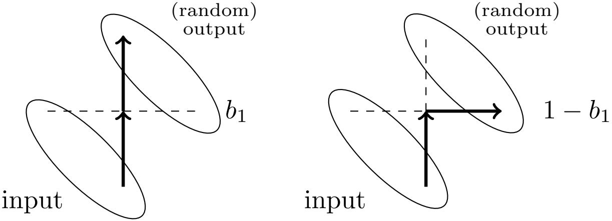
Here, the η process has particles jumping to the right, while the A process has particles jumping to the left.

This is a multi–species version of the Borodin–Corwin–Sasamoto ('12) ASEP duality.

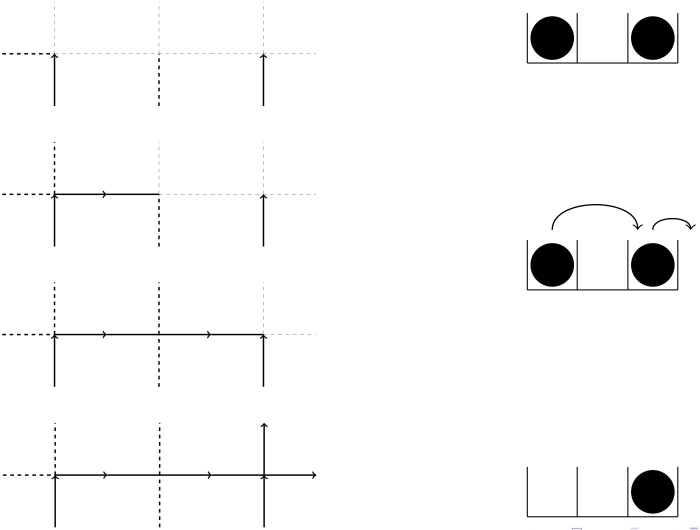
Both ASEP and q–Boson are degenerations of a stochastic vertex model. Below are the weights of the stochastic six vertex model.



The probability measure can be defined by Markov update:



The vertex model can be viewed as a discrete–time particle system (on the infinite line or with *open* boundary conditions):



In addition to quantization, there is another generalization to *affine* Lie algebras. Given a finite–dimensional simple Lie algebra $\mathfrak{g}$, then (as an infinite–dimensional vector space)

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$

Because of the additional term t, there is now a family of two–dimensional representations $\mathbb{C}^2(z)$ of $\widehat{\mathfrak{sl}_2}$, defined by letting t act as multiplication by the complex number z.

However, the Casimir element is now an infinite series, whose action will generally diverge.

It turns out that we will need another intertwining relation coming from algebra:

Because of the asymmetry in the quantum group, $\Delta \neq \Delta^{\text{rev}} := P \circ \Delta$, where P is the permutation operator

$$P(v \otimes w) = w \otimes v.$$

It is "almost" symmetric, in the sense that there exists a unique (up to a constant) invertible element $\mathcal{R}$ (called the R–matrix) such that

$$\mathcal{R} \cdot \Delta(u) = \Delta^{\text{rev}}(u) \cdot \mathcal{R}.$$

The matrix entries of $\mathcal{R}$ (in a representation) give the vertex weights.

$\mathcal{R}$ also satisfies the Yang–Baxter equation, but we will not need to use it.

In Kuniba–Mangazeev–Maruyama–Okado (16), the authors find explicit formulas for the R–matrix of $\mathcal{U}_q(\widehat{\mathfrak{sl}_{n+1}})$.
A previous paper Kuniba–Okado–Sergeev (15) gave a formula for "time reversibility" $R = VR^TV^{-1}$.

Theorem (K. (17))

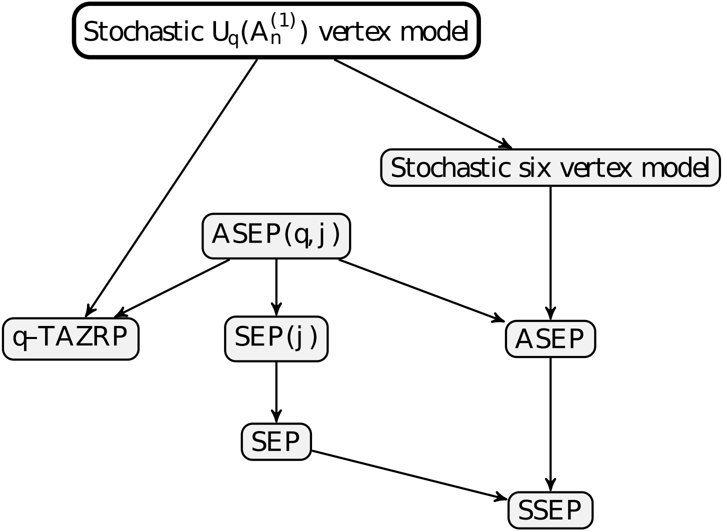
- *For a range of values of q and z, the stochastic vertex model defines an (inhomogeneous) interacting particle system on closed boundary conditions.*
- *This interacting particle system is dual to its space reversal with respect to the same duality function as multi–species ASEP(q,j) of K. ('16).*

Theorem (K. (17))

- *When there is only one species of particles, one recovers the stochastic vertex model of Corwin–Petrov ('15), Borodin–Petrov ('16).*
- *The coupling property holds: the n–species stochastic vertex model is equivalent to a coupling of n copies of the single–species stochastic vertex model.*
- *The duality results for multi–species q–Boson and multi–species ASEP are recovered as corollaries.*

The proof of duality uses the intertwining of the R–matrix $R(z) \cdot \Delta(u) = \Delta^{\mathrm{rev}}(u) \cdot R(z)$.

Here are the degenerations mentioned:



Stochastic $U_q(A_n^{(1)})$ vertex model

Stochastic six vertex model

ASEP(q,j)

q-TAZRP

SEP(j)

ASEP

SEP

SSEP

Using Bethe Ansatz, [Tracy–Widom '07]

$$P_Y(X;t) = \sum_{\sigma \in \mathbb{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i (\alpha \xi^{-1} + \beta \xi - 1)t} d\xi_1 \cdots d\xi_N$$

where $\mathcal{C}_r$ are small contours centered around 0 and

$$S_{ij} = -\frac{p + q\xi_i \xi_j - \xi_i}{p + q\xi_i \xi_j - \xi_j}$$

and

$$A_\sigma = \prod \{S_{ij} : \{i,j\} \text{ is an inversion in } \sigma\}$$

Here, $\{i,j\}$ is an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$.
Recovers determinantal formula of Schütz 97 for TASEP.

Korhonen–Lee '13 show that the q–Boson has Green's function

$$P_Y(X;t) = W_X \left(\frac{1}{2\pi i}\right)^N \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma \prod_{i=1}^N z_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{(z_i^{-1}-1)t} dz_i$$

where $\mathcal{C}_r$ are small contours and

$$A_\sigma = \prod \{S_{ij} : \{i,j\} \text{ is an inversion in } \sigma\}$$

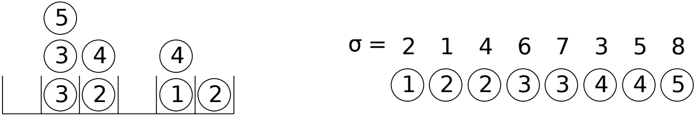
with

$$S_{ij} = -\frac{z_j - qz_i - (1-q)z_iz_j}{z_i - qz_j - (1-q)z_iz_j}, \qquad W_X = \frac{1}{\prod_{\mathbf{x}} [\eta(\mathbf{x})]_q!}.$$

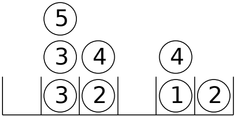
where $\eta(\mathbf{x})$ is the number of particles at lattice site $\mathbf{x}$ and $[n]_q!$ is a q–deformed factorial.

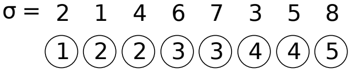
In the multi–species q–Boson, there are two natural symmetries: permuting the particles at a single lattice site, or permuting particles of the same species.

For example, both $H' = S(3) \times S(2) \times S(2) \times S(1)$ and $H = S(1) \times S(2) \times S(2) \times S(2) \times S(1)$ and preserve the particle configuration below:



Viewing H' as a left action and H as a right action, each particle configuration can be viewed as an element of a double coset $H'\sigma H$. We pick σ to be the coset representative with the fewest inversions. In this example, 24167358 represents the same configuration, but has more inversions.





Each particle configuration can be uniquely written as $(\mathbf{x},\sigma)$, where $\mathbf{x}$ lists the positions. Here, $\mathbf{x}=(2,2,2,3,3,5,5,6)$.

A measure is q–exchangeable if $\mathbb{P}(\mathbf{x},\sigma) \propto q^{\mathrm{inv}(\sigma)}$.

Theorem (K. '18)

Suppose the multi–species q–Boson starts with q–exchangeable initial conditions at $(y_1 \geq \cdots \geq y_N)$. Then

$$\mathrm{Prob}((\mathbf{x},\sigma) \textit{ at time } t) = \frac{q^{\mathrm{inv}(\sigma)}}{[N]_q^!} \cdot \left(\frac{\prod_{j=1}^n [N_j]_q^!}{\prod_{j=1}^n \prod_{i=1}^r [L_{ij}]_q^!} \right)$$
$$\times \left(\frac{1}{2\pi i}\right)^N \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \sum_{\tau \in S_N} A_\tau \prod_{j=1}^N \left[\prod_{k=y_{\tau(j)}}^{x_j} \left(\frac{1}{1-w_{\tau(j)}} \right) e^{-w_j t} \right] d\mathbf{w},$$

where

$$A_\tau = \prod_{(j,i) \textit{ an inversion of } \tau} -\frac{qw_j - w_i}{qw_i - w_j}.$$

and the $\mathcal{C}_R$ are "large contours".

The proof essentially follows from showing that q–exchangeability is preserved under the dynamics, and matching the factor W_X.

Theorem (K. '18)

Suppose the multi–species ASEP starts with q–exchangeable initial conditions at $(y_1 \geq \cdots \geq y_N)$*. Then*

$$\begin{aligned}&\mathrm{Prob}((\mathbf{x},\sigma) \text{ at time } t)\\ &= \frac{q^{\mathrm{inv}(\sigma)}}{[N]_q!} \sum_{\tau \in \mathbb{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\tau \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i (\alpha \xi^{-1} + \beta \xi - 1)t} d\xi_1 \cdots d\xi_N\end{aligned}$$

where the $\mathcal{C}_r$ *are small contours. Here*

$$A_\tau = \prod_{(j,i) \text{ an inversion of } \tau} -\frac{\alpha + \beta \xi_i \xi_j - \xi_i}{\alpha + \beta \xi_i \xi_j - \xi_j}$$

Tracy–Widom '07 prove a "miraculous" symmetrization identity

$$\sum_{\sigma\in\mathbb{S}_N} \operatorname{sgn}\sigma \left(\prod_{i<j}\left(p+q\xi_{\sigma(i)}\xi_{\sigma(j)}-\xi_{\sigma(i)}\right) \times \frac{\xi_{\sigma(2)}\xi^2_{\sigma(3)}\xi^3_{\sigma(4)}\cdots\xi^{N-1}_{\sigma(N)}}{\left(1-\xi_{\sigma(1)}\xi_{\sigma(2)}\xi_{\sigma(3)}\cdots\xi_{\sigma(N)}\right)\cdots\left(1-\xi_{\sigma(N-1)}\xi_{\sigma(N)}\right)\left(1-\xi_{\sigma(N)}\right)}\right) = p^{N(N-1)/2}\frac{\prod_{i<j}(\xi_j-\xi_i)}{\prod_j(1-\xi_j)}$$

and use it to show that

$$\mathbb{P}\left(x_1(t)=x\right)=p^{N(N-1)/2}\int_{\mathcal{C}_r}\cdots\int_{\mathcal{C}_r} I(x,Y,\xi)d\xi_1\cdots d\xi_N$$

where

$$I(x,Y,\xi)=\prod_{i<j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\frac{1-\xi_1\cdots\xi_N}{(1-\xi_1)\cdots(1-\xi_N)}\prod_i\left(\xi_i^{x-y_i-1}e^{\varepsilon(\xi_i)t}\right)$$

This probability also follows quickly (about half a page) using the Green's function for multi–species ASEP without needing a symmetrization identity.

If all $y_1 = \cdots = y_N$ (corresponding to the single–point fluctuations in the multi–species q–Boson) then [Wang–Waugh '15] uses the symmetrization identity

$$\sum_{\sigma \in S_n} A_\sigma \left(w_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(n)} \right) = [n]_q! B \left(w_1, \ldots, w_n \right)$$

where

$$B \left(w_1, \ldots, w_n \right) = \prod_{1 \leq i < j \leq n} \frac{w_i - w_j}{w_i - q w_j}$$

and then show

$$P_{(0,0,\ldots,0)}(X;t) = [n]_q! \frac{1}{W(X)} \left(\prod_{k=1}^{n} \frac{-1}{b_{x_k}} \right)$$

$$\times \int_{C_R} dw_1 \cdots \int_{C_R} dw_n B \left(w_1, \ldots, w_n \right) \prod_{j=1}^{n} \left[\prod_{k_j=0}^{\prime} \left(\frac{b_{k_j}}{b_{k_j} - w_j} \right) e^{-w_j t} \right]$$

Symmetrization for multi–species q–Boson

By the Markov projection property, it suffices to consider the case when there is exactly one particle of each species (the "rainbow" case). Let $z_i(t)$ denote the location of the i–th species particle at time t. Take the initial conditions to be $y_1 = \cdots = y_N = 0$. Then we have the "large contour" integral formula

$$\mathbb{P}(z_1(t) = x_1, \ldots, z_K(t) = x_K, z_{K+1}(t) \leq M_{N-K}, \ldots, z_N(t) \leq M_1)$$

$$= \sum_{L=K}^{N} c(q,N,K,L-K) q^{\mathrm{inv}(\sigma)} (-1)^L \left(\frac{1}{2\pi i} \right)^L \int_{\mathcal{C}_R} dw_1 \cdots \int_{\mathcal{C}_R} dw_L B(w_1, \ldots, w_L)$$

$$\times \prod_{j=1}^{L-K} \left(\frac{(1-w_j)^{-M_j-1}}{w_j} \right) \prod_{j=L-K+1}^{L} \left((1-w_j)^{-x_{\sigma(j+K-L)}-1} \right) e^{-(w_1+\cdots w_L)t},$$

where

$$c(q,N,K,m) = \sum_{K \leq i_1 < i_2 < \cdots < i_m \leq N-1} q^{i_1} q^{i_2} \cdots q^{i_m}$$

and $\sigma \in S(K)$ denotes the permutation satisfying $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(K)}$ with the fewest number of inversions.

The sum over $z_{K+1},\dots,z_N$ turns out to be a telescoping sum, because

$$B\left(w_1,\dots,w_N\right)\left(\frac{q}{w_k}-\frac{1}{w_{k+1}}\right)=\left[\prod_{\substack{1\le i<j\le N\\(i,j)\neq(k,k+1)}}\frac{w_i-w_j}{w_i-qw_j}\right]\frac{w_{k+1}-w_k}{w_kw_{k+1}},$$

making the integral anti–symmetric in w_k and w_{k+1} .
The contour only has a pole at $w_N=0$, and using that $B(w_1,\dots,w_{N-1},0)=B(w_1,\dots,w_{N-1})$, the N–fold contour integral can be written as a $(N-1)$–fold contour integral.

The sum of large contour integrals can be written as a single "small contour" integral. The deformation from large contours to small contours creates residues such that all L–fold contour integrals cancel except for the N–fold contour integral:

$$q^{N-1}q^{N-2}\cdots q^{K}q^{\mathrm{inv}(\sigma)}(-1)^N\left(\frac{1}{2\pi i}\right)^N\int_{\tilde{\mathcal{C}}^1}dw_1\cdots\int_{\tilde{\mathcal{C}}^N}dw_N B(w_1,\ldots,w_N)$$
$$\times\prod_{j=1}^{N-K}\left(\frac{(1-w_j)^{-M_j-1}}{w_j}\right)\prod_{j=N-K+1}^{N}\left((1-w_j)^{-x_{\sigma(j+K-N)}-1}\right)e^{-(w_1+\cdots w_N)t},$$

where the contour $\tilde{\mathcal{C}}^r$ contains $q\tilde{\mathcal{C}}^{r+1},\ldots,q\tilde{\mathcal{C}}^N$ and 1, but not 0.
Taking $K=0$ and a direct comparison to
[Borodin—Corwin–Sasamoto '12] finishes the proof.

Next steps / open questions:

- Prove the general "shift/flip" invariance for the multi-species q–Boson.
- Prove an analogous result for ASEP.

A math joke!

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Student: Professor Markov, I need help with a probability question. Given a non–negative random variable X and a positive number a, I need an upper bound on $\mathbb{P}(X > a)$.

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Sorry, that was a *mean* joke.