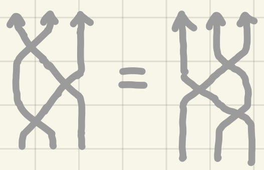
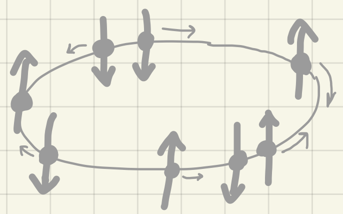
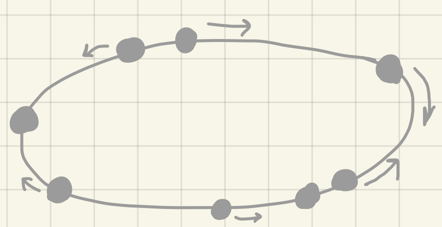
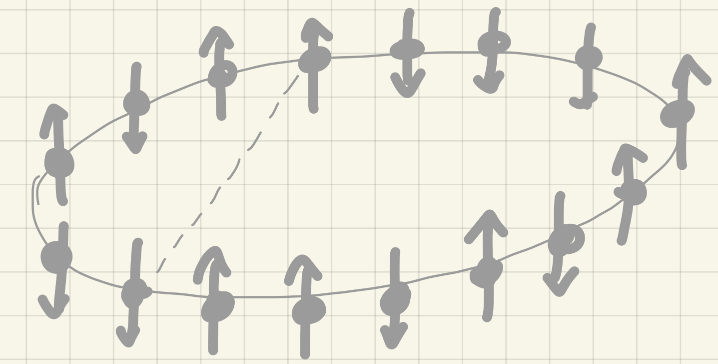
Macdonald polynomials
and
long-range spin chains









by

Jules Lamers
University of Melbourne

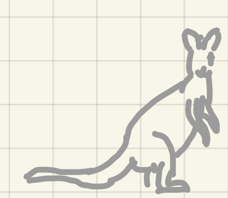
based on

JL, PRB 97 ('18) 214416
arXiv.1801 05728

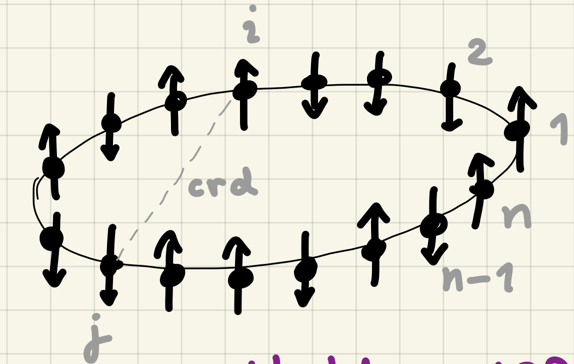
JL, V Pasquier, D Serban
arXiv 2004.13210

and ongoing

Solv latt mod sem '21 VII 14 (15)

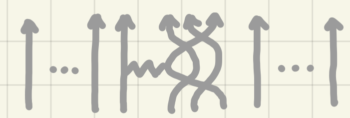


Motivation: Haldane-Shastry spin chain



the operator

$$H = \sum_{i<j}^{n} \underbrace{\frac{-z_i z_j}{(z_i-z_j)^2}}_{\frac{1}{4\sin^2(\frac{\pi}{n}(i-j))}} \times \overbrace{(1-P_{ij})}^{P_{j-1,j}\cdots P_{i+1,i+2}(1-P_{i,i+1})P_{i+1,i+2}\cdots P_{j-1,j}}$$



$= P$ permutation

$= 1-P \overset{r=2}{=} \frac{1-\vec{\sigma}\cdot\vec{\sigma}}{2}$ antisymmetriser

$z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

on $(\mathbb{C}^r)^{\otimes n}$

Haldane '88
Shastry '88

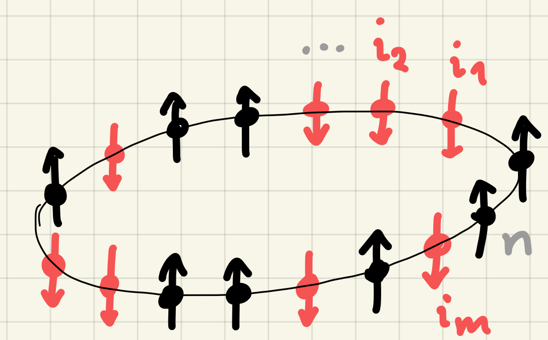
- (abelian symmetries) belongs to family of commuting operators, which — Bernard et al '93, Talstra Haldane '95
- (nonabelian symmetries) commutes with action of Yangian $Y(\mathfrak{gl}_r)$ — Haldane et al '92, Bernard et al '93
- for r=2 has exact $Y(\mathfrak{gl}_2)$ highest-weight eigenvectors ψ_λ determined by

$\mathbb{C}^2 = \mathbb{C}v_\uparrow \oplus \mathbb{C}v_\downarrow$ $\quad \ell(\lambda)=m$, $\lambda_1 \leq n-2m+1$

$$\langle \underbrace{v_\downarrow \otimes \cdots \otimes v_\downarrow}_{\#=m} \otimes \underbrace{v_\uparrow \otimes \cdots \otimes v_\uparrow}_{\#=n-m}, \psi_\lambda \rangle = \underbrace{\prod_{i<j}^{m}(z_i-z_j)^2}_{\text{square of Vandermonde}} \times \underbrace{P_\lambda^{(1/2)}(z_1,\ldots,z_m)}_{\text{Jack polyn (spherical zonal)}} =: f_\lambda(z_1,\ldots,z_m)$$

Haldane '91

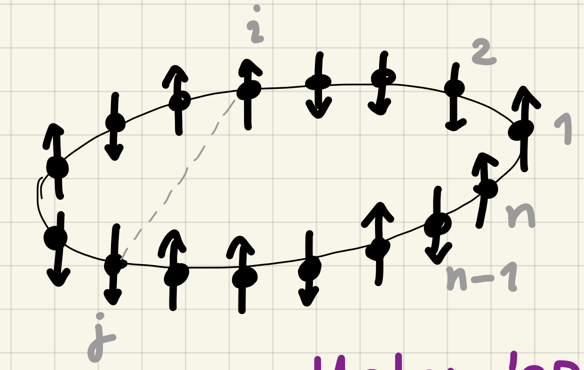
viz. $\quad \psi_\lambda = \sum_{i_1<\cdots<i_m}^{n} f_\lambda(z_{i_1},\ldots,z_{i_m}) \times \begin{pmatrix}\text{basis vec} \\ \text{with} \downarrow \text{at} \\ i_1,\ldots,i_m\end{pmatrix}$



- is lattice toy model for fractional quantum Hall effect — Haldane '91
  e.g. exhibits fractional (exclusion) statistics

Main result: t-deformed Haldane-Shastry spin chain

$\underbrace{\text{t-deformed}}$ = partially isotropic = XXZ-like (invariant under $\mathfrak{h} \subset \mathfrak{gl}_r$)



Uglov '95
JL '18

the operator

$$H = \mathrm{cst}_n \times \sum_{i<j}^{n} \frac{-t\, z_i z_j}{(t z_i - z_j)(z_i - t z_j)} \times \; z_1 \cdots z_i \cdots z_j \cdots z_n$$

$= \check{R}(u/v)$ — $U_{t^{1/2}}(L\mathfrak{gl}_r)$ R-matrix

$= -(t^{1/2} - t^{-1/2})\, \check{R}'(1)$ — t-antisymmetriser

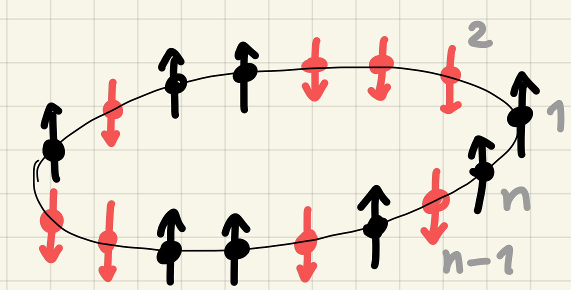
$z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

on $(\mathbb{C}^r)^{\otimes n}$

- (abelian symmetries) belongs to family of commuting operators, which — Bernard et al '93, JL et al '20
- (nonabelian symmetries) commutes with action of quantum-loop algebra $U_{t^{1/2}}(L\mathfrak{gl}_r)$ — Bernard et al '93
  $= U'_{t^{1/2}}(\widehat{\mathfrak{gl}}_r)_{c=0}$
- for $r=2$ has exact $U_{t^{1/2}}(L\mathfrak{gl}_r)$ (pseudo) highest-weight eigenvectors Ψ_λ determined by
  $\mathbb{C}^2 = \mathbb{C}v_\uparrow \oplus \mathbb{C}v_\downarrow$
  $\ell(\lambda) = m$, $\lambda_1 \leq n - 2m + 1$

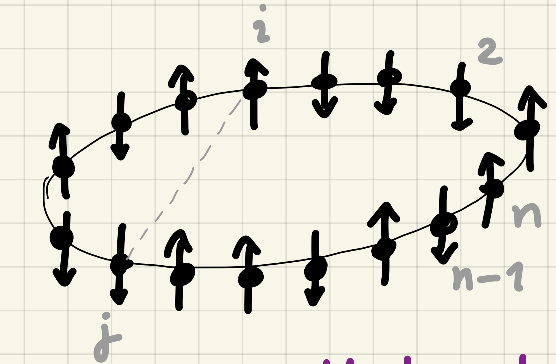
$$\langle \underbrace{v_\downarrow \otimes \cdots \otimes v_\downarrow}_{\# = m} \otimes \underbrace{v_\uparrow \otimes \cdots \otimes v_\uparrow}_{\# = n-m}, \Psi_\lambda \rangle = \underbrace{\prod_{i<j}^{m} (t z_i - z_j)(t^{-1} z_i - z_j)}_{\text{'symmetric square' of t-Vandermonde}} \times \underbrace{P_\lambda(z_1, \ldots, z_m; t, t^2)}_{\text{Macdonald (quantum spherical zonal)}}$$

JL et al '20

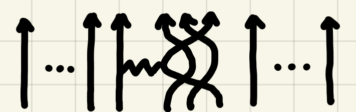


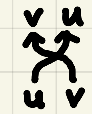
Main result: $\underbrace{\text{t-deformed}}_{}$ Haldane-Shastry spin chain

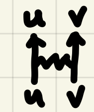
= partially isotropic = KKZ-like



Uglov '95
JL '18

$$H = \mathrm{cst}_n \times \sum_{i<j}^{n} \frac{-t\,z_i z_j}{(t z_i - z_j)(z_i - t z_j)} \times$$ 

 $= \check{R}(u/v)$ $\quad U_{t^{1/2}}(L\mathfrak{gl}_r)$ R-matrix

 $= -(t^{1/2} - t^{-1/2})\,\check{R}'(1)$ $\quad$ t-antisymmetriser

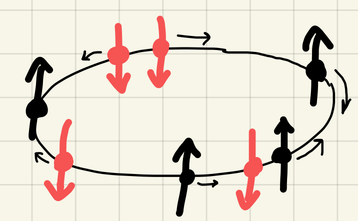
$z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

Plan: quantum-affine Schur-Weyl duality and freezing

Bernard et al '93
Talstra Haldane '95
Uglov '95
JL et al '20

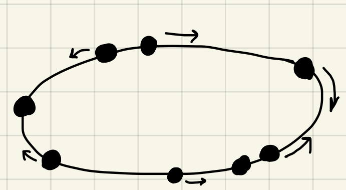
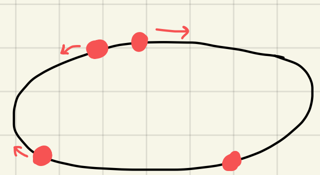
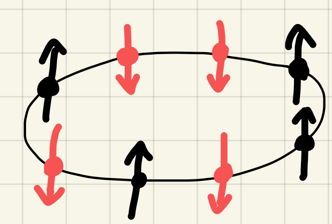
spin-Macdonald-Ruijsenaars
- abelian symmetries
- nonabelian symmetries
- exact eigenvectors

affine Hecke alg $\rightsquigarrow$ symm $(\mathbb{C}^r)^{\otimes n}$-valued polynomials



$\Big\downarrow$ symmetric polynomials

$\Big\downarrow$ freezing — semiclass lim to class equilib

  $\Longleftrightarrow$ 

Macdonald-Ruijsenaars
- abelian symmetries
- exact eigenfunctions: Macdonald polynomials

t-deformed Haldane-Shastry
- abelian symmetries
- nonabelian symmetries
- exact eigenvectors (r=2)

Algebraic background I: Hecke algebra

(Iwahori-)Hecke algebra (type A_{n-1})

$$H_n = H_n(t^{1/2}) = \left\langle \begin{matrix}\text{unital assoc alg}\\ T_1,\dots,T_{n-1}\end{matrix} \;\middle|\; \begin{matrix}\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{matrix} \right\rangle \xrightarrow{t\to 1} \mathbb{C}S_n$$

so $T_i^{-1} = T_i - (t^{1/2} - t^{-1/2})$

important representations:

- ('polynomial') on $\mathbb{C}[\underline{x}^{\pm}] := \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$
  or $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$

  via $T_i \longmapsto t^{1/2} - t^{-1/2}(t x_i - x_j)(x_i - x_j)^{-1}(1 - s_i)$ Demazure-Lusztig $\xrightarrow{t\to 1}$ $1-(1-s_i) = s_i : x_i \leftrightarrow x_{i+1}$

- ('spin') on $W = (\mathbb{C}^r)^{\otimes n}$ $r \in \mathbb{Z}_{\geq 2}$

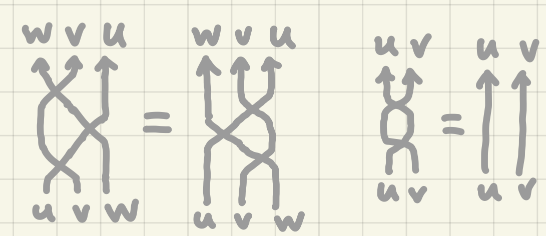
  via $T_i \longmapsto id^{\otimes(i-1)} \otimes T \otimes id^{\otimes(n-i-1)}$, $r=2$: $T = \begin{pmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & t^{1/2}-t^{-1/2} & t^{1/2} & 0 \\ 0 & t^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & t^{1/2} \end{pmatrix} \xrightarrow{t\to 1} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} = P$

  ⌈rmk (Jones)

  $$\check{R}(u/v) = t^{1/2}\,\frac{uT - vT^{-1}}{tu - v} =$$ 

  R-matrix of $\hat{\mathcal{U}}_r := \mathcal{U}_{t^{1/2}}(L\mathfrak{gl}_r)$

  braid-like form: $R(u) = P\,\check{R}(u)$

  $$t^{\pm 1/2} \lim_{u^{\pm 1}\to\infty} \check{R}(u) = T^{\pm 1}$$ ⌊

  $\lim_{t\to 1} \check{R}(t^x) = P \cdot \frac{x+P}{x+1}$ $Y(\mathfrak{gl}_r)$

this representation is reducible:

$$W = (\mathbb{C}^r)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ \ell\lambda \leq r}} V^{\lambda} \otimes V_{\lambda}$$

quantum Specht module
finite-dim H_n-irrep

multiplicity space:
W contains $\dim V_\lambda$ copies of V^λ

Algebraic background II: quantum Schur–Weyl duality

(Iwahori-)Hecke algebra (type A_{n-1})

$$H_n = H_n(t^{1/2}) = \left\langle \begin{matrix}\text{unital assoc alg}\\ T_1,\dots,T_{n-1}\end{matrix} \;\middle|\; \begin{matrix}\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{matrix} \right\rangle \xrightarrow{t\to 1} \mathbb{C}S_n$$

'spin' representation on $W = (\mathbb{C}^r)^{\otimes n}$ $\quad r \in \mathbb{Z}_{\geq 2}$

via $T_i \longmapsto \mathrm{id}^{\otimes(i-1)} \otimes T \otimes \mathrm{id}^{\otimes(n-i-1)}$, $\quad r=2:$ $T = \begin{pmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & t^{1/2}-t^{-1/2} & t^{1/2} & 0 \\ 0 & t^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & t^{1/2} \end{pmatrix} \xrightarrow{t\to 1} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} = P$

is reducible

in fact H_n-action commutes with $\mathcal{U}_r := U_{t^{1/2}}(\mathfrak{gl}_r)$ $\quad \mathbb{C}^r$ = standard rep

more precisely: the actions generate each other's commutant in End W

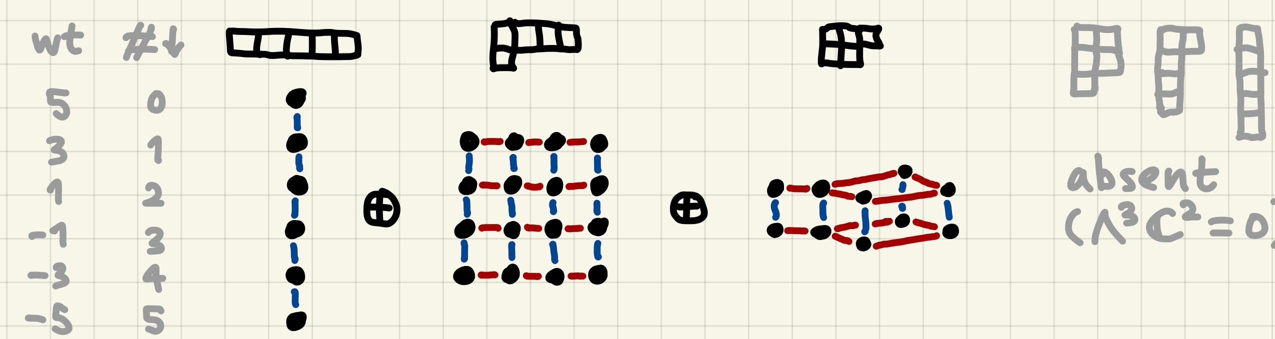
and we have

$$W = (\mathbb{C}^r)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ \ell\lambda \leq r}} V^\lambda \otimes V_\lambda$$

quantum Specht module (V^λ)
finite-dim H_n-irrep

highest-weight module (V_λ)
finite-dim $\mathcal{U}_r$-irrep

ex $n=5$, $r=2$:

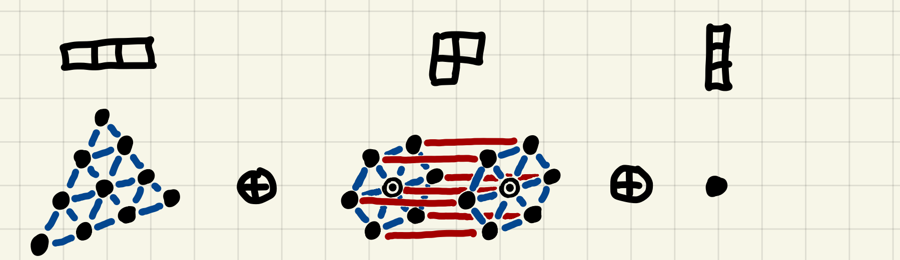


triv (6x) spin-$\frac{5}{2}$ (1x)

stand (4x) spin-$\frac{3}{2}$ (4x)

(2x) spin-$\frac{1}{2}$ (5x)

ex $n=3$, $r=3$:



trivial (10x) (1x)

standard (8x) adjoint (2x)

sign (1x) trivial (1x)

Algebraic background II: quantum Schur–Weyl duality

(Iwahori-)Hecke algebra (type A_{n-1})

$$H_n = H_n(t^{1/2}) = \left\langle \begin{smallmatrix}\text{unital assoc alg}\\ T_1,\dots,T_{n-1}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{smallmatrix} \right\rangle \xrightarrow{t\to 1} \mathbb{C}S_n$$

'spin' representation on $W = (\mathbb{C}^r)^{\otimes n}$ $\quad r \in \mathbb{Z}_{\geq 2}$

via $T_i \longmapsto id^{\otimes(i-1)} \otimes T \otimes id^{\otimes(n-i-1)}$, $\quad r=2$: $T = \begin{pmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & t^{1/2}-t^{-1/2} & t^{1/2} & 0 \\ 0 & t^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & t^{1/2} \end{pmatrix} \xrightarrow{t\to 1} \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} = P$

is reducible

in fact H_n-action commutes with $\mathcal{U}_r := U_{t^{1/2}}(\mathfrak{gl}_r)$ $\quad \mathbb{C}^r$ = standard rep

more precisely: the actions generate each other's commutant in End W

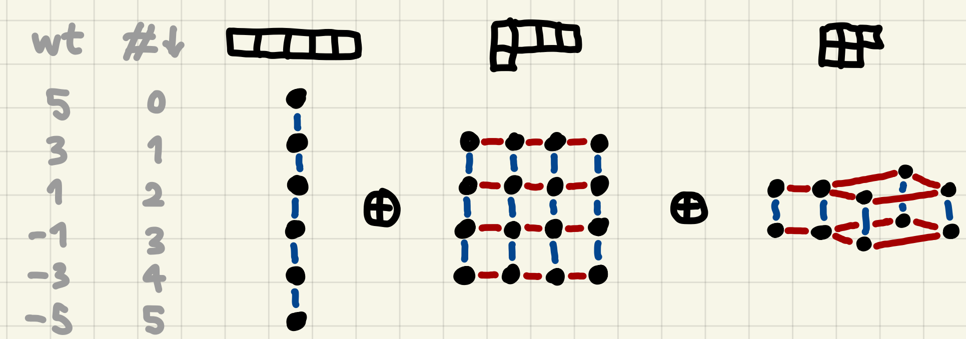
and we have

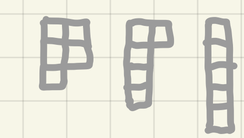
$$W = (\mathbb{C}^r)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ \ell\lambda \leq r}} V^\lambda \otimes V_\lambda$$

quantum Specht module $\quad$ highest-weight module

finite-dim H_n-irrep $\quad$ finite-dim $\mathcal{U}_r$-irrep

ex $n=5$, $r=2$:





absent ($\Lambda^3\mathbb{C}^2 = 0$)

triv (6x) $\quad$ stand (4x) $\quad$ (2x)

spin-$\frac{5}{2}$ (1x) $\quad$ spin-$\frac{3}{2}$ (4x) $\quad$ spin-$\frac{1}{2}$ (5x)

alternative perspective: quantum Schur functor

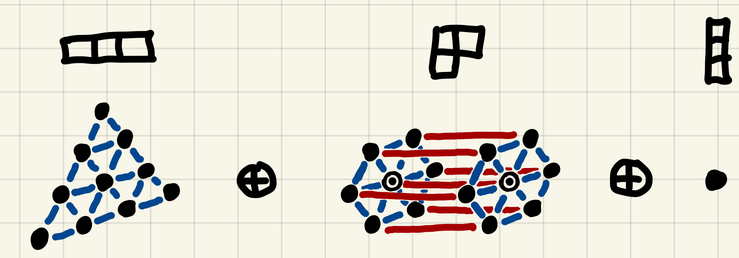
$$\begin{Bmatrix}\text{fin-dim}\\ H_n\text{-modules}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{fin-dim}\\ \mathcal{U}_r\text{-modules}\end{Bmatrix}$$

$$\begin{matrix} \Psi & & \Psi \\ M & \longmapsto & M \otimes_{H_n} W := M\otimes W \Big/ \sum_{i=1}^{n-1} \operatorname{im}(T_i \otimes 1 - 1 \otimes T_i) \end{matrix}$$

in particular,

$$V^\lambda \longmapsto \begin{cases} V_\lambda & \text{if } \ell\lambda \leq r \\ 0 & \text{else} \end{cases}$$

ex $n=3$, $r=3$:



trivial (10x) $\quad$ standard (8x) $\quad$ sign (1x)

(1x) $\quad$ adjoint (2x) $\quad$ trivial (1x)

Algebraic background III: affine Hecke algebra

$$H_n = H_n(t^{1/2}) = \left\langle \substack{\text{unital assoc alg}\\ T_1,\dots,T_{n-1}} \;\middle|\; \substack{\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0} \right\rangle \xrightarrow{t\to 1} \mathbb{C}S_n$$

affine Hecke algebra (type GL_n, Bernstein-Zelevinsky presentation)

$$\hat{H}_n = \hat{H}_n(t^{1/2}) = H_n \otimes \underbrace{\mathbb{C}[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]}_{\text{maximal abelian subalgebra}} \text{ with cross relations } \begin{matrix} T_i Y_i T_i = Y_{i+1}, \; 1 \le i < n \\ T_i Y_j = Y_j T_i, \; j \ne i, i+1 \end{matrix} \xrightarrow{t\to 1} \text{degenerate affine Hecke algebra}$$

important representations:

- on $M_{\underline{z}} := \mathrm{Ind}_{\mathbb{C}[\underline{Y}^{\pm}]}^{\hat{H}_n} (\mathbb{C} v_{\underline{z}}) = \hat{H}_n \otimes_{\mathbb{C}[\underline{Y}^{\pm}]} \mathbb{C} v_{\underline{z}}$, $\quad Y_i v_{\underline{z}} = z_i v_{\underline{z}}$

  $\underline{z} \in (\mathbb{C}^\times)^n$

  $\dim M_{\underline{z}} = n!$ basis $T_w v_{\underline{z}}$, $w \in S_n$

  irreducible for $\underline{z} \in (\mathbb{C}^\times)^n$ generic $z_i \ne t z_j \;\forall i \ne j$

- on $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$

  via $T_i \mapsto t^{1/2} - t^{-1/2}(t x_i - x_j)(x_i - x_j)^{-1}(1 - s_i)$ Demazure-Lusztig $\xrightarrow{t\to 1}$ $1 - (1 - s_i) = s_i : x_i \leftrightarrow x_{i+1}$

  $q \in \mathbb{C}^\times$ $\quad Y_j \mapsto T_{j-1} \cdots T_1 \, \rho \, T_{n-1}^{-1} \cdots T_j^{-1}$ Cherednik $\quad Y_i|_{q = t^\alpha} = 1 + (t-1) d_i + O(t-1)^2$ Dunkl

  $(\rho f)(x_1, \dots, x_n) = f(x_2, \dots, x_n, q^{-1} x_1)$

  which is reducible with fin-dim $\hat{H}_n$-irreps:

  $$\mathbb{C}[\underline{x}] = \bigoplus_{\ell \lambda \le n} \mathrm{span}_{\mathbb{C}} \{ E_\mu(\underline{x}; q, t), \; \mu \in S_n \cdot \lambda \}$$

  nonsymm Macdonald polyn: $\begin{cases} Y_j E_\mu(\underline{x}; q, t) = t^{\cdots} q^{-\mu_j} E_\mu(\underline{x}; q, t) \\ E_\mu(\underline{x}; q, t) = \underline{x}^\mu + \text{lower monomials} \end{cases}$

  $E_\mu(\underline{x}; t^\alpha, t) \xrightarrow{t\to 1} E_\mu^{(\alpha)}(\underline{x})$ nonsymm Jack

Algebraic background IV: quantum affine Schur–Weyl duality

$$H_n = H_n(t^{1/2}) = \left\langle \begin{matrix}\text{unital assoc alg}\\ T_1,\dots,T_{n-1}\end{matrix} \,\middle|\, \begin{matrix}\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{matrix} \right\rangle \xrightarrow{t\to 1} \mathbb{C}S_n$$

$$\hat{H}_n = \hat{H}_n(t^{1/2}) = H_n \otimes \underbrace{\mathbb{C}[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]}_{\text{maximal abelian subalgebra}} \text{ with cross relations } \begin{matrix} T_i Y_i T_i = Y_{i+1}, & 1 \le i < n \\ T_i Y_j = Y_j T_i, & j \ne i, i+1 \end{matrix} \xrightarrow{t\to 1} \text{degenerate affine Hecke algebra}$$

quantum-affine Schur functor

$$\begin{Bmatrix}\text{fin-dim}\\ \hat{H}_n\text{-modules}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{fin-dim}\\ \hat{\mathcal{U}}_r\text{-modules}\end{Bmatrix}$$

$$M \longmapsto M \otimes_{H_n} W := M \otimes W \Big/ \sum_{i=1}^{n-1} \mathrm{im}(T_i \otimes 1 - 1 \otimes T_i)$$

Drinfeld '86

with explicit action of $\hat{\mathcal{U}}_r := \mathcal{U}_{t^{1/2}}(L\mathfrak{gl}_r)$

idea: upgrade W to tensor product of evaluation modules and formally replace evaluation parameters by $Y_1, \dots, Y_n$ ('quantised inhomogeneities')

ex $r = 2$: $\mathcal{U}_2$-action on $M \otimes_{H_n} (\mathbb{C}^2)^{\otimes n}$ extends to action of $\hat{\mathcal{U}}_2$

$e := \begin{pmatrix}0&1\\0&0\end{pmatrix} = \sigma^+$

$f := \begin{pmatrix}0&0\\1&0\end{pmatrix} = \sigma^-$

$k := \begin{pmatrix}t^{1/2}&0\\0&t^{-1/2}\end{pmatrix} = t^{\sigma^z/2}$

$$E_1 = \mathrm{id} \otimes \sum_{i=1}^n k_1 \cdots k_{i-1} e_i \qquad E_0 = \sum_{i=1}^n Y_i \otimes k_1^{-1} \cdots k_{i-1}^{-1} f_i$$

$$F_1 = \mathrm{id} \otimes \sum_{i=1}^n f_i k_{i+1}^{-1} \cdots k_n^{-1} \qquad F_0 = \sum_{i=1}^n Y_i^{-1} \otimes e_i k_{i+1} \cdots k_n$$

$$K_1 = \mathrm{id} \otimes k_1 \cdots k_n \qquad K_0 = \mathrm{id} \otimes k_1^{-1} \cdots k_n^{-1} = K_0^{-1}$$

Drinfeld-Jimbo presentation — Chari Pressley '96

expansion in $u^{\pm 1}$ of Gauss decomp

equivalently: on $M[[u]] \otimes_{H_n} (\underbrace{\mathbb{C}^2 \otimes W}_{\cong (\mathbb{C}^2)^{\otimes (n+1)}})$

$\cong (\mathbb{C}^2)^{\otimes(n+1)}$ H_{n+1}-module $T_0, \dots, T_{n-1}$

$$L_0(u) = R_{0n}(uY_n^{-1}) \cdots R_{01}(uY_1^{-1})$$
$$= P_{(01\dots n)} \check{R}_{n-1,n}(uY_n^{-1}) \cdots \check{R}_{01}(uY_1^{-1})$$

Faddeev-Reshetikhin-Takhtajan presentation — Bernard et al '93

where $\check{R}_{i-1,i}(uY_i^{-1}) = t^{1/2} \dfrac{u - Y_i}{tu - Y_i} \otimes T_{i-1} + \dfrac{(t-1)Y_i}{tu - Y_i} \otimes 1$

via power series expansion

Algebraic background V: the center

$$H_n = H_n(t^{1/2}) = \left\langle \begin{matrix}\text{unital assoc alg} \\ T_1, \ldots, T_{n-1}\end{matrix} \;\middle|\; \begin{matrix}\text{braid relations} \\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{matrix} \right\rangle$$

$$\widehat{H}_n = \widehat{H}_n(t^{1/2}) = H_n \otimes \underbrace{\mathbb{C}[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]}_{\text{maximal abelian subalgebra}} \text{ with cross relations } \begin{matrix} T_i Y_i T_i = Y_{i+1}, & 1 \le i < n \\ T_i Y_j = Y_j T_i, & j \ne i, i+1 \end{matrix}$$

quantum-affine Schur functor

$$\left\{\begin{matrix}\text{fin-dim} \\ \widehat{H}_n\text{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{fin-dim} \\ \widehat{\mathcal{U}}_r\text{-modules}\end{matrix}\right\}$$

$$M \longmapsto M \underset{H_n}{\otimes} W := M \otimes W \Big/ \sum_{i=1}^{n-1} \operatorname{im}(T_i \otimes 1 - 1 \otimes T_i)$$

with explicit action of $\widehat{\mathcal{U}}_r := U_{t^{1/2}}(L\mathfrak{gl}_r)$ qdet commute!

plus action of center $Z_n := Z(\widehat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$

Algebraic background V: the center

$H_n = H_n(t^{1/2}) = \left\langle \begin{smallmatrix}\text{unital assoc alg}\\ T_1,\dots,T_{n-1}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{braid relations}\\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\end{smallmatrix} \right\rangle$

$\widehat{H}_n = \widehat{H}_n(t^{1/2}) = H_n \otimes \underbrace{\mathbb{C}[Y_1, Y_1^{-1}, \dots, Y_n, Y_n^{-1}]}_{\text{maximal abelian subalgebra}}$ with cross relations $T_i Y_i T_i = Y_{i+1}$, $1 \le i < n$; $T_i Y_j = Y_j T_i$, $j \ne i, i+1$

quantum-affine Schur functor $\quad \left\{\begin{smallmatrix}\text{fin-dim}\\ \widehat{H}_n\text{-modules}\end{smallmatrix}\right\} \longrightarrow \left\{\begin{smallmatrix}\text{fin-dim}\\ \widehat{\mathcal{U}}_r\text{-modules}\end{smallmatrix}\right\}$

$M \longmapsto M \underset{H_n}{\otimes} W := M \otimes W \Big/ \sum_{i=1}^{n-1} \mathrm{im}(T_i \otimes 1 - 1 \otimes T_i)$

with explicit action of $\widehat{\mathcal{U}}_r := U_{t^{1/2}}(L\mathfrak{gl}_r)$) qdet commute!

plus action of center $Z_n := Z(\widehat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$

important representations:

- on $M_{\underline{z}} := \mathrm{Ind}_{\mathbb{C}[\underline{Y}^{\pm}]}^{\widehat{H}_n}(\mathbb{C} v_{\underline{z}})$, $\underline{z} \in (\mathbb{C}^\times)^n$, $Y_i v_{\underline{z}} = z_i v_{\underline{z}}$ $\longmapsto$ $M_{\underline{z}} \underset{H_n}{\otimes} W \cong \mathbb{C}^r[z_1, z_1^{-1}] \otimes \cdots \otimes \mathbb{C}^r[z_n, z_n^{-1}]$ tensor prod of evaluation modules Chari Pressley '96

  $L_0(u) = R_{0n}(u/z_n) \cdots R_{01}(u/z_1)$ interesting: $\mathrm{tr}_0 L_0(u)$ transfer matrix of inhomog XXZ
  r=2: six-vertex model

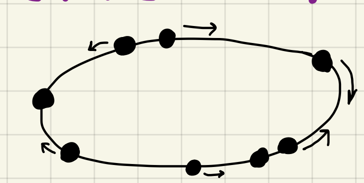
  $Z_n = \mathbb{C}$ scalars boring

- on $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$ $\xmapsto{r=1}$ $\mathbb{C}[\underline{x}] \underset{H_n}{\otimes} \mathbb{C} \cong \mathbb{C}[\underline{x}]^{S_n}$ Cherednik '92

  via $T_i \mapsto t^{1/2} - t^{-1/2}(t x_i - x_j)(x_i - x_j)^{-1}(1 - s_i)$

  $q \in \mathbb{C}^\times$ $\quad Y_j \mapsto T_{j-1} \cdots T_1\, \rho\, T_{n-1}^{-1} \cdots T_j^{-1}$ cherednik

  $(\rho f)(x_1, \dots, x_n) = f(x_2, \dots, x_n, q^{-1} x_1)$



  $L_0(u) = 1$ boring

  $Z_n = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$ interesting Macdonald operators

  $e_k(Y_1^{-1}, \dots, Y_n^{-1}) = D_n^k$ on $\mathbb{C}[\underline{x}]^{S_n}$ e.g. $D_n^1 = \sum_{j=1}^{n} \underset{\text{rational function}}{A_j(\underline{x}; t)}\, \underset{x_j \mapsto q x_j}{T_{q, x_j}}$

  reducible with fin-dim $\widehat{H}_n$-irreps

  $\mathrm{span}_{\mathbb{C}}\{E_\mu(\underline{x}; q, t),\ \mu \in S_n \cdot \lambda\}$ $(\lambda \vdash n)$ $\xmapsto[\sum_{w \in S_n} t^{\ell w/2} T_w]{}$ $\mathbb{C}\, P_\lambda(\underline{x}; q, t)$ symmetric Macdonald polyn

  $P_\lambda(\underline{x}; t^\alpha, t) \xrightarrow{t \to 1} P_\lambda^{(\alpha)}(\underline{x})$ Jack

Spin-Macdonald-Ruijsenaars

for $r \geq 2$ the 'physical space'

$$\widetilde{W} := \underbrace{\mathbb{C}[\underline{x}]}_{\widehat{H}_n\text{-module}} \otimes_{H_n} W \cong \left(\mathbb{C}^r[x_1] \otimes \cdots \otimes \mathbb{C}^r[x_n]\right)^{S_n}$$

t-deformed bosons

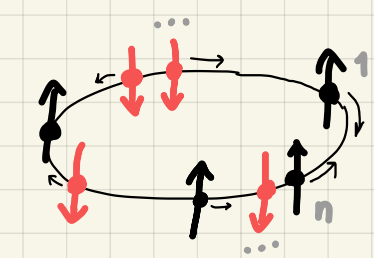
bosons

invariants under $s_i \mapsto (s_i \otimes 1)\check{R}_{i,i+1}(x_i/x_{i+1}) \xrightarrow{t \to 1} s_i \otimes P_{i,i+1}$

$\widehat{H}_n$-module $\quad (q \in \mathbb{C}^\times)$

via $Y_j \mapsto T_{j-1} \cdots T_1 \, \rho \, T_{n-1}^{-1} \cdots T_j^{-1}$

$(\rho f)(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, q^{-1} x_1)$



comes with

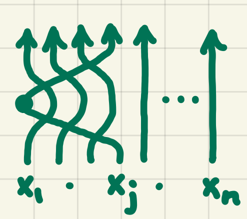
- (nonabelian symmetries) action of quantum-loop algebra $\widehat{\mathcal{U}}_r = U_{t^{1/2}}(L\mathfrak{gl}_r)$ — Bernard et al '93

$$\widetilde{L}_0(u) = R_{0n}(u Y_n^{-1}) \cdots R_{01}(u Y_1^{-1})$$
$$= P_{(01\cdots n)} \check{R}_{n-1,n}(u Y_n^{-1}) \cdots \check{R}_{01}(u Y_1^{-1})$$

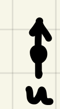
qdet

- (abelian symmetries) family of commuting spin-Macdonald operators — Cherednik '94, JL et al '20

$$Z_n = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$$
$$\psi$$
$$e_k(Y_1^{-1}, \ldots, Y_n^{-1}) = \widetilde{D}_n^k \text{ on } \widetilde{W}$$

e.g. $\widetilde{D}_n^1 = \sum_{j=1}^{n} A_j(\underline{x};t) \times$ 

$= \check{R}(u/v)$



$= T_{q,u} : u \mapsto qu$

$$= A_1(\underline{x};t)\, T_{q,x_1} + A_2(\underline{x};t)\, \check{R}_{12}(x_2/x_1)\, T_{q,x_1}\, \check{R}_{12}(x_1/x_2)$$
$$+ A_3(\underline{x};t)\, \check{R}_{23}(x_3/x_2)\, \check{R}_{12}(x_3/x_1)\, T_{q,x_3}\, \check{R}_{12}(x_1/x_3)\, \check{R}_{23}(x_2/x_3) + \ldots$$

for $q = t^\alpha$, $t \to 1$ yields spin-Calogero-Sutherland

$$\widetilde{H}^{nr} = \frac{1}{2} \sum_{j=1}^{n} (x_j \partial_{x_j})^2 + \sum_{i<j}^{n} \frac{-x_i x_j}{(x_i - x_j)^2} \frac{1}{\alpha}\left(\frac{1}{\alpha} - P_{ij}\right)$$

Spin-Macdonald-Ruijsenaars

for $r \geq 2$ the 'physical space'

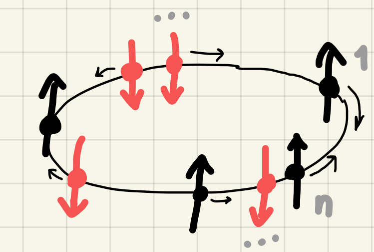
$$\widetilde{W} := \underbrace{\mathbb{C}[\underline{x}]}_{} \underset{H_n}{\otimes} W \cong \left(\mathbb{C}^r[x_1]\otimes\cdots\otimes\mathbb{C}^r[x_n]\right)^{S_n}$$ t-deformed bosons $\qquad$ bosons

$\widehat{H}_n$-module $\qquad (q \in \mathbb{C}^\times)$

via $Y_j \mapsto T_{j-1}\cdots T_1\,\rho\,T_{n-1}^{-1}\cdots T_j^{-1}$

$(\rho f)(x_1,\dots,x_n) = f(x_2,\dots,x_n,q^{-1}x_1)$

invariants under $s_i \mapsto (s_i\otimes 1)\check{R}_{i,i+1}(x_i/x_{i+1}) \xrightarrow{t\to 1} s_i\otimes P_{i,i+1}$



comes with

- (nonabelian symmetries) action of quantum-loop algebra $\widehat{U}_r = U_{t^{1/2}}(L\mathfrak{gl}_r)$ $\qquad$ Bernard et al '93

$$\widetilde{L}_0(u) = R_{0n}(uY_n^{-1})\cdots R_{01}(uY_1^{-1})$$
$$= P_{(01\dots n)}\check{R}_{n-1,n}(uY_n^{-1})\cdots\check{R}_{01}(uY_1^{-1})$$

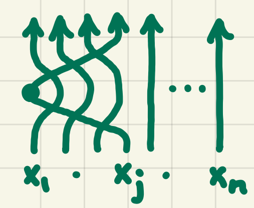
qdet

- (abelian symmetries) family of commuting spin-Macdonald operators $\qquad$ Cherednik '94, JL et al '20

$Z_n = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$

ψ

$e_k(Y_1^{-1},\dots,Y_n^{-1}) = \widetilde{D}_n^k$ on $\widetilde{W}$

e.g. $\widetilde{D}_n^1 = \sum_{j=1}^n A_j(\underline{x};t)\times$ 

$\check{R}$-crossing $= \check{R}(u/v)$, $\qquad$ arrow $= T_{q,u} : u \mapsto qu$

$$= A_1(\underline{x};t)\,T_{q,x_1} + A_2(\underline{x};t)\,\check{R}_{12}(x_2/x_1)\,T_{q,x_1}\,\check{R}_{12}(x_1/x_2)$$
$$+ A_3(\underline{x};t)\,\check{R}_{23}(x_3/x_2)\check{R}_{12}(x_3/x_1)\,T_{q,x_3}\,\check{R}_{12}(x_1/x_3)\check{R}_{23}(x_2/x_3) + \dots$$

for $q = t^\alpha$, $t\to 1$ yields spin-Calogero-Sutherland

$$\widetilde{H}^{nr} = \frac{1}{2}\sum_{j=1}^n (x_j\partial_{x_j})^2 + \sum_{i<j}\frac{-x_i x_j}{(x_i-x_j)^2}\,\frac{1}{\alpha}\left(\frac{1}{\alpha} - P_{ij}\right)$$

- exact $\widehat{U}_r$-(pseudo) highest-weight eigenvectors $\qquad$ cf Takemura Uglov '97

obtained from partial symmetrisations of nonsymmetric Macdonald polyn

ex $r=2$ $\quad W = (\mathbb{C}^2)^{\otimes n} = \bigoplus_{m=0}^n W_m$ $\quad \mathfrak{sl}_2$-weight spaces $W_m : \#\downarrow = m$

$\mathbb{C}^2 = \mathbb{C}v_\uparrow \oplus \mathbb{C}v_\downarrow$

likewise $\quad \widetilde{W} = \bigoplus_{m=0}^n \widetilde{W}_m \qquad \widetilde{W}_m = \mathbb{C}[\underline{x}]\underset{H_n}{\otimes} W_m \cong \mathbb{C}[\underline{x}]^{S_m\times S_{n-m}}$

$$\psi \longmapsto \langle \underbrace{v_\downarrow\otimes\cdots\otimes v_\downarrow}_{\#=m}\otimes\underbrace{v_\uparrow\otimes\cdots\otimes v_\uparrow}_{\#=n-m},\psi\rangle$$

$$\sum_{w\in S_n/S_m\times S_{n-m}} t^{\ell(w)/2}\,T_w f \otimes \begin{pmatrix}\text{basis vec}\\ \text{with}\downarrow\text{at}\\ w1,\dots,wm\end{pmatrix} \longleftarrow f$$

Extracting a spin chain via freezing

Talstra Haldane '95
Uglov '95
JL et al '20

$$\widetilde{W} := \underbrace{\mathbb{C}[\underline{x}]\otimes_{H_n} W}_{\widehat{H}_n\text{-module}} \cong \left(\mathbb{C}^r[x_1]\otimes\cdots\otimes\mathbb{C}^r[x_n]\right)^{S_n}$$

t-deformed bosons $\qquad$ bosons

$\widehat{H}_n$-module $\quad (q\in\mathbb{C}^\times)$

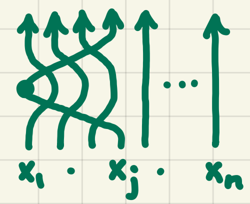
via $Y_j \mapsto T_{j-1}\cdots T_1\,\rho\,T_{n-1}^{-1}\cdots T_j^{-1}$

invariants under

$s_i \mapsto (s_i\otimes 1)\check{R}_{i,i+1}(x_i/x_{i+1}) \xrightarrow{t\to 1} s_i\otimes P_{i,i+1}$

$$\widetilde{L}_0(u) = R_{0n}(uY_n^{-1})\cdots R_{01}(uY_1^{-1})$$
$$= P_{(01\dots n)}\check{R}_{n-1,n}(uY_n^{-1})\cdots\check{R}_{01}(uY_1^{-1})$$

$$Z_n = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$$
$$\Psi$$
$$e_k(Y_1^{-1},\dots,Y_n^{-1}) = \widetilde{D}_n^k \text{ on } \widetilde{W}$$

e.g. $\widetilde{D}_n^1 = \sum_{j=1}^n A_j(\underline{x};t)\times$  $= A_1(\underline{x};t)\,T_{q,x_1} + A_2(\underline{x};t)\,\check{R}_{12}(x_2/x_1)\,T_{q,x_1}\,\check{R}_{12}(x_1/x_2)$
$+ A_3(\underline{x};t)\,\check{R}_{23}(x_3/x_2)\check{R}_{12}(x_3/x_1)\,T_{q,x_3}\,\check{R}_{12}(x_1/x_3)\check{R}_{23}(x_2/x_3)$
$+\dots$

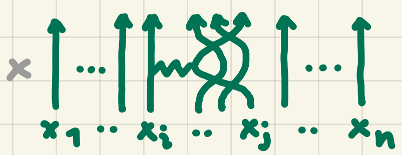
$= T_{q,u} : u\mapsto qu$

$= \check{R}(u/v)$

$= -(t^{1/2}-t^{-1/2})\,\check{R}'(1)$
$= t^{1/2} - T$
t-antisymm

i) semiclassical limit: from difference to differential

$\mathcal{O}(q^0)$: $\quad \widetilde{D}_n^k|_{q=1} = D_n^k|_{q=1} = \begin{bmatrix} n\\ k\end{bmatrix}_{t^{1/2}}$ constant

$\mathcal{O}(q^1)$: $\quad \delta := \frac{d}{dq}\Big|_{q=1} \qquad \delta\widetilde{D}_n^1 = \sum_{j=1}^n A_j(\underline{x};t)\,x_j\partial_{x_j} + (t^{1/2}-t^{-1/2})\sum_{j=1}^n A_j(\underline{x};t)\sum_{i=1}^n \frac{-t\,x_i x_j}{(t x_i - x_j)(x_i - t x_j)}\times$ 

$T_{q,x} = 1 + (q-1)\,x\partial_x + \mathcal{O}(q-1)^2$

nonabelian symmetries $\qquad [\widetilde{L}_0(u)|_{q=1}, \delta\widetilde{D}_n^k] = 0, \qquad \widetilde{L}_0(u)|_{q=1} = R_{0n}(uY_n^{-1}|_{q=1})\cdots R_{01}(uY_1^{-1}|_{q=1})$ nontrivial

abelian symmetries $\qquad [\delta\widetilde{D}_n^k, \delta\widetilde{D}_n^\ell] = 0$

Extracting a spin chain via freezing

Talstra Haldane '95
Uglov '95
JL et al '20

$\widetilde{W} := \underbrace{\mathbb{C}[\underline{x}]}_{\widehat{H}_n\text{-module}} \otimes_{H_n} W \cong \left(\mathbb{C}^r[x_1]\otimes\cdots\otimes\mathbb{C}^r[x_n]\right)^{S_n}$ t-deformed bosons   bosons

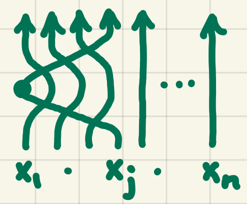
$(q\in\mathbb{C}^\times)$

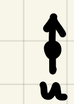
via $Y_j \mapsto T_{j-1}\cdots T_1\, \rho\, T_{n-1}^{-1}\cdots T_j^{-1}$

invariants under
$s_i \mapsto (s_i\otimes 1)\check{R}_{i,i+1}(x_i/x_{i+1}) \xrightarrow{t\to 1} s_i\otimes P_{i,i+1}$

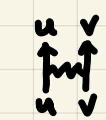
$\widetilde{L}_0(u) = R_{0n}(uY_n^{-1})\cdots R_{01}(uY_1^{-1})$
$\quad = P_{(01\cdots n)}\check{R}_{n-1,n}(uY_n^{-1})\cdots\check{R}_{01}(uY_1^{-1})$

$Z_n = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$
$\quad\Psi$
$e_k(Y_1^{-1},..,Y_n^{-1}) = \widetilde{D}_n^k$ on $\widetilde{W}$

e.g. $\widetilde{D}_n^1 = \sum_{j=1}^n A_j(\underline{x};t)\times$  $= A_1(\underline{x};t)T_{q,x_1} + A_2(\underline{x};t)\check{R}_{12}(x_2/x_1)T_{q,x_1}\check{R}_{12}(x_1/x_2)$
$\quad + A_3(\underline{x};t)\check{R}_{23}(x_3/x_2)\check{R}_{12}(x_3/x_1)T_{q,x_3}\check{R}_{12}(x_1/x_3)\check{R}_{23}(x_2/x_3)$
$\quad + \ldots$

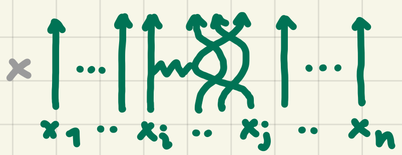
 $= T_{q,u}: u\mapsto qu$

 $= \check{R}(u/v)$

 $= -(t^{1/2}-t^{-1/2})\check{R}'(1)$
$\quad = t^{1/2} - T$
t-antisymm

i) semiclassical limit: from difference to differential

$\mathcal{O}(q^0)$: $\widetilde{D}_n^k|_{q=1} = D_n^k|_{q=1} = \begin{bmatrix} n \\ k\end{bmatrix}_{t^{1/2}}$ constant

$\mathcal{O}(q^1)$: $\delta := \frac{d}{dq}\Big|_{q=1}$ $\quad \delta\widetilde{D}_n^1 = \sum_{j=1}^n A_j(\underline{x};t)\,x_j\partial_{x_j} + (t^{1/2}-t^{-1/2})\sum_{j=1}^n A_j(\underline{x};t)\sum_{i=1}^n \frac{-t\,x_i x_j}{(tx_i-x_j)(x_i-tx_j)}\times$ 

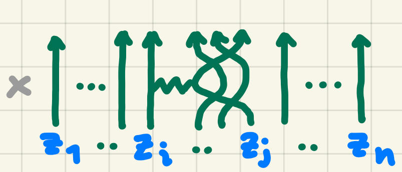
$T_{q,x} = 1 + (q-1)\,x\partial_x + \mathcal{O}(q-1)^2$

nonabelian symmetries $\quad [\widetilde{L}_0(u)|_{q=1}, \delta\widetilde{D}_n^k] = 0$, $\quad \widetilde{L}_0(u)|_{q=1} = R_{0n}(uY_n^{-1}|_{q=1})\cdots R_{01}(uY_1^{-1}|_{q=1})$ nontrivial
abelian symmetries $\quad [\delta\widetilde{D}_n^k, \delta\widetilde{D}_n^\ell] = 0$

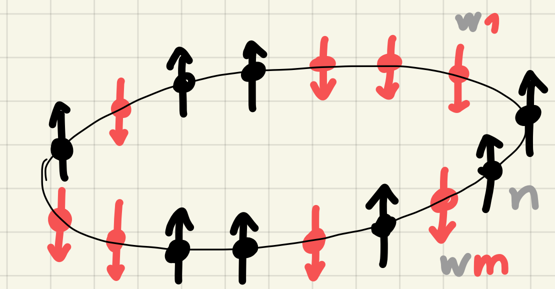
ii) remove derivatives

$\widetilde{D}_n^n = D_n^n = T_{q,x_1}\cdots T_{q,x_n}$ has $\delta\widetilde{D}_n^n = \sum_{j=1}^n x_j\partial_{x_j}$

Q: $?\exists\, \underline{z}\in(\mathbb{C}^\times)^n$ s.t. $A_j(\underline{z};t) = cst_n$ independent of j
A: yes, $\underline{z} = (\omega,\omega^2,\ldots,\omega^n)$ $\quad \omega := e^{2\pi i/n}$

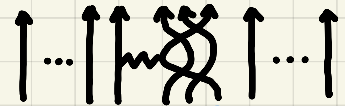
so $H := \frac{1}{t^{1/2}-t^{-1/2}}\left(\delta\widetilde{D}_n^1 - cst_n\times\delta\widetilde{D}_n^n\right)_{\underline{x}=\underline{z}} = cst_n\times\sum_{i<j}^n V(z_i,z_j)\times$  $\quad z_j = \omega^j$

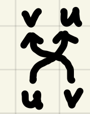
Main result: t-deformed Haldane-Shastry spin chain



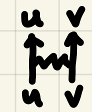
Uglov '95
JL '18

the operator

$$H_n^1 = cst_n \times \sum_{i<j}^{n} \frac{-t z_i z_j}{(t z_i - z_j)(z_i - t z_j)} \times$$ 



$= \check{R}(u/v)$ $\qquad \hat{\mathcal{U}}_r$ R-matrix



$= -(t^{1/2} - t^{-1/2})\, \check{R}'(1)$ $\qquad$ t-antisymmetriser

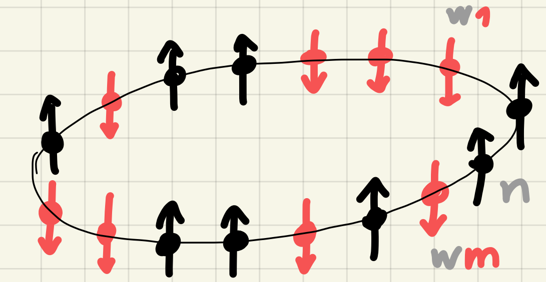
$= t^{1/2} - T$

on $(\mathbb{C}^r)^{\otimes n} \cong \left(\mathbb{C}[\underline{x}] / \langle p_1, \dots, p_{n-1}, p_n - n \rangle\right) \underset{H_n}{\otimes} (\mathbb{C}^r)^{\otimes n}$ $\qquad z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

- (abelian symmetries) belongs to family of commuting operators, which $\qquad$ Bernard et al '93, JL et al '20
  obtained from $Z_n := Z(\hat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$ via freezing $\qquad H_n^k = \frac{1}{t^{1/2} - t^{-1/2}} \left(\delta \tilde{D}_n^k - cst_n^k \times \delta \tilde{D}_n^n\right)_{\underline{x} = \underline{z}}$

- (nonabelian symmetries) commutes with action of quantum-loop algebra $\hat{\mathcal{U}}_r := U_{t^{1/2}}(L\mathfrak{gl}_r)$
  $L_0(u) := R_{0n}(u Y_n^{-1}|_{q=1}) \cdots R_{01}(u Y_1^{-1}|_{q=1}) \bmod \langle p_1, \dots, p_{n-1}, p_n - n \rangle$ $\qquad$ Bernard et al '93

Main result: t-deformed Haldane-Shastry spin chain



Uglov '95
JL '18

the operator

$$H_n^1 = cst_n \times \sum_{i<j}^{n} \frac{-t\, z_i z_j}{(t z_i - z_j)(z_i - t z_j)} \times \Big| \cdots \Big| \cdots \Big| \cdots \Big|_{z_1 \,\cdots\, z_i \,\cdots\, z_j \,\cdots\, z_n}$$

$= \check{R}(u/v)$ $\quad$ $\hat{\mathfrak{U}}_r$ R-matrix

$= -(t^{1/2} - t^{-1/2})\,\check{R}'(1)$ $\quad$ t-antisymmetriser

$= t^{1/2} - T$

on $(\mathbb{C}^r)^{\otimes n} \cong \left(\mathbb{C}[\underline{x}]/\langle p_1,\dots,p_{n-1},p_n - n\rangle\right) \otimes_{H_n} (\mathbb{C}^r)^{\otimes n}$ $\quad$ $z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

- (abelian symmetries) belongs to family of commuting operators, which $\quad$ Bernard et al '93, JL et al '20

  obtained from $\mathcal{Z}_n := Z(\hat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$ via freezing $\quad H_n^k = \frac{1}{t^{1/2} - t^{-1/2}}\left(\delta \tilde{D}_n^k - cst_n^k \times \delta \tilde{D}_n^n\right)_{\underline{x} = \underline{z}}$

- (nonabelian symmetries) commutes with action of quantum-loop algebra $\hat{\mathfrak{U}}_r := U_{t^{1/2}}(L\mathfrak{gl}_r)$

  $L_0(u) := R_{0n}(u Y_n^{-1}|_{q=1}) \cdots R_{01}(u Y_1^{-1}|_{q=1})$ mod $\langle p_1,\dots,p_{n-1},p_n - n\rangle$ $\quad$ Bernard et al '93

- for $r=2$ has exact $\hat{\mathfrak{U}}_2$-(pseudo) highest-weight eigenvectors $\Psi_\lambda = \sum_{w \in S_n/S_m \times S_{n-m}} t^{\ell w/2}\, T_w\, f_\lambda \otimes \begin{pmatrix}\text{basis vec}\\ \text{with} \downarrow \text{at}\\ w_1,\dots,w_m\end{pmatrix}$ determined by

  $\mathbb{C}^2 = \mathbb{C}v_\uparrow \oplus \mathbb{C}v_\downarrow$

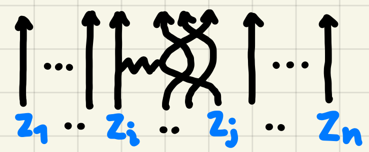
  $$f_\lambda := \langle \underbrace{v_\downarrow \otimes \cdots \otimes v_\downarrow}_{\# = m} \otimes \underbrace{v_\uparrow \otimes \cdots \otimes v_\uparrow}_{\# = n-m}, \Psi_\lambda\rangle = \underbrace{\prod_{i<j}^{m} (t x_i - x_j)(t^{-1} x_i - x_j)}_{\text{'symmetric square' of t-Vandermonde}} \times \underbrace{P_\lambda(x_1,\dots,x_m; t, t^2)}_{\text{Macdonald (quantum spherical zonal)}} = m_\mu + \text{lower terms (i.e. } \mu = \lambda + 2\delta_m)$$ $\quad$ JL et al '20

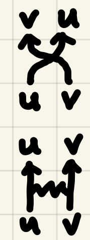
  $\ell(\lambda) = m$, $\lambda_1 \leq n - 2m + 1$

  $$\in \mathbb{C}[x_1,\dots,x_m]^{S_m}/\langle\dots\rangle \cong \mathbb{C}[x_1,\dots,x_n]^{S_m \times S_{n-m}}/\langle p_1,\dots,p_{n-1},p_n - n\rangle \cong \left(\mathbb{C}[\underline{x}]/\langle p_1,\dots,p_{n-1},p_n - n\rangle\right) \otimes_{H_n} W_m \cong W_m : \#\downarrow = m$$

  Drinfeld polynomial $\prod_{i(\notin \mu, \mu+1)}^{n} (1 - t^{(n-2i+1)/2} u)$, $\frac{H_n^1 + H_n^{n-1}}{2}$-eigenvalue $\sum_{i \in \mu} \frac{1}{2} [i]_{t^{1/2}} [n-i]_{t^{1/2}} \longrightarrow \sum_{i \in \mu} \frac{1}{2} i(n-i)$

Summary: t-deformed Haldane-Shastry spin chain

$H^1_n = cst_n \times \sum_{i<j}^{n} \frac{-t z_i z_j}{(t z_i - z_j)(z_i - t z_j)} \times$ 

 $= \check{R}(u/v)$ $\qquad \hat{\mathfrak{U}}_r$ R-matrix

$= -(t^{1/2}-t^{-1/2})\,\check{R}'(1)$ $\qquad$ t-antisymmetriser

$z_j := \omega^j, \quad \omega := e^{2\pi i/n}$

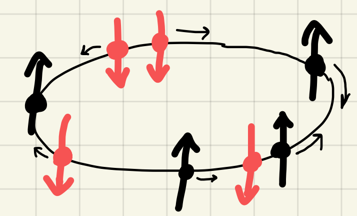
Uglov '95
JL '18

from quantum-affine Schur-Weyl duality and freezing

Drinfeld '86
Bernard et al '93
Talstra Haldane '95
Uglov '95
JL et al '20

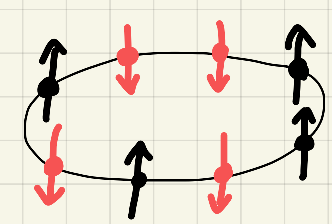
affine Hecke alg $\hat{H}_n$ $\xrightarrow[\text{symm } (\mathbb{C}^r)^{\otimes n}\text{-valued polynomials}]{\tilde{W} = \mathbb{C}[\underline{x}] \otimes_{H_n} (\mathbb{C}^r)^{\otimes n} \cong (\mathbb{C}^r[x_1] \otimes \cdots \otimes \mathbb{C}^r[x_n])^{S_n}}$ spin-Macdonald-Ruijsenaars

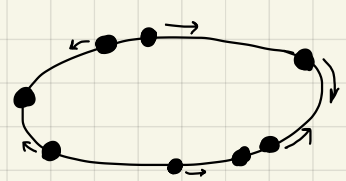
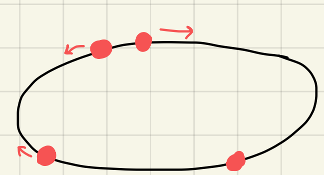
- abelian symmetries $\qquad \tilde{D}^k_n \in Z_n = Z(\hat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$
- nonabelian symmetries $\qquad \tilde{L}_0(u) = R_{0n}(u Y_n^{-1}) \cdots R_{01}(u Y_1^{-1})$
- exact eigenvectors $\qquad \hat{\mathfrak{U}}_r = \mathfrak{U}_{t^{1/2}}(L\mathfrak{gl}_r)$



$\downarrow$ freezing $\quad$ semiclass lim to class equilib $\quad$ expansion in $q \approx 1$: $x_j = z_j$, i.e. mod $\langle p_1, \dots, p_{n-1}, p_n - n \rangle$

$\downarrow$ symmetric polynomials $\quad \mathbb{C}[\underline{x}] \otimes_{H_n} V^{\text{triv}} = \mathbb{C}[\underline{x}]^{S_n}$



 $\Longleftrightarrow$ 

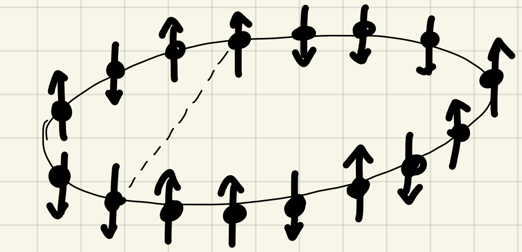
Macdonald-Ruijsenaars

- abelian symmetries $\qquad D^k_n \in Z_n = Z(\hat{H}_n) = \mathbb{C}[\underline{Y}^{\pm}]^{S_n}$
- exact eigenfunctions:
  Macdonald polynomials

t-deformed Haldane-Shastry

- abelian symmetries $\qquad H^k_n = (t^{1/2}-t^{-1/2})^{-1}(\delta\tilde{D}^k_n - cst^k_n \times \delta\tilde{D}^n_n)_{\underline{x}=\underline{z}}$
- nonabelian symmetries $\qquad L_0(u) = \tilde{L}_0(u)|_{q=1}$ mod $\langle p_1, \dots, p_{n-1}, p_n - n \rangle$
- exact eigenvectors (r=2)

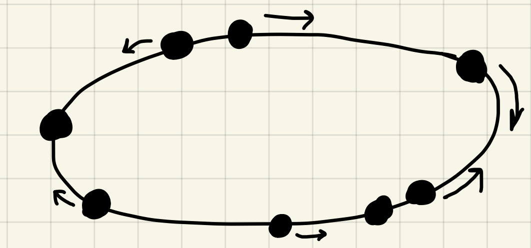
Outlook landscape of long-range spin chains (r=2)



interaction range →

↓ degree of spin symmetry	**nearest neighbour** contact potential	**intermediate range** elliptic potential	**long range** trig potential
anisotropic elliptic quantum alg	**Heisenberg XYZ** Sutherland '70, Baxter '73	?	?
partially isotropic quantum-loop alg, affine Hecke alg, trig quantum alg	**Heisenberg XXZ** Orbach '58, Yang-Yang '66	?	**t-deformed HS** Uglov '95, JL '18, JL Pasquier Serban '20
isotropic Yangian, degenerate AHA, rational quant alg	**Heisenberg XXX** Heisenberg '28, Bethe '31	**Inozemtsev** Inozemtsev '90, Inozemtsev '90–'00, Klabbers JL '20	**Haldane-Shastry** Haldane '88, Shastry '88, Haldane '91, Bernard et al '93

Outlook landscape of quantum many-body systems



interaction range

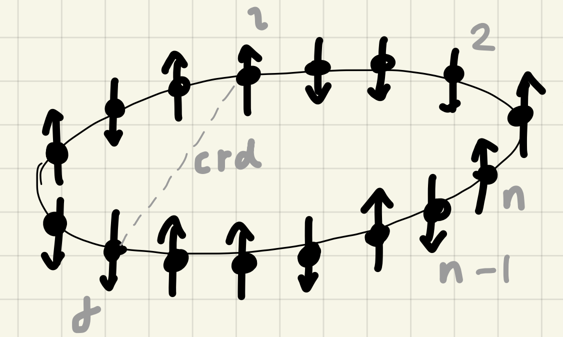
	nearest neighbour contact (variables)	intermediate range elliptic (variables)	long range trig (variables)
(?) elliptic (derivatives)	?	'DELL'	?
relativistic trig (derivatives)	?	ell Ruijsenaars	trig Ruijsenaars-Macdonald
non-rlt rational (derivatives)	~Lieb-Liniger?	ell Cal-Sut	trig Cal-Sut

towards **grand unified theory** for
quantum-integrable long-range spin chains?

Bonus: isotropic level (r=2)

$$H=\sum_{i<j}^{n} V(i,j)\,(1-P_{(i,j)}) \quad \text{on } (\mathbb{C}^2)^{\otimes n}$$

isotropic — $\mathfrak{sl}_2$-invariant
homogeneous — $\mathbb{Z}/n\mathbb{Z}$-invariant



	Heisenberg XXX '28	Inozemtsev '90	Haldane '88 – Shastry '88
	$V_{Heis}(i,j)=\delta_{d(i,j),1}$ $\xleftarrow[\kappa\to\infty]{}$	$V_{Ino}(i,j)\sim\wp(i-j)$ $(n,\frac{i\pi}{\kappa})\in\mathbb{Z}_{\geq 2}\times i\mathbb{R}_{>0}$	$\xrightarrow[\kappa\to 0]{}$ $V_{HS}(i,j)=\dfrac{1}{4\sin^2(\frac{\pi}{n}(i-j))}=\dfrac{1}{\mathrm{crd}^2}$
exact solvability (spectrum)	up to solving 'Bethe equations' ⟵ Bethe '31	up to solving 'Bethe equations' via connection to ell Calogero–Sutherland Inozemtsev '90, '95, '00 Klabbers JL '20	⟶ in closed form via connection to trig Calogero–Sutherland ($\alpha^*=\frac{1}{2}$ zonal spherical) Haldane '91
	↑ Borel subalg		↑ nonsymm theory + freezing
quantum integrab	Yangian structure Faddeev et al, late '70s–'80s	unknown	degenerate affine Hecke alg Yangian symmetry Bernard et al '93
	↓ abelian subalgebra		↓ centre + freezing
higher Ham's	known	conjectured partial proof Dittrich Inozemtsev '08	known Bernard et al '93 Talstra Haldane '95