Introduction to Compression

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Main point

Non-symmetric Macdonald polynomial (DAHA).

Main Result

Thm 1.1
$$\begin{cases} ① \ E_\mu^z = \sum_{p \in AW_\mu^z} wt(p) \quad (\text{Step by step recursion}) \\ ② \ E_\mu^z = \sum_{T \in NAF_\mu^z} wt(T) \quad (\text{box by box recursion}) \end{cases}$$

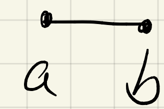
Example box by box recursion.

Arun Ram

Double affine Heck algebra (DAHA)

$ab = ba$ $\quad$ a b

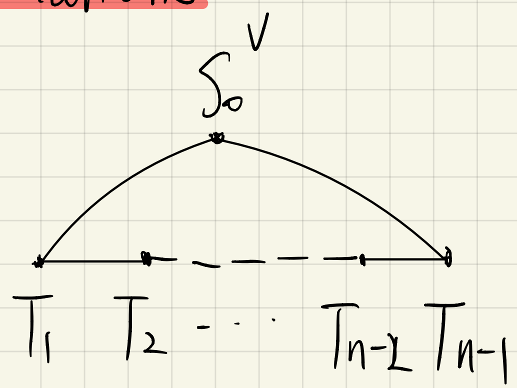
$aba = bab$

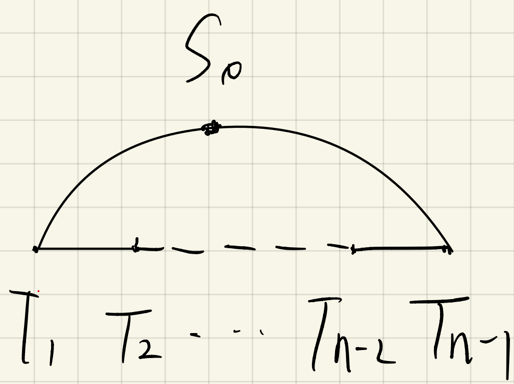


Generators

$$q,\ g,\ g^\vee,\ S_0,\ S_0^\vee,\ T_1, \dots, T_{n-1}$$

Relations





$$(T_i + t^{-\frac{1}{2}})(T_i - t^{\frac{1}{2}}) = 0.$$

\+ More relations.

Operators (elements in DAHA).

$$Y_1 = g^\vee T_{n-1} \cdots T_1$$

$$Y_{i+1} = T_i^{-1} Y_i T_i^{-1}$$

for $i \in \{1, \dots, n-1\}$.

$Y_i^{-1} Y_{i+1}$

$$\tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}} = T_i^{-1} + \frac{t^{\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}}$$

$$\tau_\pi^\vee = g^\vee.$$

AHA and polynomial representation

$$AHA = \langle Y_1, \dots, Y_n, T_1, \dots, T_{n-1} \rangle \subseteq DAHA.$$

Polynomial representation.

$$\mu = (\mu_1, \mu_2, \dots, \mu_n).$$

$$\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = \mathrm{Ind}_{AHA}^{DAHA} \mathbb{1} = \mathbb{C}\text{-span}\{X^\mu \mathbb{1} \mid \mu \in \mathbb{Z}^n\}.$$

Non-symmetric Macdonald polynomial.

$$E_{(0,\dots,0)} = \mathbb{1}$$

simultenous eigenvectors of $Y_1, \dots, Y_n$ on polynomial representation.

$$E_\mu = t^{-\frac{1}{2}(\ell(v_\mu^{-1}))} \tau_{u_\mu}^\vee \mathbb{1}.$$

$$E_{\pi\mu} = t^{(v_\mu(n)-1)+\frac{1}{2}(n-1)} \tau_\pi^\vee \mathbb{1}.$$

$$E_{s_i\mu} = \left(T_i + \frac{(1-t)\, t^{-\frac{1}{2}}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \right) E_\mu.$$

$$\pi(\mu_1, \dots, \mu_n) = (\mu_n + 1, \mu_1, \dots, \mu_{n-1}).$$

Two elements v_μ and u_μ.

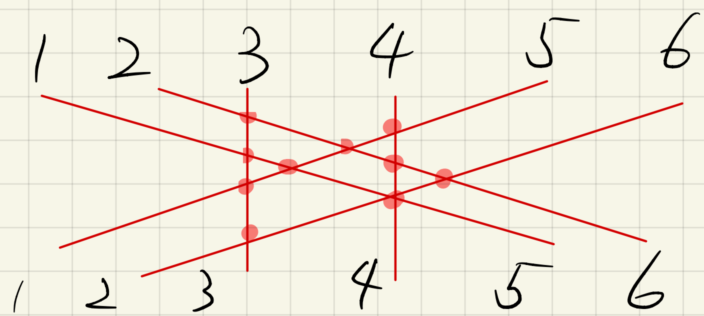
$$\boxed{E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})}\, \tau^\vee_{u_\mu} \mathbb{1}}$$

$$v_\mu \in S_n := \begin{cases} ① \text{ minimal length} \quad (\#\text{ inversion}) \\ ② \; v_\mu \cdot \mu \text{ is weakly increasing.} \end{cases}$$

$\ell(v_\mu) = \ell(v_\mu^{-1}) = 12$

$(1,1,0,0,2,2)$

Example For $\mu = (2,2,1,1,0,0)$.

$v_\mu = (563412) =$  $= s_2 s_1 s_3 s_2$

$s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2$

Two elements v_μ and u_μ.

$$\boxed{E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})}\, \tau_{u_\mu}^\vee \mathbb{1}.}$$

Affine Weyl group (Presentation 1)

$w(i+n) = w(i) + n.$

$t_\mu(i) = i + n\mu_i. \quad i \in \{1, \dots, n\}.$

$$W = \mathbb{Z}^n \rtimes S_n = \{ t_\mu \cdot v \mid \mu \in \mathbb{Z}^n,\ v \in S_n \}.$$

$t_\mu = t_{\mu_1 \varepsilon_1^\vee} \cdot t_{\mu_2 \varepsilon_2^\vee} \cdots t_{\mu_n \varepsilon_n^\vee}.$

$\mu = (\mu_1, \dots, \mu_n)$

Affine Weyl group (Presentation 2)

Generators

$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}$

$\pi, s_0, s_1, \dots, s_{n-1}$

Relations

$\pi s_i \pi^{-1} = s_{i+1}$ for $i \in \{0, 1, \dots, n-2\}$, $\quad \pi s_{n-1} \pi^{-1} = s_0.$

Two elements v_μ and u_μ.

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \check{T}_{u_\mu} \mathbb{1}.$$

Affine Weyl group (n-periodic permutations)

$$W = \langle \pi, s_1, s_2, \ldots, s_{n-1} \rangle$$

belongs to

The element

$$u_\mu = t_\mu v_\mu^{-1} = s_{i_1} \cdots s_{i_r}.$$

① minimal length

② $v_\mu \cdot \mu$ weakly increasing.

for $i_k \in \{\pi, 1, \ldots, n-1\}$ $s_\pi = \pi$.

Interesting choice (box greedy).

$$u_\mu^{\square} = \prod_{u \in dg(\mu)} \pi \, s_{u_\mu(i,j)} \cdots s_1$$

$$u_\mu(i,j) = n-1-\#\mathrm{attack}_\mu(i,j).$$

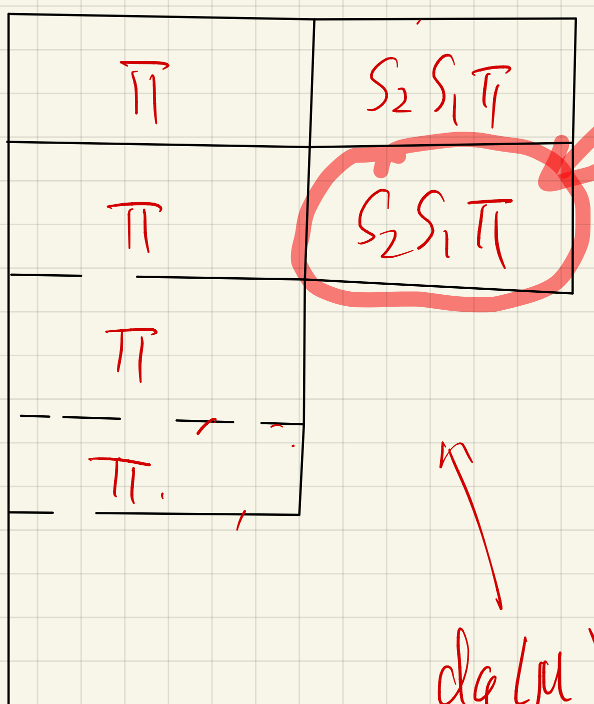
Box, diagram and attacking boxes.

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})}\, \tau_{u_\mu}^\vee \mathbb{1}.$$

For $\mu=(2,2,1,1,0,0)$,

$$(2,2,1,1,0,0) \xrightarrow{\pi^{-1}} (2,1,1,0,0,1) \xrightarrow{\pi^{-1}} (1,1,0,0,1,1) \xrightarrow{\pi^{-1}} (1,0,0,1,1,0)$$

$$\xrightarrow{\pi^{-1}} (\underline{0},0,1,1,0,0) \xrightarrow{\pi^{-1}s_1s_2} (0,0,1,0,0,0) \xrightarrow{\pi^{-1}s_1s_2} (0,0,0,0,0,0)$$



2^j

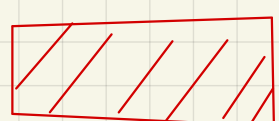
$j-1$

$n=6$

$dg(\mu)$

$$u_\mu^{\square} = \pi\pi\pi\pi(s_2 s_1 \pi)(s_2 s_1 \pi)$$

$$\tau_{u_\mu^{\square}}^\vee = (\tau_\pi^\vee \tau_\pi^\vee \tau_\pi^\vee \tau_\pi^\vee (\tau_2^\vee \tau_1^\vee \tau_\pi^\vee)(\tau_2^\vee \tau_1^\vee \tau_\pi^\vee)$$



attacking boxes.

Summary

Non-symmetric Macdonald polynomials

$Y_1, \ldots, Y_n$

simultenous eigenvectors

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbb{1}.$$

Permutated basement Macdonald polynomial.

$z \in S_n$ $\qquad E_\mu^z = t^{-\frac{1}{2}(\ell(z v_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu.$

So when $z = Id$, $\quad E_\mu^z = E_\mu.$

Main point

Non-symmetric Macdonald polynomial (DAHA)

Main result

$Th^{\underline{m}}$ 1.1

① $E_\mu^z = \sum_{p \in AW_\mu^z} wt(p)$ (step by step recursion).

② $E_\mu^z = \sum_{T \in NAF_\mu^z} wt(T)$ (box by box recursion)

Example of box by box recursion.

Main result (Th^{m} 1.1)

Relative Macdonald polynomial (Permuted basement).

$$E_\mu^z = t^{-\frac{1}{2}(\ell(z v_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu.$$

$u_\mu^{\square} = s_{i_1} \cdots s_{i_r}$

Th^{m} 1.1

(a) $E_\mu^z = \sum_{p \in AW_\mu^z} wt(p)$ $\qquad wt(p) = \prod_{k=1}^{r} wt_p(k).$

(b) $E_\mu^z = \sum_{T \in NAP_\mu^z} wt(T)$, $\qquad wt(T) = \prod_{b \in dg(\mu)} wt(b).$

Main result (Th$^{\underline{m}}$ 1.1)

Th$^{\underline{m}}$ 1.1

(a) $E_\mu^z = \sum_{p \in AW_\mu^z} wt(p)$ $\qquad wt(p) = \prod_{k=1}^{r} wt_p(k)$.

$AW_\mu^z = \{$ alcove walks of type $(z, u_\mu^{\square})$. $\}$.

$z \in S_n$.

Alcove walk $(z, u_\mu^{\square}) \longrightarrow s_{i_1} \cdots s_{i_r}$.

affine Weyl group.

$(p_0, p_1, \ldots, p_r)$ where $p_i \in W$

$p_0 = z$ and $p_k = p_{k-1} \cdot \pi$ if $s_{i_k} = \pi$

$p_k \in \{ p_{k-1}, p_{k-1} \cdot s_{i_k} \}$ if $s_{i_k} \neq \pi$.

Main result (Th^{m} 1.1)

Th^{m} 1.1 (Step by step recursion)

(a) $$E_{\mu}^{z} = \sum_{p \in AW_{\mu}^{z}} wt(p) \qquad wt(p) = \prod_{k=1}^{r} wt_p(k).$$

$$wt_p(k) = \begin{cases} \dfrac{1-t}{1 - q^{sh(-\beta_k^\vee)}\, t^{ht(-\beta_k^\vee)}}\, t^{norm(p_k)} & \text{if } \begin{array}{l} p_k = p_{k-1} \\ p_{k-1} s_{i_k} < p_{k-1} \end{array} \\[2ex] \dfrac{(1-t)\left(q^{sh(-\beta_k^\vee)}\, t^{ht(-\beta_k^\vee)}\right)}{1 - q^{sh(-\beta_k^\vee)}\, t^{ht(-\beta_k^\vee)}}\, t^{norm(p_k)} & \text{if } \begin{array}{l} p_k = p_{k-1} \\ p_{k-1} s_{i_k} > p_{k-1} \end{array} \\[2ex] 1 & \text{if } p_k = p_{k-1} s_{i_k} \\[1ex] X_{z_{k-1}}(1) & \text{if } p_k = p_{k-1} \pi \end{cases}$$

$$wt_p(k) = \begin{cases} \dfrac{1-t}{1-q^{sh(-\beta_k^\vee)}t^{ht(-\beta_k^\vee)}} t^{norm(p_k)} & \text{if } p_k = p_{k-1},\ p_{k-1}s_{i_k} < p_{k-1} \\ \dfrac{(1-t)(q^{sh(-\beta_k^\vee)}t^{ht(-\beta_k^\vee)})}{1-q^{sh(-\beta_k^\vee)}t^{ht(-\beta_k^\vee)}} t^{norm(p_k)} & \text{if } p_k = p_{k-1},\ p_{k-1}s_{i_k} > p_{k-1} \\ 1 & \text{if } p_k = p_{k-1}s_{i_k} \\ X_{z_{k-1}}(1) & \text{if } p_k = p_{k-1}\pi \end{cases}$$

Prop 4.1

a) If $\mu_1 \neq 0$, then

$$E_\mu^z = X_{z(1)} E^{zC_n}_{(\mu_2, \mu_3, \ldots, \mu_{n-1}, \mu_1 - 1)}$$

b) Let $i \in \{1, \ldots, n-1\}$ such that $\mu_i < \mu_{i+1}$

$$E_\mu^z = \begin{cases} \mathbb{1} E^{zs_i}_{s_i\mu} + \left(\dfrac{1-t}{1-q^{sh(-\beta^\vee)}t^{ht(-\beta^\vee)}}\right) t^{norm^z_\mu(i)} E^z_{s_i\mu} & \text{if } z(i) < z(i+1) \\ \mathbb{1} E^{zs_i}_{s_i\mu} + \left(\dfrac{(1-t)\, q^{sh(-\beta^\vee)}t^{ht(-\beta^\vee)}}{1-q^{sh(-\beta^\vee)}t^{ht(-\beta^\vee)}}\right) t^{norm^z_\mu(i)} E^z_{s_i\mu} & \text{if } z(i) > z(i+1) \end{cases}$$

Main result (Th^{m} 1.1)

Relative Macdonald polynomial (Permuted basement).

$$E_\mu^z = t^{-\frac{1}{2}(\ell(z v_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu .$$

Th^{m} 1.1

(a) $$E_\mu^z = \sum_{p \in AW_\mu^z} wt(p) \qquad wt(p) = \prod_{k=1}^{r} wt_p(k).$$

(b) $$E_\mu^z = \sum_{T \in NAF_\mu^z} wt(T) , \qquad wt(T) = \prod_{b \in dg(\mu)} wt(b)$$

Main result (Th^m 1.1)

Th^m 1.1 (box by box recursion)

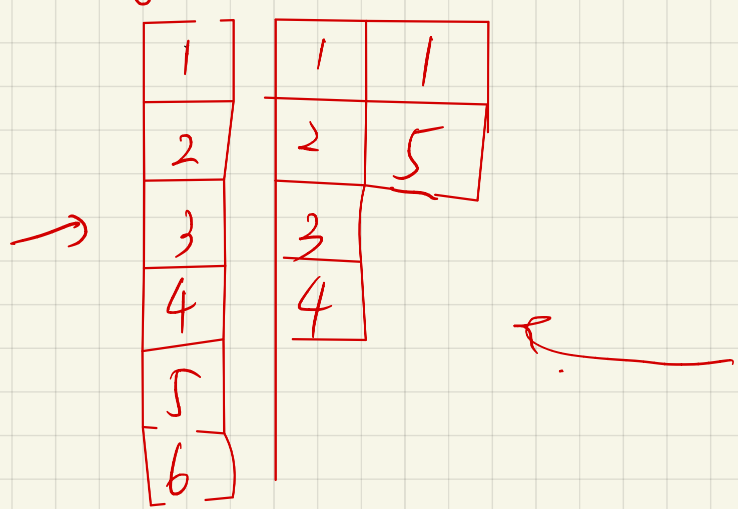
(b) $\quad E^z_\mu = \sum_{T \in NAF^z_\mu} wt(T)$, $\quad wt(T) = \prod_{b \in dg(\mu)} wt(b)$

Non-attacking filling $(z, u^\square_\mu)$

① $T(i,0) = z(i)$ (basement)

② if $b \in dg(\mu)$ and $a \in attack_\mu(b)$ then $T(a) \neq T(b)$

Example



$z = Id \in S_6$

$\mu = (2,2,1,1,0,0)$.

Main result (Th^{m} 1.1)

Th^{m} 1.1 (box by box recursion)

(b) $$E_\mu^z = \sum_{T \in NAF_\mu^z} wt(T), \qquad wt(T) = \prod_{b \in dg(\mu)} wt(b)$$

$$wt_p(b) = \begin{cases} \dfrac{1-t}{1-q^{\#Nleg_\mu(b)+1}\, t^{\#Narm_\mu(b)+1}}\, t^{\#bwn_T(b)}\, x_{T(b)} & \text{if } T(b\text{-}n) > T(b) \\[2ex] \dfrac{(1-t)\left(q^{\#Nleg_\mu(b)+1}\, t^{\#Narm_\mu(b)+1}\right)}{1-q^{\#Nleg_\mu(b)+1}\, t^{\#Narm_\mu(b)+1}}\, t^{-(-\#bwn_T(b))}\, x_{T(b)} & \text{if } T(b\text{-}n) < T(b) \\[2ex] x_{T(b)} & \text{if } T(b\text{-}n) = T(b). \end{cases}$$

$$
\mathrm{wt}_p(b) = \begin{cases} \dfrac{1-t}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}\, t^{\#\mathrm{Narm}_\mu(b)+1}}\, t^{\#\mathrm{bwn}_T(b)}\, x_{T(b)} & \text{if } T(b-n) > T(b) \\[2ex] \dfrac{(1-t)\left(q^{\#\mathrm{Nleg}_\mu(b)+1}\, t^{\#\mathrm{Narm}_\mu(b)+1}\right)}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}\, t^{\#\mathrm{Narm}_\mu(b)+1}}\, t^{-(1-\#\mathrm{bwn}_T(b))}\, x_{T(b)} & \text{if } T(b-n) < T(b) \\[2ex] x_{T(b)} & \text{if } T(b-n) = T(b). \end{cases}
$$

$\boxed{\text{Prop 4.3}}$ Let

$$\mu = (0, 0, \ldots, 0, \mu_j, \ldots, \mu_n)$$

$$\gamma = (0, 0, \ldots, 0, \mu_{j+1}, \ldots, \mu_n, \mu_j - 1)$$

removed a box.

Then

$$E^{z}_{\mu} =$$

$$x_{y(j)} E_{\gamma}^{z C_j^{-1} C_n} + \frac{1-t}{1-q^{\mu_j}\, t^{\nu_\mu(j)-(j-1)}} \sum_{a=1}^{j-1} q^{\mathrm{maj}^z_\mu(a)}\, t^{\mathrm{coinv}^z_\mu(a)}\, x_{z(a)} E_{\gamma}^{z C_a^{-1} C_n}.$$

$$
wt_p(b) = \begin{cases} \dfrac{1-t}{1-q^{\#Nleg_\mu(b)+1}t^{\#Narm_\mu(b)+1}}\, t^{\#bwn_T(b)}\, x_{T(b)} & \text{if } T(b-n) > T(b) \\[2ex] \dfrac{(1-t)\left(q^{\#Nleg_\mu(b)+1}t^{\#Narm_\mu(b)+1}\right)}{1-q^{\#Nleg_\mu(b)+1}t^{\#Narm_\mu(b)+1}}\, t^{-1-\#bwn_T(b)}\, x_{T(b)} & \text{if } T(b-n) < T(b) \\[2ex] x_{T(b)} & \text{if } T(b-n) = T(b). \end{cases}
$$

$$
maj_\mu^z(a) = \begin{cases} 0 & \text{if } z(j) > z(a) \\ \mu_j & \text{if } z(j) < z(a) \end{cases}
$$

$$
E_\mu^z =
$$

$$
x_{y(j)} E_\gamma^{z c_j^{-1} c_n} + \frac{1-t}{1-q^{\mu_j}t^{\nu_\mu(j)-(j-1)}} \sum_{a=1}^{j-1} q^{maj_\mu^z(a)} t^{covid_\mu^z(a)} x_{z(a)} E_\gamma^{z c_a^{-1} c_n}.
$$

Main point

Non-symmetric Macdonald polynomial (DAHA)

Main result

Th^{m} 1.1 $\begin{cases} ① \displaystyle\sum_{p \in AW^{z}_{\mu}} wt(p) & \text{(step by step recursion)} \\ ② \displaystyle\sum_{T \in NAF^{z}_{\mu}} wt(T) & \text{(box by box recursion)} \end{cases}$

Example of box by box recursion.

Example $(j=1)$

$\mu=(2,2,1,1,0,0)$ $\quad z=\mathrm{Id}=(1\,2\,3\,4\,5\,6)$

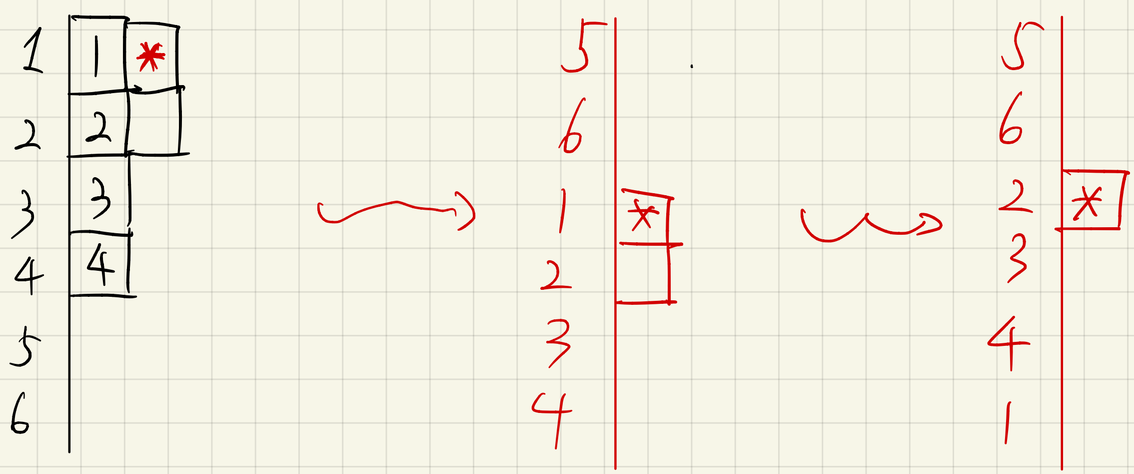
$$E_\mu^{z} = x_{z(j)} E_\gamma^{z\, c_j^{-1} c_n} + \frac{1-t}{1-q^{\mu_j} t^{\nu_\mu(j)-(j-1)}} \sum_{a=1}^{j-1} q^{\mathrm{maj}_\mu^z(a)} t^{\mathrm{coinv}_\mu^z(a)} x_{z(a)} E_\gamma^{z\, c_a^{-1} c_n}$$

$j=1$.

$$E_{(2,2,1,1,0,0)}^{(1\,2\,3\,4\,5\,6)} = x_1 E_{(2,1,1,0,0,1)}^{(2\,3\,4\,5\,6\,1)} = x_1 x_2 E_{(1,1,0,0,1,1)}^{(3\,4\,5\,6\,1\,2)} = x_1x_2x_3 E_{(1,0,0,1,1,0)}^{(4\,5\,6\,1\,2\,3)}$$

$$= x_1x_2x_3x_4 E_{(0,0,1,1,0,0)}^{(5\,6\,1\,2\,3\,4)}$$

Picture



Example $(j=3)$ $\quad \mu=(0,0,1,1,0,0) \quad z=(5\,6\,1\,2\,3\,4)$

$$E_\mu^z = x_{z(j)} E_\gamma^{z c_j^{-1} c_n} + \frac{1-t}{1-q^{\mu_j} t^{\nu_\mu(j)-(j-1)}} \sum_{a=1}^{j-1} q^{\mathrm{maj}^z_{\mu(a)}} t^{\mathrm{covid}^z_{\mu(a)}} x_{y(a)} E_\gamma^{y c_a^{-1} c_n}.$$

$$E^{(561234)}_{(0,0,1,1,0,0)} = x_1 E^{(562341)}_{(0,0,1,0,0,0)} + \left(\frac{1-t}{1-qt^{5-2}}\right)\left(q t^0 x_5 E^{(612345)}_{(0,0,1,0,0,0)} + q t^0 x_6 E^{(523416)}_{(0,0,1,0,0,0)}\right)$$

$$E^{(562341)}_{(0,0,1,0,0,0)} = x_2 + \left(\frac{1-t}{1-qt^{6-2}}\right)\left(qt\, x_5 + qt\, x_6\right)$$

$$E^{(562341)}_{(0,0,1,0,0,0)} = x_2 + \left(\frac{1-t}{1-qt^{6-2}}\right)\left(qt\,x_5 + qt\,x_6\right)$$

$$E^{(123456)}_{(2,2,1,1,0,0)} = x_1x_2x_3x_4\,x_1x_2 + x_1x_2x_3x_4\,x_1\left(\frac{1-t}{1-qt^4}\right)qt\,x_5$$

$$+ x_1x_2x_3x_4x_1\left(\frac{1-t}{1-qt^4}\right)qt\,x_6 + \cdots$$

So

$$\begin{array}{c|cc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array} \qquad \begin{array}{c|cc} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array} \qquad \begin{array}{c|cc} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$