

# A vertex model for LLT polynomials

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## Coauthors

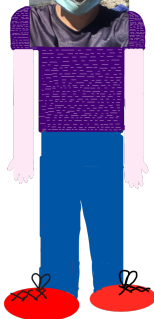
arXiv:2012.02376



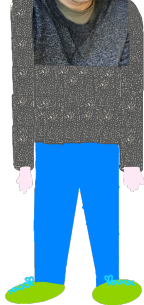
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# Overview

## Plan:

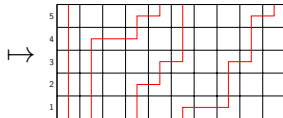
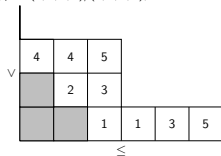
- 1 Review a vertex model for Schur polynomials.
- 2 Introduce our vertex model for coinversion LLT polynomials.
- 3 Sketch the proofs of some results about the LLT polynomials in the vertex model framework.
- 4 Discuss possible future work involving super-symmetric LLT polynomials.

## Part 1: A vertex model for Schur polynomials

## A vertex model for Schur polynomials

There is a well-known bijection between semistandard skew Young tableaux and nonintersecting up-right lattice paths.

$$\lambda/\mu = (6, 3, 3, 0)/(2, 1, 0, 0), n = 5$$

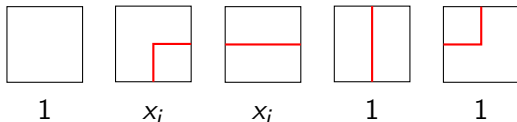


In general,

- 1 The partition  $\lambda = (\lambda_1, \dots, \lambda_m)$  gives the top boundary condition. The  $i$ -th path exits in column  $\lambda_i + m - i$ .
- 2 The partition  $\mu = (\mu_1, \dots, \mu_m)$  gives the bottom boundary condition. The  $i$ -th path enters in column  $\mu_i + m - i$ .
- 3 The number of rows is  $n$ .

## Local weights

We can consider this as a vertex model with local weights given by



where row  $i$  has the variable  $x_i$ . The weight of the configuration is the product of the weights of the vertices.

$$w \left( \begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline 4 & & & & & \\ \hline 3 & & & & & \\ \hline 2 & & & & & \\ \hline 1 & & & & & \\ \hline \end{array} \right) = x_1^2 x_2 x_3^2 x_4^2 x_5^2$$

The **partition function** of the vertex model is

$$Z_{\lambda/\mu}(x_1, \dots, x_n) := \sum_{\text{config. } C} w(C) = s_{\lambda/\mu}(x_1, \dots, x_n).$$

## Part 2: A vertex model for LLT polynomials

## A brief history of LLT polynomials

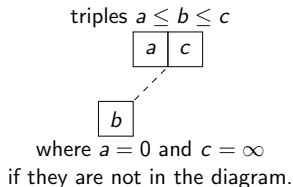
- 1 LLT polynomials were first defined by Lascoux, Leclerc, and Thibon (1997). They serve as a generating function for semistandard ribbon tableaux of shape  $\lambda$ ,  $q$ -counting a spin statistic.
- 2 Haglund, Haiman, and Loehr (2005) gave an alternate description of the LLT polynomials as a generating function for tuples of semistandard Young tableaux,  $q$ -counting an inversion statistic.
- 3 Blasiak, Haiman, Morse, Pun, and Seelinger (2021) gave a formulation in terms of coinversions.
- 4 We formulate them as collection of up-right lattice paths in bijection with the coinversion LLT polynomials.

## Coinversion LLT polynomials

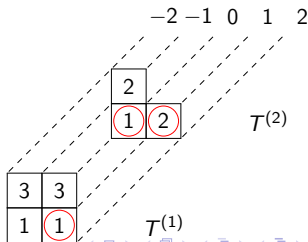
Let  $\lambda/\mu = (\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(k)}/\mu^{(k)})$  be a tuple of skew partitions.  
 Define the coinversion LLT polynomial

$$\mathcal{L}_{\lambda/\mu}(X; t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} t^{\text{coinv}(T)} X^T$$

where **coinversion triples** are given by



For example:



## Other definitions of/approaches to LLT polynomials

The inversion LLT polynomial is defined by

$$\mathcal{G}_{\lambda/\mu}(X; t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} t^{\text{inv}(T)} X^T$$

where **inversion triples** are triples that are not coinversion triples.  
Note that

$$\mathcal{L}_{\lambda/\mu}(X; t) = t^m \mathcal{G}_{\lambda/\mu}(X; t^{-1})$$

where  $m$  is the total number of triples in the diagram.

## Other definitions of/approaches to LLT polynomials

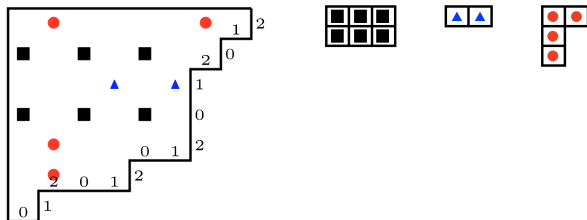
- Our definition of LLT polynomials is in terms of tuples of SSYT, but the original definition of LLT polynomials by Lascoux, Leclerc, and Thibon (1997) was in terms of semistandard ribbon tableaux.
- The Littlewood quotient map gives a bijective correspondence

$$\{\text{semistandard } k\text{-ribbon tableaux of shape } \lambda\} \leftrightarrow \{k\text{-tuples of SSYT of shapes } (\lambda^{(1)}, \dots, \lambda^{(k)})\}$$

where  $(\lambda^{(1)}, \dots, \lambda^{(k)})$  is the  $k$ -quotient of  $\lambda$ .

# The $k$ -quotient map

the  $k$ -quotient  $(\lambda^{(0)}, \dots, \lambda^{(k-1)})$  of a partition  $\lambda$ :  
 label the Maya diagram on the boundary of  $\lambda$  with  $0, \dots, k-1$

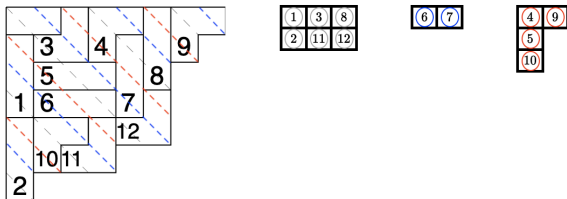


picture adapted from <https://www2.math.upenn.edu/~peal/polynomials/borderStripTableaux.htm>

# The Littlewood quotient map

the Littlewood quotient map:

look at the minimum content of each ribbon modulo  $k$



1	3	8
2	11	12

6	7
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4	9
5	
10	

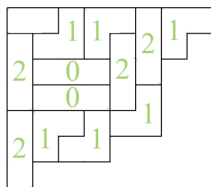
picture adapted from <https://www2.math.upenn.edu/~peal/polynomials/borderStripTableaux.htm>

## Other definitions of/approaches to LLT polynomials

- Lascoux, Leclerc, and Thibon (1997) define LLT polynomials using semistandard ribbon tableaux and the spin statistic.
- Given  $T \in \text{SSRT}_k(\lambda/\mu)$ , we define

$$\text{spin}(T) = \sum_{\text{ribbons } r \text{ in } T} (\text{height}(r) - 1).$$

As an example:



$$\text{spin}(T) = 14$$

## Other definitions of/approaches to LLT polynomials

- Curran, Yost-Wolff, Zhang, and Zhang (2019) define LLT polynomials by

$$\tilde{\mathcal{G}}_{\lambda/\mu}^{(k)}(X; t) = \sum_{T \in \text{SSRT}_k(\lambda/\mu)} t^{\text{spin}(T)} X^T.$$

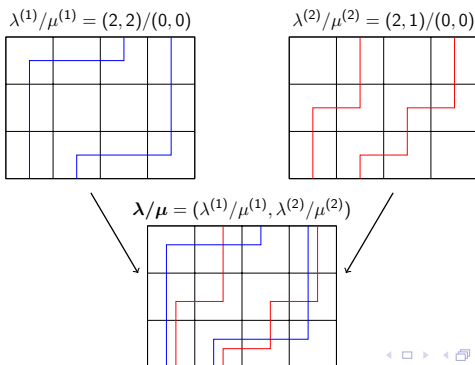
- $\mathcal{L}_{\lambda/\mu}(X; t) = t^e \tilde{\mathcal{G}}_{\lambda/\mu}^{(k)}(X; t^{1/2})$  for some integer  $e$ , where  $\lambda/\mu \leftrightarrow \boldsymbol{\lambda}/\boldsymbol{\mu}$  via the Littlewood quotient map
- Starting with this definition, they construct a vertex model of a similar flavor as ours.

## From Schur to LLT

Let  $\lambda/\mu = (\lambda^{(1)}/\mu^{(1)}, \dots, \lambda^{(k)}/\mu^{(k)})$ . It is known that when  $t = 1$  the LLT polynomial is a product of Schur polynomials

$$\mathcal{L}_{\lambda/\mu}(X; 1) = s_{\lambda^{(1)}/\mu^{(1)}}(X) \dots s_{\lambda^{(k)}/\mu^{(k)}}(X).$$

We can superimpose the paths:



## Local weights

Let  $i, j, k, l \in \{0, 1\}$ . Recall the weights for one color:

$$\begin{array}{cccccc}
 \begin{array}{c} k \\ \square \\ i \end{array} & \begin{array}{c} 0 \\ \square \\ 0 \end{array} & \begin{array}{c} 0 \\ \square \\ 1 \end{array} & \begin{array}{c} 0 \\ \square \\ 0 \end{array} & \begin{array}{c} 1 \\ \square \\ 1 \end{array} & \begin{array}{c} 1 \\ \square \\ 0 \end{array} \\
 L_x^{(1)}(i, j, k, l) & 1 & x & x & 1 & 1
 \end{array}$$

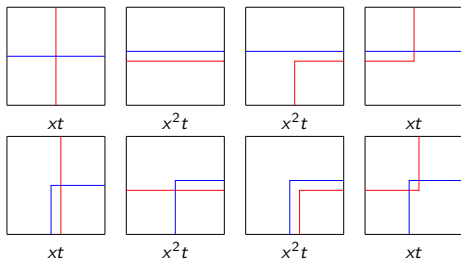
Let  $I, J, K, L \in \{0, 1\}^k$ ,  $\mathbf{A} = (A_1, \dots, A_k)$  for  $\mathbf{A} \in \{I, J, K, L\}$ .  
 We introduce  $t$  into the weights for  $k$  colors

$$\begin{array}{c} K \\ \square \\ J \quad L \\ I \end{array}, \quad L_x^{(k)}(I, J, K, L) = \prod_{i=1}^k L_{xt^{\delta_i}}^{(1)}(I_i, J_i, K_i, L_i)$$

where  $\delta_i = \#$  colors greater than  $i$  that appear at the vertex.  
 In other words, we pick up a factor of  $t$  when a color exits right  
 and a larger color is present.

# LLT polynomials as multicolored collection of paths

With **blue** as color 1 and **red** as color 2, the faces that contribute a factor of  $t$  are



As an example:

$$w \left( \begin{array}{|c|c|c|c|} \hline \text{[Complex Path]} \\ \hline \end{array} \right) = x_1^3 x_2^2 x_3^2 t^3$$



## Part 3: Yang-Baxter integrability + other results

# Yang-Baxter equation

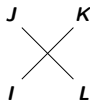
## Theorem

*This vertex model is Yang-Baxter integrable.*

The Yang-Baxter equation is given by

$$\sum_{\text{interior paths}} w \left( \begin{array}{c} \begin{array}{ccc} & J_3 & \\ & \boxed{\begin{array}{c} y \\ x \end{array}} & \\ I_1 & & J_2 \\ & & \\ I_2 & & J_1 \\ & & \\ & & I_3 \end{array} \end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{c} \begin{array}{ccc} & J_3 & \\ & \boxed{\begin{array}{c} x \\ y \end{array}} & \\ I_1 & & J_2 \\ & & \\ I_2 & & J_1 \\ & & \\ & & I_3 \end{array} \end{array} \right)$$

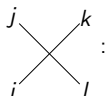





for any choice of boundary condition  $I_1, I_2, I_3$  and  $J_1, J_2, J_3$ . The weights



are the  $R$ -matrix for the model.

# Yang-Baxter equation

The weights of the one-color  $R$ -matrix are

	Type 1	Type 2	Type 3	Type 4	Type 5
					
$R_{y/x}^{(1)}(i, j, k, l):$	$1 - y/x$	$y/x$	$1$	$y/x$	$1$


The weights of the  $k$ -color  $R$ -matrix are given by

$$R_{y/x}^{(k)}(I, J, K, L) = \prod_{i=1}^k R_{y/(xt^{\epsilon_i})}^{(1)}(I_i, J_i, K_i, L_i)$$

where  $\epsilon_i = \#$  colors greater than  $i$  of type 1.


# Yang-Baxter equation

With **blue** as color 1 and **red** as color 2



$(1 - y/x)1$

and



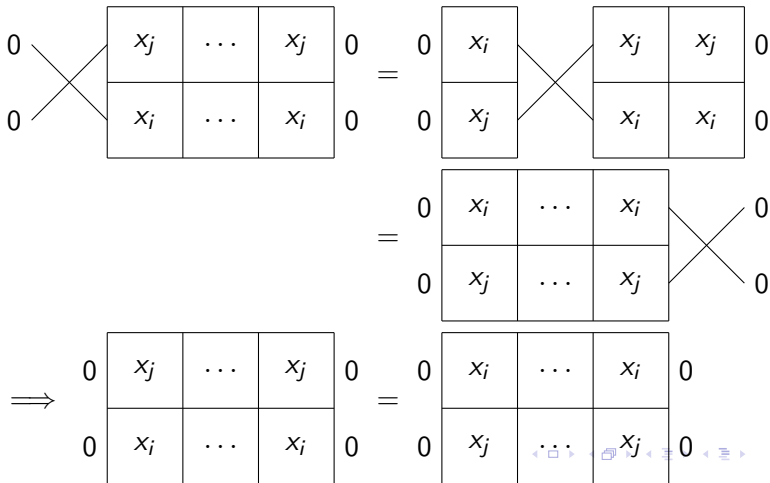
$(1 - y/(xt))(1 - y/x)$

$$\begin{aligned}
 w \left( \begin{array}{c} \text{blue line over red line} \\ \text{box with } y \text{ on top, } x \text{ on bottom} \\ \text{red line under blue line} \end{array} \right) &= w \left( \begin{array}{c} \text{blue line over red line} \\ \text{box with } x \text{ on top, } y \text{ on bottom} \\ \text{red line under blue line} \end{array} \right) + w \left( \begin{array}{c} \text{blue line over red line} \\ \text{box with } x \text{ on top, } y \text{ on bottom} \\ \text{red line under blue line} \end{array} \right) \\
 x^2 t (1 - y/x) &= xy (1 - y/x) + x^2 t (1 - y/x) (1 - y/(xt))
 \end{aligned}$$

The proof of the YBE follows by a (slightly complicated) induction argument on the number of colors.

# Symmetry

The YBE implies that the polynomials are symmetric. Let 0 indicate “no paths.” Then using the YBE repeatedly gives



## Results

With this formulation of the vertex model, we can give a combinatorial proof of a Cauchy identity

$$\sum_{\lambda} t^{d(\lambda)} \mathcal{L}_{\lambda}(X_n; t) \mathcal{L}_{\lambda}(Y_n; t) = \prod_{i,j=1}^n \prod_{m=0}^{k-1} (1 - x_i y_j t^m)^{-1}.$$

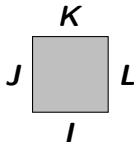
## The $L^*$ -matrix

Before proving the Cauchy identity, we introduce the  $L^*$ -matrix, with weights given by

$$L_{\bar{x}}^*(I, J; K, L) = x^k t^{\binom{k}{2}} L_{\bar{x}}(I, J; K, L)$$

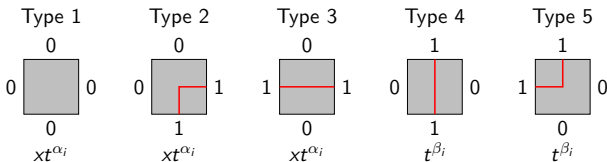
$$\text{where } \bar{x} = \frac{1}{xt^{k-1}}.$$

We represent the  $L^*$ -matrix with a gray box.



## The $L^*$ -matrix

Here's one way to think about the weights of the  $L^*$ -matrix. Similarly as before, we can think of each color as contributing a multiplicative factor to the overall weight. The contribution of color  $i$  is given by



where

$\alpha_i = \#$  colors greater than  $i$  of Type 1, 4, or 5,  
 $\beta_i = \#$  colors greater than  $i$  of Type 4 or 5.

## Cauchy identity

The Cauchy identity

$$\sum_{\lambda} t^{d(\lambda)} \mathcal{L}_{\lambda}(X_n; t) \mathcal{L}_{\lambda}(Y_n; t) = \prod_{i,j=1}^n \prod_{m=0}^{k-1} (1 - x_i y_j t^m)^{-1}$$

is equivalent to the following identity, which follows from repeated use of the YBE:

$$\sum_{\text{interior paths}} w \left( \begin{array}{c} \text{Diagram 1: A grid with a shaded bottom half. Inputs } x_1, \dots, x_n \text{ on the left and outputs } y_1, \dots, y_n \text{ on the right.} \end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{c} \text{Diagram 2: A grid with a shaded top half. Inputs } x_1, \dots, x_n \text{ on the left and outputs } y_1, \dots, y_n \text{ on the right.} \end{array} \right)$$

- $d(\lambda)$  is an integer that depends only on  $\lambda$
- the lattice has infinitely many columns

# Cauchy identity

$$\begin{array}{l}
 \begin{array}{|c|} \hline \phantom{x} \\ \hline \end{array} = 1 \quad \Rightarrow \quad \begin{array}{|c|} \hline \circ \quad \circ \quad \circ \quad \dots \\ \hline \circ \quad \circ \quad \circ \quad \circ \end{array} = 1 \\
 \begin{array}{|c|} \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} = x^k t^{\binom{k}{2}} \quad \Rightarrow \quad \begin{array}{|c|} \hline \circ \quad \circ \quad \circ \quad \dots \\ \hline \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \circ \quad \circ \quad \circ \quad \circ \end{array} = 0 \\
 \begin{array}{|c|} \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} = x^k t^{\binom{k}{2}} \quad \Rightarrow \quad \begin{array}{|c|} \hline \circ \quad \circ \quad \circ \quad \dots \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} = 0 \\
 \begin{array}{|c|} \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} = 1 \quad \Rightarrow \quad \begin{array}{|c|} \hline \circ \quad \circ \quad \circ \quad \dots \\ \hline \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \circ \quad \circ \quad \circ \quad \circ \end{array} = 1
 \end{array}$$

# Cauchy identity

$$\text{LHS} = w \left( \begin{array}{c} \begin{array}{c} \bar{x}_n \\ \bar{x}_i \\ y_n \\ y_1 \end{array} \\ \begin{array}{c} \text{grid} \\ \text{shaded grid} \end{array} \end{array} \right) = w \left( \begin{array}{c} \text{shaded grid} \\ \text{grid} \end{array} \right) = (x^\rho)^k t_{(2)}^{(n)}(k)$$

where  $x^\rho = x_1^{n-1} x_2^{n-2} \dots x_{n-1}^1 x_n^0$

# Cauchy identity

$$\begin{aligned}
 \text{RHS} &= \sum_{\lambda} w \left( \begin{array}{c} \text{Diagram: A grid with a shaded top-right region of size } \lambda \text{ and a crossing pattern of lines.} \end{array} \right) \\
 &= \sum_{\lambda} w \left( \begin{array}{c} \text{Diagram: A grid with a shaded bottom-left region of size } \lambda \text{ and a crossing pattern of lines.} \end{array} \right) w \left( \begin{array}{c} \text{Diagram: A grid with a shaded top-right region of size } \lambda \text{ and a crossing pattern of lines.} \end{array} \right) w \left( \begin{array}{c} \text{Diagram: A crossing pattern of lines.} \end{array} \right) \\
 &= \sum_{\lambda} \mathcal{L}_{\lambda}(Y_n; t) \cdot (x^{\rho})^k t^{\binom{n}{2}} \binom{k}{2} t^{d(\lambda)} \mathcal{L}_{\lambda}(X_n; t) \cdot \prod_{i,j=1}^n \prod_{m=0}^{k-1} (1 - x_i y_j t^m)
 \end{aligned}$$

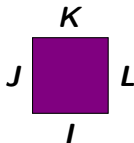
## Part 4: Purple vertices + super-symmetric LLT

## Generalizing the vertex model

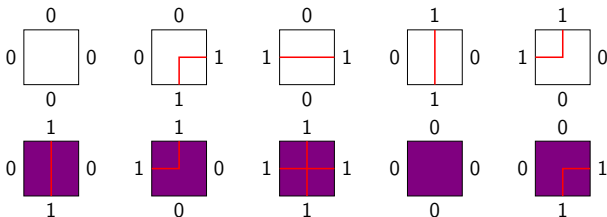
- Aggarwal, Borodin, and Wheeler (2021) independently derived many of the results in this presentation. They used a more general vertex model, which has the  $L$  and  $R$  matrices as specializations.
- Their vertex model has other interesting specializations, including the  $L'$  and  $R'$  matrices (to be introduced shortly), which can be used to prove results about LLT polynomials and super-symmetric LLT polynomials.

## The $L'$ -matrix

We introduce the  $L'$ -matrix, which we represent with a purple box.



To get the  $L'$ -matrix from the  $L$ -matrix, we invert the vertical parts of the paths and then reflect the paths over the vertical axis.



To be more precise, the weights for one color are

$$L_x^{(1)}(i, j, k, l):$$

and the weights for  $k$  colors are

$$L_x^{(k)}(I, J, K, L) = \prod_{i=1}^k L_{xt^{\delta'_i}}^{(1)}(I_i, J_i, K_i, L_i)$$

where  $\delta'_i = \#$  colors greater than  $i$  that are vertical.

## The partition function of the $L'$ -matrix

Proposition (Aggarwal, Borodin, and Wheeler (2021))

Let  $\lambda/\mu$  be a tuple of skew partitions. Let  $Z'_{\lambda/\mu}$  be the partition function associated to the  $L'$ -matrix. Then

$$Z'_{\lambda/\mu}(X_n; t) = t^{-\frac{1}{2}(d(\lambda)+d(\lambda')-d(\mu)-d(\mu'))} \mathcal{L}_{\lambda'/\mu'}(t^{\frac{k-1}{2}} X_n; t^{-1})$$

where  $\lambda' = (\lambda^{(k)'}, \dots, \lambda^{(1)'})$  (and similarly for  $\mu'$ ).

# Yang-Baxter equation

Theorem (Aggarwal, Borodin, and Wheeler (2021))

The  $L$  and  $L'$  matrices satisfy the Yang-Baxter equation

$$\sum_{\text{interior paths}} w \left( \begin{array}{ccc} & J_3 & \\ l_1 & \begin{array}{c} y \\ x \end{array} & J_2 \\ l_2 & \begin{array}{c} x \\ y \end{array} & J_1 \\ & l_3 & \end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{ccc} & J_3 & \\ l_1 & \begin{array}{c} x \\ y \end{array} & J_2 \\ l_2 & \begin{array}{c} y \\ x \end{array} & J_1 \\ & l_3 & \end{array} \right)$$

for any choice of boundary condition  $l_1, l_2, l_3$  and  $J_1, J_2, J_3$ . The weights



are the  $R'$ -matrix for the model.

# Yang-Baxter equation

The weights of the one-color  $R'$ -matrix are

$$\begin{array}{c}
 \begin{array}{ccc}
 j & & k \\
 & \diagdown & / \\
 & \text{yellow square} & \\
 & / & \diagdown \\
 i & & l
 \end{array}
 : &
 \begin{array}{c} \diagdown \\ \text{yellow square} \\ / \end{array} &
 \begin{array}{c} \diagdown \\ \text{yellow square} \\ / \end{array} &
 \begin{array}{c} \diagdown \\ \text{yellow square} \\ / \end{array} &
 \begin{array}{c} \diagdown \\ \text{white square} \\ / \end{array} &
 \begin{array}{c} \diagdown \\ \text{yellow square} \\ / \end{array}
 \end{array}$$

$$R'_{y/x}(i, j, k, l) : \quad \frac{1}{1+y/x} \quad \frac{y/x}{1+y/x} \quad \frac{1}{1+y/x} \quad \frac{y/x}{1+y/x} \quad 1$$

The weights of the  $k$ -color  $R'$ -matrix are given by

$$R'_{y/x}(I, J, K, L) = \prod_{i=1}^k R'_{y/(xt^{\epsilon'_i})}(l_i, j_i, k_i, l_i)$$

where  $\epsilon'_i = \#$  colors greater than  $i$  that are present at the vertex.

## A dual Cauchy identity

Aggarwal, Borodin, and Wheeler (2021) use this YBE to prove a dual Cauchy identity

$$\sum_{\lambda} t^{-\frac{1}{2}(d(\lambda)-d(\lambda'))} \mathcal{L}_{\lambda}(X_n; t) \mathcal{L}_{\lambda'}(Y_n; t^{-1}) = \prod_{i,j=1}^n \prod_{m=0}^{k-1} \left(1 + x_i y_j t^{\frac{k-1}{2}} t^m\right).$$

# Super-symmetric LLT polynomials

A **super ribbon tableaux** is a ribbon tableaux with ribbons labelled by the alphabet  $1 < \dots < n < 1' < \dots < m'$  such that

- 1 across each row, the primed ribbons strictly increase
- 2 down each column, the un-primed ribbons strictly increase.

The super-symmetric LLT polynomial is defined by

$$\mathcal{L}_{\lambda/\mu}^{S(k)}(X_n, Y_{m'}; t) = \sum_{T \in SRT_k(\lambda/\mu)} t^{\text{spin}(T)} x^{\text{wt}(T)} y^{\text{wt}'(T)}.$$

We define

$$\mathcal{L}_{\lambda/\mu}^S(X_n, Y_{m'}; t) = \mathcal{L}_{\lambda/\mu}^{S(k)}(X_n, Y_{m'}; t)$$

when  $\lambda/\mu \leftrightarrow \boldsymbol{\lambda}/\boldsymbol{\mu}$  via the Littlewood quotient map.

# A vertex model for the super-symmetric LLT polynomials

The  $L$  and  $L'$  matrices form a vertex model for the super-symmetric LLT polynomials, up to a power of  $t$ .

## Theorem

For some integer  $e$ ,

$$\frac{1}{t^e} \mathcal{L}_{\lambda/\mu}^S(X_n, Y_m; t) = \sum_{\nu} \mathcal{L}_{\nu/\mu}(X_n; t) Z'_{\lambda/\nu}(Y_m; t)$$

$$= \sum_{\nu} W \left( \begin{array}{c} \lambda \\ y_m \\ \vdots \\ y_1 \\ x_n \\ \vdots \\ x_1 \end{array} \begin{array}{c} \mu \\ \nu \end{array} \right) = \sum_{\text{interior paths}} W \left( \begin{array}{c} \lambda \\ y_m \\ \vdots \\ y_1 \\ x_n \\ \vdots \\ x_1 \end{array} \begin{array}{c} \mu \end{array} \right).$$



# End!

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# Thank You!