Whittaker functions through polymer models & geometric (Kirillov's) RSK

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Notational dictionary of Whittaker fct.

A. Number Theory

$$\hat{W}^{A}_{n,a}(y) = \pi^{-|a|/2} \int \hat{W}^{A}_{n-1,\tilde{a}} \left(y_2 \sqrt{\frac{t_2}{t_1}}, \ldots, y_{n-1}\sqrt{\frac{t_{n-1}}{t_{n-2}}}, \sqrt{\frac{1}{t_{n-1}}} \right)$$

$$\times \prod_{j=1}^{n-1} \exp\left\{ -(\pi y_j)^2 t_j - \frac{1}{t_j} \right\} (\pi y_j)^{\frac{n-j}{n-1} a_1} \; t_j^{\frac{n a_1}{2(n-1)}} \frac{dt_j}{t_j}$$

with $a = (a_1, \ldots, a_n)$

$\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_{n-1})$ s.t. $\tilde{a}_i = a_{i+1} + \frac{a_1}{n-1}$

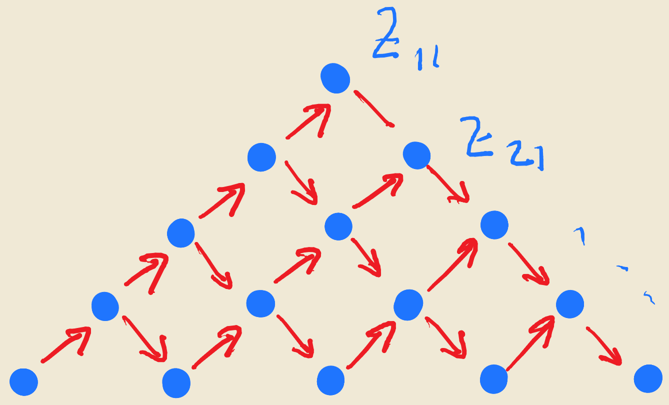
& $$\hat{W}^{A}_{2,(a_1,a_2)}(y_1, y_2) = 2\, y_1^{|a|/2}\, y_2^{|a|}\, K_{\frac{a_1-a_2}{2}}(2\pi y_1)$$

$$K_\nu(x) = \frac{1}{2} \int_{\mathbb{R}_+} z^{\nu}\, e^{-\frac{x}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{z}$$

B. In probability & integrable systems

(see e.g. Givental,
Gerasimov-Lebedev-Oblezin)

Definition (Gelfand-Tsetlin patterns)



$$\equiv \{ z_{ij} \,:\, 1 \le i \le j \le N \}$$

$$\mathrm{wt}\left(z_{ij} \searrow z_{i+1,j} \right) = \exp\left\{ - \frac{z_{ij}}{z_{i+1,j}} \right\}$$

* In the p-adic (Schur) case $z_{ij} \searrow z_{i+1,j}$ mean $\mathbb{1}_{z_{i,j} \le z_{i+1,j}}$

gl_N-Whittaker representation (Givental)

$$\Psi_\alpha^{gl_N}(x) := \int_{\mathbb{R}_+^{\frac{N(N+1)}{2}}} \prod_{i=1}^N \left(\frac{\mathrm{prod}_i(Z)}{\mathrm{prod}_{i-1}(Z)} \right)^{-\alpha_i} \mathrm{wt}(Z) \prod_{i,j} \frac{dz_{ij}}{z_{ij}}$$

with $\mathrm{prod}_i(Z) := \prod_{j=1}^{i} z_{ij}$

Number Theory $\longleftrightarrow$ Probability

$$\widehat{W}^{A}_{N,a}(y) = \pi^{-(N+1)|a|} \, \Psi^{\mathfrak{gl}_N}_{-\alpha}(x)$$

for $\quad a_i = 2\alpha_{N-i+1} \quad$ &

$$y_1 = \frac{1}{\pi}\sqrt{\frac{x_2}{x_1}}, \quad \ldots\ldots, \quad y_{N-1} = \frac{1}{\pi}\sqrt{\frac{x_N}{x_{N-1}}}, \quad y_N = \frac{1}{\pi}\sqrt{\frac{1}{x_N}}$$

Bump–Stade identity

Number Theory notation

$$\int_{\mathbb{R}_+^{N-1}} \hat{W}^{A}_{N,a}(y)\, \hat{W}^{A}_{N,b}(y) \prod_j (\pi y_j)^{2js} (2y_j)^{-j(n-j)} \prod_j \frac{dy_j}{y_j} =$$

$$= \Gamma(ns)^{-1} \prod_{j,k} \Gamma(s + a_j + b_k)$$

Prob. notation

$$\int_{\mathbb{R}_+^N} e^{-ux_1}\, \Psi^{gl_N}_{-\alpha}(x)\, \Psi^{gl_N}_{-\beta}(x) \prod_i \frac{dx_i}{x_i} = u^{-\sum(\alpha_i+\beta_i)} \prod_{i,j} \Gamma(\alpha_i + \beta_j)$$

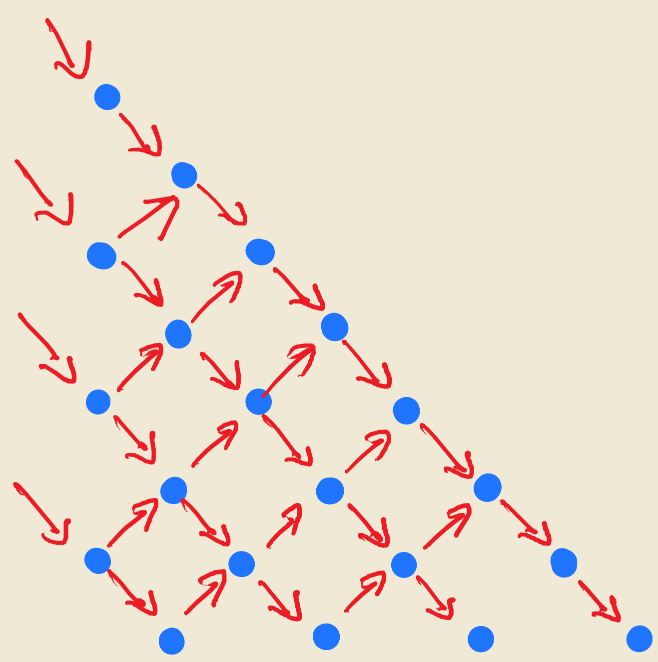
with
$$a_j = \alpha_j - \frac{1}{N}\sum_{i=1}^N \alpha_i$$

$$b_j = \beta_j - \frac{1}{N}\sum_{i=1}^N \beta_i$$

$$s = \frac{1}{N}\sum_{j=1}^N (\alpha_j + \beta_j)$$

Orthogonal Whittaker & Ishii-Stade identities

Symplectic Gelfand-Tsetlin



$$\mathrm{wt}\left(z_a \searrow z_b\right) = e^{-z_a/z_b}$$

$$\mathrm{wt}\left(\searrow z_b\right) = e^{-1/z_b}$$

Orthogonal Whittaker

$$\Psi^{SO_{2N+1}}_{\alpha}(x) = \int_{g \in spGT(x)} \prod_{i=1}^{N} \left(\frac{\mathrm{prod}_i(Z)}{\mathrm{prod}_{i-1}(Z)} \right)^{\alpha_i - \alpha_{i-1} - \alpha_{i+1}} \mathrm{wt}(Z) \prod_{i,j} \frac{dz_{ij}}{z_{ij}}$$

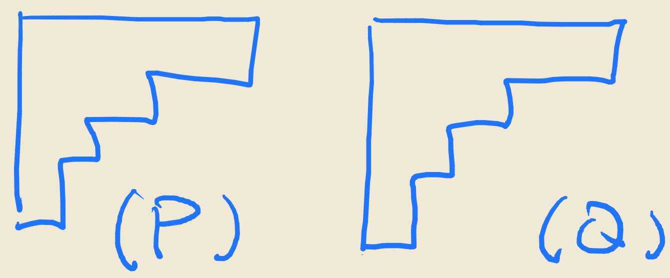
Ishii-Stade identity

$$\int_{\mathbb{R}_+^N} \Psi^{gl_N}_{-\alpha}(x)\, \Psi^{SO_{2N+1}}_{\beta}(x) \prod_{i=1}^{N} \frac{dx_i}{x_i} = \frac{\prod_{i,j} \Gamma(\alpha_i + \beta_j)\,\Gamma(\alpha_i - \beta_j)}{\prod_{i<j} \Gamma(\alpha_i + \alpha_j)}$$

Robinson-Schensted-Knuth (RSK)
& geometric / Kirillov's RSK

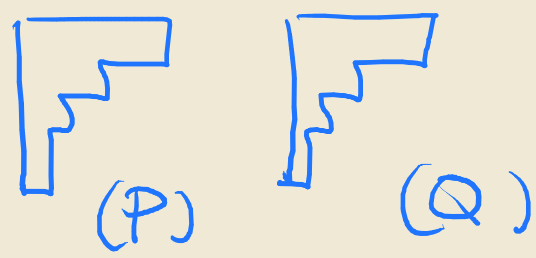
RS-correspondence

$$\begin{pmatrix} 1 & 2 & \cdots & N \\ \sigma(1) & \sigma(2) & & \sigma(N) \end{pmatrix} \longrightarrow$$

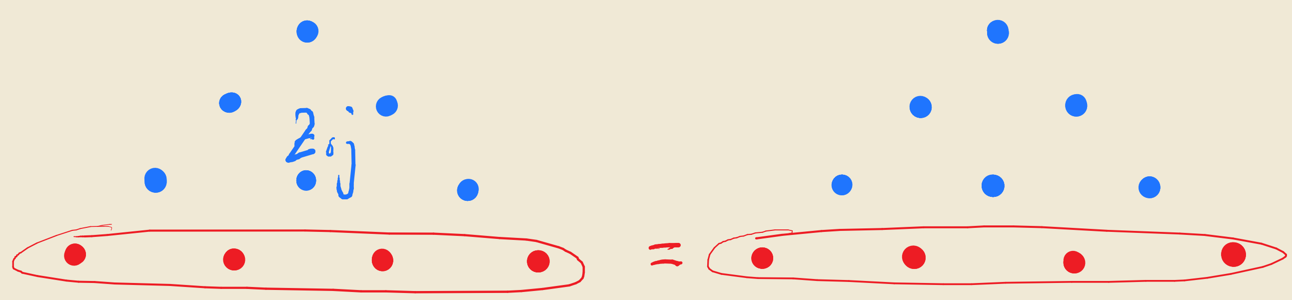


RSK-correspondence

$$W = (w_{ij})_{\substack{i \le n \\ j \le N}} \longrightarrow$$



$\cong$



shape

$Z \cong P$ $\qquad$ $Z' \cong Q$

GT-Young tableau correspondence

$$z_{ij} = \left.\begin{array}{l} \text{\# of entries up to } i \\ \text{in row } j \end{array}\right\}$$

RSK via piecewise linear transformations

By "sticking" P & Q along their common shape RSK is a bijection between matrices

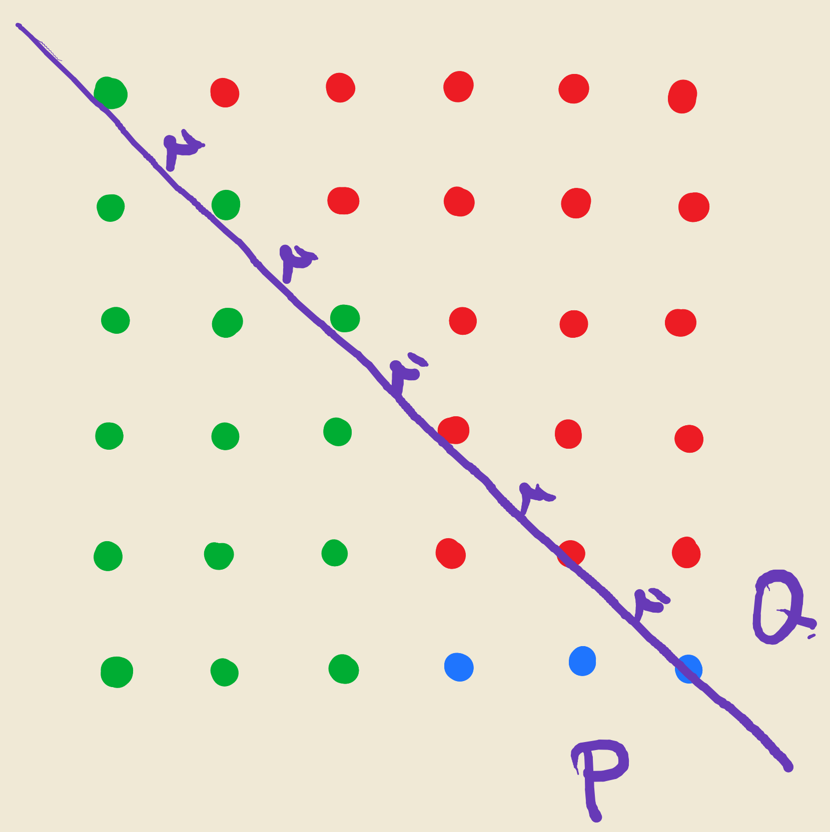
$$\begin{pmatrix} w_{ij} \end{pmatrix} \xrightarrow{RSK} \begin{pmatrix} t_{ij} \end{pmatrix} = \begin{pmatrix} Z \setminus Z' \end{pmatrix}$$

Basic "local transformation"

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} b\wedge c - a & b \\ c & d + b\vee c \end{pmatrix}$$

RSK in a cartoon:

$W \rightarrow$



red part: matrix after first $(N-1)$ rows have been inserted

green part: matrix after bottom green entries have been inserted via sequence of local transforms

blue part: to be inserted.

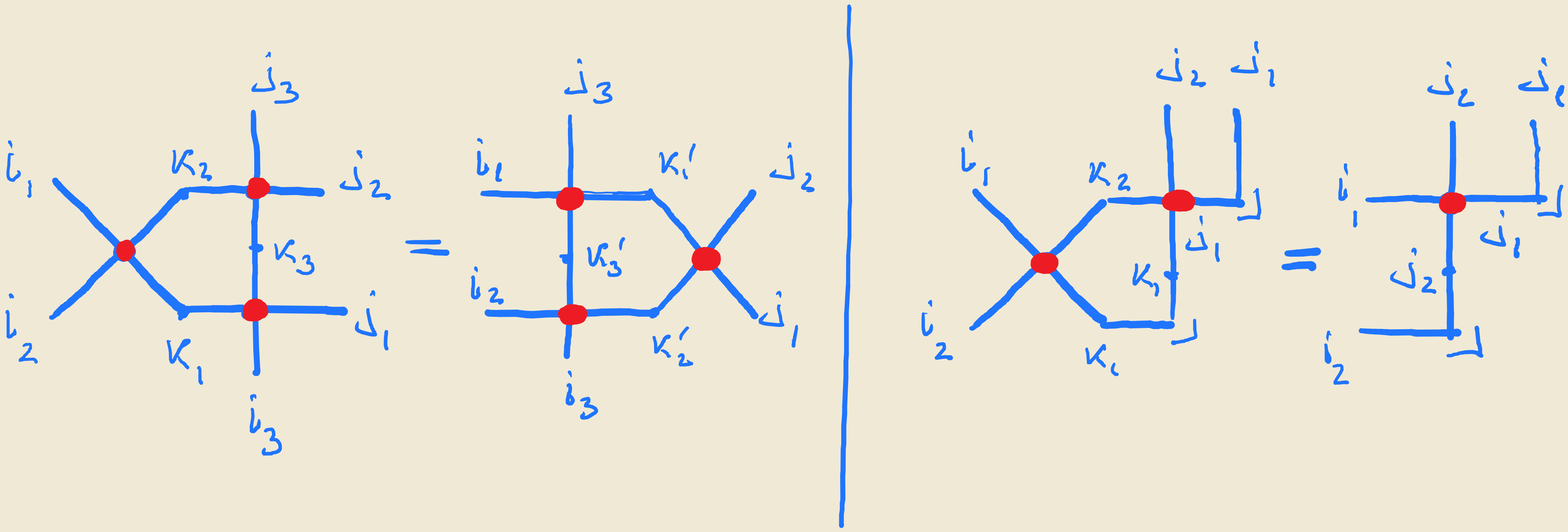
Summary: If $\hat{a}_{ij}$ denotes local transformation (*) applied at the (i,j)-entry then row-insertion of w_{ij} is encoded via

$$\hat{\rho}_{ij} = \hat{a}_{i-i\wedge j+1,\, j-i\wedge j+1} \circ \cdots \circ \hat{a}_{i-1,j-1} \circ \hat{a}_{ij}$$

[Berenstein – Kirillov analogue of Fomin growth diagrams]

Petrov's - Mucciconi's RSK - vertex model

(degeneration of spin - q - Whittaker)



$$W_x^{+}(i_1, j_1; i_2, j_2) = \mathbb{1}_{i_1 + j_1 = i_2 + j_2} \, \mathbb{1}_{i_1 \geq j_2} \, x^{j_2}$$

$$W_x^{\lrcorner}(i_1, j_1; i_2, j_2) = \mathbb{1}_{i_1 + j_1 = i_2 + j_1} \, \mathbb{1}_{i_1 \geq j_2} \, y^{j_2}$$

$$R_{x,y}(i_1, j_1; i_2, j_2) = \mathbb{1}_{i_2 + j_1 = i_2 + j_2} \, (xy)^{j_1 \wedge j_2}$$

Prop (Petrov - Mucciconi '20)

The above vertex model recovers RSK local transform description

geometric / Kirillov's RSK & polymers

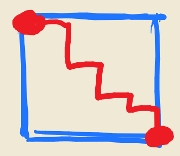
Local transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{bc}{a(b+c)} & b \\ c & d(b+c) \end{pmatrix}$$

RSK local transformation up to $(\max, +) \to (+, \times)$

Polymer model

$$Z_N := \sum_{\pi} \prod_{(i,j)\in\pi} w_{ij}$$

π: 

$$\mathbb{P}(\{w_{ij}\}) = \prod_{i,j} \frac{1}{\Gamma(\alpha_i+\beta_j)} w_{ij}^{-\alpha_i-\beta_j} e^{-\frac{1}{w_{ij}}} \frac{dw_{ij}}{w_{ij}}$$

<u>Thm</u> (Corwin-O'Connell-Seppäläinen-Z
O'Connell-Seppäläinen-Z)

$$\mathbb{E}\left[e^{-u Z_N}\right] = \frac{1}{\prod_{i,j}\Gamma(\alpha_i+\beta_j)} \int_{\mathbb{R}_+^N} e^{-ux_1 - \frac{1}{x_N}} \Psi_\alpha^{gl_N}(x)\, \Psi_\beta^{gl_N}(x) \frac{dx}{x}$$

<u>Corollary</u>

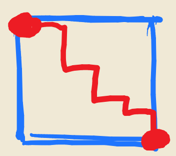
Set $u=0 \Rightarrow$

$$\int_{\mathbb{R}_+^N} e^{-\frac{1}{x_N}} \Psi_\alpha^{gl_N}(x)\, \Psi_\beta^{gl_N}(x) \frac{dx}{x} = \prod_{i,j}\Gamma(\alpha_i+\beta_j)$$

Properties of Kirillov's RSK following OSZ'14

Assume $gRSK(W) = (Z, Z')$ then

- [Kirillov] $$\sum_{\pi} \prod_{(i,j)\in\pi} w_{ij} = Z_{N1} = Z'_{N1}$$

  π: 

- $$\prod_{i} w_{ij} = \frac{\mathrm{prod}_j(Z)}{\mathrm{prod}_{j-1}(Z)} \quad \& \quad \prod_{j} w_{ij} = \frac{\mathrm{prod}_i(Z')}{\mathrm{prod}_{i-1}(Z')}$$

- $$\sum_{i,j} \frac{1}{w_{ij}} = \sum_{a\to b} \frac{Z_a}{Z_b} + \sum_{a\to b} \frac{Z'_a}{Z'_b} + \frac{1}{Z_{N1}}$$

- $$\mathrm{Jac}\left\{ \log w_{ij} : 1\le i,j \le N \right\} \xrightarrow{gRSK} \left\{ \log Z_{ij}, \log Z'_{ij} : i \le j \le N \right\}$$

Proof of Bump-Stade identity

$$1 = \int_{\mathbb{R}_+^{N^2}} \mathbb{P}(\{w_{ij}\}) \prod dw_{ij} =$$

$$= \frac{1}{\prod_{ij} \Gamma(\alpha_i + \beta_j)} \int \prod_j \left(\prod_i w_{ij}\right)^{-\beta_j} \prod_i \left(\prod_j w_{ij}\right)^{-\alpha_i} e^{-\sum \frac{1}{w_{ij}}} \frac{dw}{w}$$

$$= \frac{1}{\prod_{ij} \Gamma(\alpha_i + \beta_j)} \int \prod_j \left(\frac{\mathrm{prod}_j(Z)}{\mathrm{prod}_{j-1}(Z)}\right)^{-\beta_j} \prod_i \left(\frac{\mathrm{prod}_i(Z')}{\mathrm{prod}_{i-1}(Z')}\right)^{-\alpha_i}$$

$$e^{-\frac{1}{Z_{N1}}} \; e^{-\sum_{a\to b} Z_a/Z_b} \; e^{-\sum_{a\to b} Z'_a/Z'_b}$$

$$\prod_{i\le j} \frac{dZ_{ij}}{Z_{ij}} \prod_{i\le j} \frac{dZ'_{ij}}{Z'_{ij}}$$

$$= \frac{1}{\prod_{ij} \Gamma(\alpha_i + \beta_j)} \int e^{-\frac{1}{x_N}} \Psi_\alpha^{gl_N}(x) \, \Psi_\beta^{gl_N}(x) \prod_i \frac{dx_i}{x_i}$$

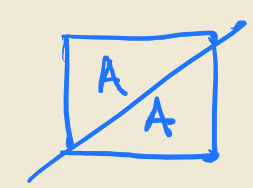
On the volume preserving property

- Assume $gRSK(W) = (Z, Z')$

$$\mathrm{Jac}\left\{ \log w_{ij} : i,j \le N \longrightarrow (\log z_{ij}, \log z'_{ij}) : 1 \le i \le j \le N \right\} = 1$$

- Assume $W = W^T \iff gRSK(W) = (Z, Z)$

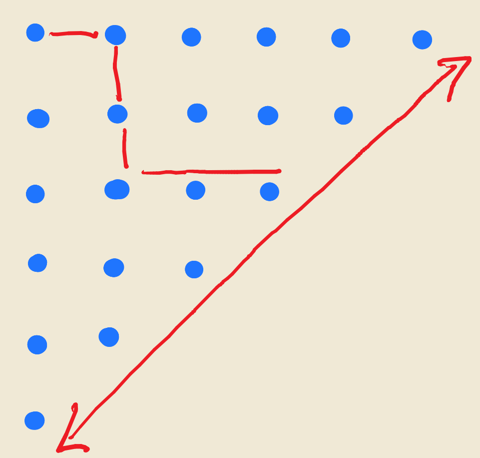
$$\mathrm{Jac}\left\{ \log w_{ij} : 1 \le i \le j \le N \longrightarrow \log z_{ij} : 1 \le i \le j \le N \right\} = 1$$

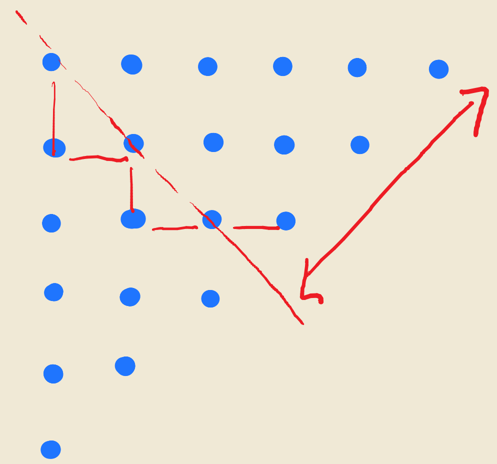
- Assume W persymmetric i.e 

$$\iff gRSK(W) = (Z, Z^{\text{Schützenberger}})$$

$$\mathrm{Jac}\left\{ \log w_{ij} : i+j \le N+1 \longrightarrow \log z_{ij} : 1 \le i \le j \le N \right\} = 1$$

Other geometries

$Z_N^{\text{plane}} :=$ 

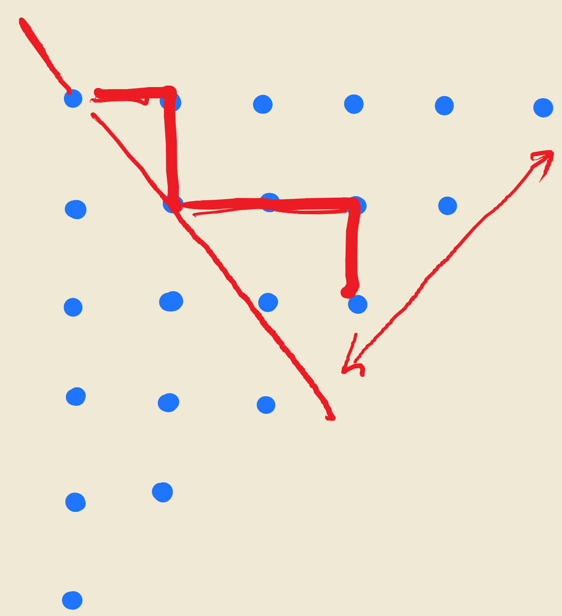
$Z_N^{\text{h-plane}} :=$ 

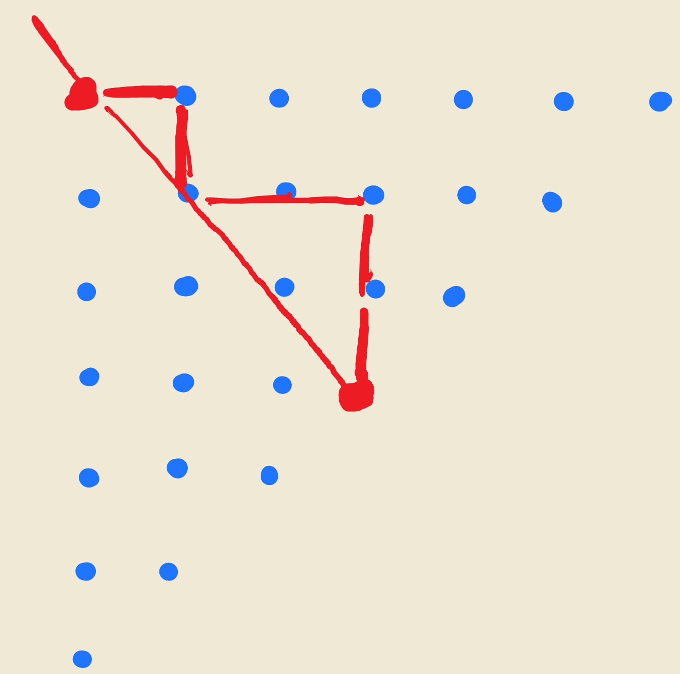
Then

- $\mathbb{E}\left[e^{-u Z_{2N}^{\text{plane}}}\right] = \dfrac{u^{\sum_{i=1}^{N}(\alpha_i+\beta_i)}}{\prod_{i,j\le N}\Gamma(\alpha_i+\beta_j)\prod_{i<j}\Gamma(\alpha_i+\alpha_j)\Gamma(\beta_i+\beta_j)} \times \displaystyle\int_{\mathbb{R}_+^N} e^{-u x_1}\,\Psi_\alpha^{SO_{2N+1}}(x)\,\Psi_\beta^{SO_{2N+1}}(x)\,\frac{dx}{x}$

- $\mathbb{E}\left[e^{-u Z_{2N}^{\text{h-plane}}}\right] = \dfrac{u^{\sum_{i=1}^{N}(\alpha_i+\beta_i)}}{\prod_{i,j\le N}\Gamma(\alpha_i+\beta_j)\prod_{i<j}\Gamma(\alpha_i+\alpha_j)} \times \displaystyle\int_{\mathbb{R}_+^N} e^{-u x_1}\,\Psi_\alpha^{SO_{2N+1}}(x)\,\Psi_\beta^{gl_N}(x)\,\frac{dx}{x}$

Polymers & Littlewood type identities

$Z_N^{\text{r-plane}} :=$ 

$Z_N^{\text{restr.}} :=$ 

$$\mathbb{E}\left[e^{-u Z_{2N}^{\text{r-plane}}}\right] = \frac{u^{\sum_{i=1}^{N}\alpha_i}}{\prod_{i=1}^{N}\Gamma(\alpha_i)\prod_{i,j\le N}\Gamma(\alpha_i+\alpha_j)}\int_{\mathbb{R}_+^N} e^{-u x_1}\,\Psi_\alpha^{so_{2N+1}}(x)\,\frac{dx}{x}$$

$$\mathbb{E}\left[e^{-u Z_{2N}^{\text{restr.}}}\right] = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)\prod_{i<j}\Gamma(\alpha_i+\alpha_j)}\int_{\mathbb{R}_+^N} e^{-\frac{1}{x_N}}\,\Psi_\alpha^{gl_N}(x)\,\frac{dx}{x}$$

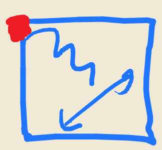
Remark: Setting $u=0$ in the last leads to the Bump-Friedberg-Stade identity

0-temperature or unramified situations

$$\mathbb{E}\left[e^{-u Z_{2N}^{\text{plane}}}\right] = \frac{u^{\sum \alpha_i+\beta_i}}{\Gamma_{\alpha,\beta}} \int_{\mathbb{R}_+^N} e^{-u x_1}\, \Psi_\alpha^{SO_{2N+1}}(x)\, \Psi_\beta^{SO_{2N+1}}(x)\, \frac{dx}{x}$$

$(\alpha,\beta) \to \varepsilon(\alpha,\beta)$, $u \to u/\varepsilon$

$x \to e^{x/\varepsilon}$

$\mathbb{P}\Big(\max_{\pi} \sum$  $\leq u \Big)$

$$\overset{\text{Bisi-Z '17}}{=\!=\!=} \frac{\prod_{i<j}(1+\alpha_i\alpha_j)(1+\beta_i\beta_j)\prod_{i,j}(1+\alpha_i\beta_j)}{u^{\sum \alpha_i+\beta_i}} \times \sum_{\lambda_N \leq \cdots \leq \lambda_1 \leq u} sp_\lambda^{(2N)}(\alpha)\, sp_\lambda^{(2N)}(\beta)$$

Baik-Rains '00 $\Big\|$ $\qquad\qquad$ $\Big\|$ Bisi-Z '19

$$C_{\alpha,\beta} \sum_{\mu_{2N} \leq \cdots \leq \mu_1 \leq u} s_{2\mu}(\alpha_1 \ldots \alpha_N, \beta_1, \ldots, \beta_N) \underset{\text{Bisi-Z '19}}{=\!=\!=} sp_{(u,\ldots,u)}^{(4N)}(\alpha, \beta)$$

Some references

Baik – Rains

Algebraic aspects of increasing subsequences

Bisi – O'Connell – Zygouras

The geometric Burge correspondence & the partition function of polymer replicas

Bisi – Zygouras

Point-to-line polymers & orthogonal Whittaker functions

Bisi – Zygouras

Transition between characters of classical groups, decomposition of Gelfand-Tetlin patterns & last passage percolation

Bump

Automorphic forms on $GL(3,\mathbb{R})$

Ishii – Stade

Archimedian zeta integrals on $GL_n \times GL_n$ & $SO_{2n+1} \times GL_n$

A.N. Kirillov

Introduction to tropical combinatorics

Mucciconi – Petrov

Spin q-Whittaker polynomials & deformed quantum Toda

O'Connell – Seppäläinen – Zygouras

Geometric RSK correspondence, Whittaker functions & symmetrised random polymers

Okada

Applications of minor summation formulas to rectangular shaped representations of classical groups

Stade

Archimedian L-factors on $GL(n) \times GL(n)$ & generalized Barnes integrals

Thanks