Vertex Operators and Solvable Lattice Models

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Report on Brubaker, Buciumas, Bump and Gustafsson

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History

- The Boson Fermion Correspondence had two origins in mathematical physics, dual resonance theory (ancestor of string theory) and Soliton theory.
- Dual resonance theory (Skyrme, Mandelstam, ...) led to the basic representation of affine Lie algebras and Virasoro Lie algebra, with concrete realization by vertex operators.
- Kac-Kazhdan-Lepowsky-Wilson, Frenkel-Kac, ... gave vertex operator representations of affine Lie algebras.
- In a parallel development research in Soliton Theory on the KDV equation and KP heierarchy let to similar algebraic structures. (Sato, Jimbo, Date, Miwa, Kashiwara ...)
- The algebraic formulation treated here was presented by Kac and Raina with an important variant by Kashiwara-Miwa-Stern (KMS) and Lascoux-Leclerc-Thibon (LLT) axiomatized by Thomas Lam.

The Heisenberg Lie algebra

Let $\mathfrak{H}$ be the Heisenberg Lie algebra spanned by B_k ($k \in \mathbb{Z}$) and I such that B_0, I are central and

$$[B_k, B_l] = k\delta_{k,-l} I.$$

Let $\mathfrak{Z} = \mathbb{C}B_0 \oplus \mathbb{C}I$ be the center of $\mathfrak{H}$. Let ℓ be a linear functional on $\mathfrak{Z}$ such that $\ell(I) = 1$. A vector v is called a highest weight vector if $B_k v = 0$ when $k > 0$. The representation is called highest weight if it is generated by a highest weight vector.

Theorem (Algebraic Stone-VN)

$\mathfrak{H}$ has a unique irreducible highest weight representation such that $\mathfrak{Z}$ acts by ℓ.

The Bosonic Fock Space

The ring Λ of symmetric functions has particular elements e_k (the elementary symmetric functions) such that $\Lambda = \mathbb{Z}[e_1, e_2, \cdots]$. As symmetric polynomials (in an infinite number of variables $x_1, x_2, \cdots$)

$$e_k(x_1, x_2, \cdots) = \sum_{i_1 < i_2 < \cdots i_k} x_{i_1} \cdots x_{i_k}.$$

Also let $p_k = \sum x_i^k$ be the power sum symmetric functions and s_λ for partitions λ be the Schur functions.

$\mathbb{C} \otimes \Lambda$ is a polynomial ring in the p_k and an $\mathfrak{H}$-module:

$$B_{-k} = \textit{multiplication by } p_k \ (k > 0), \qquad B_k = k\,\frac{\partial}{\partial p_k}.$$

The action of B_0 may be chosen at convenience.

The Dirac Sea

The Dirac equation for the electron is a Hamiltonian for a spin 1/2 particle. The energy levels are quantized, but can be arbitrarily small. This is a problem, since theoretically the electron could radiate an arbitrary amount of energy by dropping into lower and lower energy levels.

Dirac proposed that all sufficiently negative energy states are occupied. This prevents the problem since the electron is a fermion, so no state can be occupied by more than one particle.

$$\begin{array}{ccccccccccccc} \cdots & \circ & \circ & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \bullet & \cdots \\ & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & \end{array}$$

unoccupied states ($\circ$) occupied states ($\bullet$)

The Fermionic Fock Space

The Hilbert space modeling this situation is the fermionic Fock space. Let V be a vector space with basis u_i ($i \in \mathbb{Z}$) representing states of energy i. Then the Fock space $\mathfrak{F}_m$ of charge m consists of semi-infinite monomials

$$u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots$$

where $i_k = k$ for sufficiently negative k.

This is interpreted as zero if any vector u_i appears twice. It is understood that the sign changes if we switch two basis vectors so we may assume that

$$i_m > i_{m-1} > \cdots .$$

States are indexed by partitions

If λ is a partition, let

$$u_\lambda = |\lambda\rangle := u_{\lambda_1+m} \wedge u_{\lambda_2+m-1} \wedge \cdots .$$

These give a basis of $\mathfrak{F}_m$. We represent this as follows:

	Energy level	5	4	3	2	1	0	-1	-2	-3	-4	-5
$\|(1,1)\rangle =$		$\circ$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\bullet$

This is a vector in $\mathfrak{F}_{-1}$.

Vacua and operators

If $i_k = k$ for all k we get the vacuum vector

$$|m\rangle = u_m \wedge u_{m-1} \wedge \cdots$$

$$
\begin{array}{cccccccccccccc}
|0\rangle = & \cdots & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
 & & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 &
\end{array}
$$

More energetic states can be obtained from the vacuum by applying boson operators $\mathfrak{F}_m \to \mathfrak{F}_m$ that move a particle to a higher energy level.

Furthermore there are fermionic creation and annihilation operators $\mathfrak{F}_m \to \mathfrak{F}_{m\pm 1}$ that create or destroy a particle.

Boson operators

Let k be a positive or negative integer. Let B_k be an operator that adds $-k$ to the energy of a state.

$$B_k(u_i) = u_{i-k}.$$

Applying this to a semiinfinite monomial gives

$$B_k(u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots) = \sum_i u_{i_m} \wedge \cdots \wedge B_k(u_{i_{m-i}}) \wedge \cdots \ .$$

For all but finitely many i, the term on the right has a repeated factor, so this is a finite sum. For example:

$$B_{-2}|0\rangle = u_2 \wedge u_{-1} \wedge u_{-2} \wedge \cdots - u_1 \wedge u_0 \wedge u_{-2} \wedge \cdots$$

The Boson-Fermion correspondence

We make B_0 act by the scalar m on both $\mathfrak{F}_m$ and the bosonic Fock space $\mathfrak{B}$; with this $\mathfrak{H}$-module action we denote $\mathfrak{B}$ as $\mathfrak{B}_m$. By the algebraic SVN theorem there is a unique isomorphism

$$\sigma_m : \mathfrak{F}_m \to \mathfrak{B}_m$$

that sends $|0\rangle$ to $1 \in \Lambda = \mathfrak{B}_m$. This is the Boson-Fermion Correspondence of charge m.

Theorem (See Kac-Raina, Chapter 6)

We have $\sigma(u_\lambda) = s_\lambda$.

Creation and annihilation operators

If $v \in V$ define $\hat{v} : \mathfrak{F}_m \to \mathfrak{F}_{m+1}$:

$$\hat{v} : u_{i_m} \wedge u_{i_{m-1}} \wedge = v \wedge u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots .$$

Similarly let $v^* \in V^*$, and assume that $v^*(u_i) = 0$ for all but finitely many i. Define $\check{v} : \mathfrak{F}_m \to \mathfrak{F}_{m-1}$:

$$\check{v}^*(u_{i_m} \wedge u_{i_{m-1}} \wedge) = \sum_{k=0}^{\infty} (-1)^k u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \wedge v^*(u_{i-k}) \wedge \cdots .$$

The other part of the BFC identifies the effect of these operators on the Bosonic Fock space. To formulate these elegantly we package the vectors u_i in a formal series:

$$X(t) = \sum_{i \in \mathbb{Z}} t^j \hat{u}_j, \qquad X^*(t) = \sum_{i \in \mathbb{Z}} t^{-j} \check{u}_j^*$$

where u_j^* is the dual basis of V^*.

Vertex operators

Transferring the operators $X(t)$ and $X^*(t)$ to maps $\mathfrak{B}_m \to \mathfrak{B}_{m\pm 1}$ by the BFG gives operators

$$t^{m+1} \exp\left(\sum_{j=1}^{\infty} t^j p_j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{t^{-j}}{j} \frac{\partial}{\partial p_j}\right)$$

$$t^{-m} \exp\left(\sum_{j=1}^{\infty} -t^j p_j\right) \exp\left(\sum_{j=1}^{\infty} \frac{t^{-j}}{j} \frac{\partial}{\partial p_j}\right)$$

Operators like this are called vertex operators. Call

$$\exp\left(\sum_{j=1}^{\infty} t^j p_j\right), \qquad \exp\left(-\sum_{j=1}^{\infty} \frac{t^{-j}}{j} \frac{\partial}{\partial p_j}\right)$$

half-vertex operators. (Only the second is really an operator.)

Problem of noncommutativity

These "operators" are avatars of

$$\exp\left(\sum_{j=1}^{\infty} t^j B_{-j}\right), \qquad \exp\left(-\sum_{j=1}^{\infty} t^{-j} B_j\right).$$

If the B_j commuted, we could write

$$\exp\left(\sum_{j=-\infty}^{\infty} t^{-j} B_j\right).$$

However the operators B_j only almost commute:

$$[B_j, B_k] = j\delta_{j,-k} I.$$

So the operators only commute "up to scalar." There are problems of convergence here, hence the quotation marks.

Normal ordering

When we write

$$\exp\left(\sum_{j=1}^{\infty} t^j B_{-j}\right) \exp\left(-\sum_{j=1}^{\infty} t^{-j} B_j\right)$$

we are following the normal order practice of applying annihilation operators B_j before creation operators B_{-j}. This is common in quantum mechanics, where changing a Hermitian operator by a scalar does not change its meaning, but where trying to make the scalar explicit leads to divergences. The solution is to arrange the terms in an infinite expression in such a way such that for all but finitely many terms, annihilation operators are done before creation operators.

Convergence issue

Do

$$\exp\left(\sum_{j=1}^{\infty} t^j B_{-j}\right) \qquad \text{and} \qquad \exp\left(-\sum_{j=1}^{\infty} t^{-j} B_j\right)$$

make sense as operators? The second clearly does. Since the operators B_j ($j > 0$) commute, we may formally expand the exponential, resulting in an infinite number of terms, each involving a product of B_j. All but finitely many annihilate $|\lambda\rangle$ so

$$\exp\left(-\sum_{j=1}^{\infty} t^{-j} B_j\right) |\lambda\rangle$$

is a well defined state.

Convergence issue (continued)

We denote by

$$\langle\mu|\exp\left(-\sum_{j=1}^{\infty}t^{-j}B_j\right)|\lambda\rangle$$

the coefficient of $|\mu\rangle$ in $\exp(-\sum t^{-j}B_j)|\lambda\rangle$. By contrast

$$\exp\left(\sum_{j=1}^{\infty}t^{j}B_{-j}\right)|\lambda\rangle$$

is an infinite sum that is not a state. But

$$\langle\mu|\exp\left(\sum_{j=1}^{\infty}t^{j}B_{-j}\right)|\lambda\rangle$$

still does make sense.

Clifford algebra

Denote $\psi_i^* = \hat{v}_i$, $\psi_i = \check{v}_i^*$. These fermionic operators satisfy

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$$

while

$$\psi_i \psi_j + \psi_j \psi_i = 0,$$

$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0.$$

hence form an infinite-dimensional Clifford algebra. It is possible to either make this central in the theory or to avoid them. The B_k in terms of the Clifford algebra:

$$B_k = \sum_{i \in \mathbb{Z}} \psi_k^* \psi_{i+k}, \qquad k \neq 0,$$

$$B_0 = \sum_i : \psi_i^* \psi_i := \sum_{i>0} \psi_i^* \psi_i - \sum_{i \leqslant 0} \psi_i \psi_i^*.$$

Generalizations

- Kashiwara, Miwa and Stern defined a q-Fock space that is a module for $U_q(\widehat{sl}_n)$. There is a Boson-Fermion Correspondence. LLT used the q-Fock space to define ribbon symmetric functions (also called LLT polynomials. These are related to metaplectic Whittaker functions (BBBG).
- Thomas Lam gave an axiomatic theory that applies to Hall-Littlewood, Macdonald and LLT polynomials. His theory includes a Cauchy identity and a Pieri rule.

In the q-deformed theory (KMS) the Clifford algebra has been so far avoided. BBBG used the creation operators ψ_i^* but avoided the ψ_i.

Vertex Algebras

Inspired by the Frenkel-Kac description of the basic representation of an affine Lie algebra by vertex operators, Borcherds gave a purely algebraic context for vertex operators in vertex operator algebras. The theory was streamlined by Kac and others.

Vertex algebras give a rigorous and purely algebraic setting for the results of conformal field theory.

There are vertex algebras associated with the Virasoro algebra, The Heisenberg algebra and other setups. The ones most resembling the operators encountered here are lattice vertex algebras. The lattice in this case is $\mathbb{Z}$.

The basic representation

Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra, Λ_i the fundamental weights. The affine fundamental weight Λ_0 is the highest weight of an integrable representation, the basic representation. It arises in different contexts, from string theory to the modular representation theory of the symmetric group.

- The basic representation of an affine Lie algebra appeared in string theory and soliton theory.
- A quantum version was studied by Hayashi, Misra-Miwa.
- The KMS Fock space was developed by Kashiwara, Miwa and Stern. They produced an action of $U_q(\mathfrak{sl}_n)$ and Boson operators B_n that are $U_q(\mathfrak{sl}_n)$-module homomorphisms. This is a realization of the basic representation.
- LLT connected the crystal of the basic representation to the modular representations of the symmetric group. Meanwhile Jing-Misra describe similar vertex operators.

The Kashiwara-Miwa-Stern Fock Space

The KMS Fock space is a generalization of the fermionic Fock space. It depends on a positive integer n and a deformation parameter v. It is a module for the quantum group $U_{v^{1/2}}(\widehat{sl}_n)$.

$$J_k(u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots) = (u_{i_m - nk} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots)$$

$$+(u_{i_m} \wedge u_{i_{m-2}-nk} \wedge u_{i_{m-2}} \wedge \cdots) + \cdots \ .$$

We have to explain how to interchange $u_l \wedge u_m$ when $l < m$ since this is more complicated than just a sign.

Interchanging vectors

We slightly generalize the KMS setup by introducing parameters $g(a)$ that are periodic mod n such that $g(0) = -v$ and $g(a)g(-a) = v$ if $a \not\equiv 0$ mod n. In an application to metaplectic Whittaker functions $g(a)$ is a Gauss sum. If $k < m$ define $u_l \wedge u_m$ to be

$$-u_m \wedge u_l$$

if $l \equiv m$ modulo n,

$$g(l-m)u_m \wedge u_l + (v-1)(u_{m-i} \wedge u_{l+i} + g(l-m)u_{m-n} \wedge u_{l+n}$$

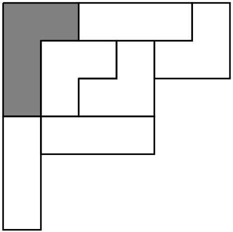
$$+vu_{m-n-i} \wedge u_{l+n+i} + v\,g(l-m)u_{m-2n} \wedge u_{l+2n} + \cdots)$$

otherwise. Here i is the unique value with $0 < i < n$ and $m - i \equiv l$ modulo n. The summation continues as long as the terms are of the form $u_a \wedge u_b$ with $a > b$; this is a finite sum.

Ribbons and *n*-cores

An *n*-ribbon is a connected skew shape λ/μ of size *n* that contains no 2×2 square. Given a partition λ, we may remove *n*-ribbons until no more removals are possible. The resulting shape δ is called the *n*-core of λ. If $\lambda = \delta$ then λ is called an *n*-core.

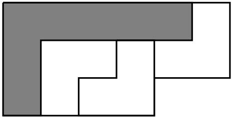
$$\begin{aligned} n &= 3 \\ \lambda &= (6,6,4,4,1,1), \\ \delta &= (2,1,1) \end{aligned}$$



The spin of an *n*-ribbon strip is its height (number of rows) minus 1.

Horizontal *n*-ribbon strips

A skew shape λ/mu is called a horizontal *n*-ribbon strip if it can be decomposed into *n*-ribbon strips such that the top-most right box of each lies on the top row of the skew shape λ/μ. Thus if $n = 3$ then $\lambda = (6, 6, 4)$ and $\mu = (5, 1, 1)$ works for this definition.

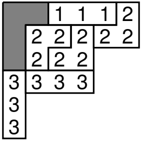


Ribbon skew tableaux

An *n-ribbon skew tableau T* of shape λ/μ is a sequence of partitions

$$\mu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda, \tag{1}$$

where α^{i+1}/α^i is a horizontal *n*-ribbon strip. We may associate with such data a tableau in which the strip α^{i+1}/α^i is filled with *i*'s. The weight will then be $(\nu_1, \cdots, \nu_r)$ where ν_i is α^{i+1}/α^i divided by *n*. The spin is the sum of the spins of the constituent ribbons.



Here is a 3-ribbon tableau with spin 5 and weight $(1, 3, 2)$.

LLT Polynomials

Define the LLT or ribbon symmetric function by

$$\mathcal{G}^{n}_{\lambda/\mu}(\mathbf{z}) = \mathcal{G}^{n}_{\lambda/\mu}(\mathbf{z}; q) = \sum_{T} q^{s(T)} \mathbf{z}^{\mathrm{weight}(T)}.$$

They have Cauchy identities and Pieri formulas.

LLT Polynomials as 2-point functions for half-vertex operators

The operators J_k satisfy

$$[J_k, J_l] = \delta_{k,-l} \cdot k \frac{1 - v^{n|k|}}{1 - v^{|k|}}.$$

Fix $\mathbf{z} = (z_1, \cdots, z_r)$. It may be shown that

$$\mathcal{G}^n_{\lambda/\mu}(\mathbf{z}) = \langle \mu | e^{L_+(\mathbf{z})} | \lambda \rangle, \qquad L_+(\mathbf{z}) = \sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{z}) \, J_k,$$

$$|\lambda\rangle = u_{\lambda_1} \wedge u_{\lambda_2 - 1} \wedge u_{\lambda_3 - 2} \wedge \cdots \in \mathfrak{F}_0.$$

(As before $\mathfrak{F}_0$ is the charge 0 part of the KMS Fock space.) If $\mu = 0$ and if we suppress the $g(a)$, then $\mathcal{G}_\lambda$ is called a ribbon symmetric function or LLT polynomial. It can be expressed as a sum of tableaux and is zero unless λ is partition with empty n-core .

A variant

We modify the operators $L_+(\mathbf{z}) = \sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{z})\, J_k$ and define

$$H_+(\mathbf{z}) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - v^k) p_{nk}(\mathbf{z}) J_k.$$

Note that $p_{nk}(\mathbf{z}) = p_k(\mathbf{z}^n)$, so the change from p_k to p_{nk} is unimportant. What is new here is the factor $1 - v^k$. We can consider $\langle\mu|e^{H_+(\mathbf{z})}|\lambda\rangle$, and these produce polynomials that do fit in Lam's framework. They are a special case of what he calls super ribbon functions.

The operators $H_\pm(z)$

We may factor

$$H_+(\mathbf{z}) = \sum H_+(z), \qquad H_+(z) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - v^k) z^{nk} J_k.$$

and similarly

$$H_-(z) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - v^k) z^{-nk} J_{-k}.$$

Since the J_k ($k > 0$) commute, the operators $H_+(z)$ form a commuting family as z varies, as do the operators $H_-(z)$. However $H_+(z)$ does not commute with $H_-(w)$.

The edge paradigm

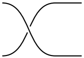
The usual paradigm in lattice models is to associate to each edge in a planar graph a module over a quantum group. Then there will be plenty of Yang-Baxter equations to prove integrability of the model.

We consider the models whose partition functions include spherical metaplectic Whittaker functions, found by Brubaker, Buciumas and Bump (2016). We will reinterpret these as colored models as in recent papers.

The R-matrices tell us that the quantum group in this case should be the affine super group $U_q(\widehat{gl}(1|n))$. Indeed, the horizontal edges of the model can be identified with the $n+1$-dimensional graded standard modules, which depend on a spectral parameter $z \in \mathbb{C}^\times$. This accounts for the YBE.

The mystery of the vertical edges

But the R-matrix only attaches to a pair of horizontal edges.



So the R-matrix does not “see” the vertical edges.

However the horizontal edges do see the vertical edges, and we would like to interpret the vertices of the grid themselves as R-matrices.

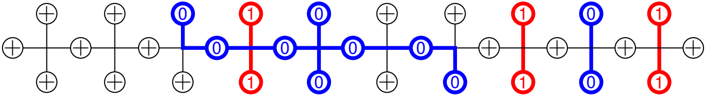
What is missing is interpretation of the vertical edges of the model as modules for a quantum group. We will propose a kind of solution to this dilemma by interpreting the sequence of vertical edges in an infinitely wide grid as the KMS Fock space. A priori this is a $U_q(\widehat{gl}(n))$ module only.

Precursors

The key will be a relationship between the half vertex operators and row transfer matrices for a lattice model. These operators are q-deformations of $\widehat{\mathfrak{gl}}(1|n)$ Hamiltonians. It is probably possible to upgrade this story to $\widehat{\mathfrak{gl}}(r|n)$. Corresponding undeformed vertex operators were studied by Kac and van de Leur.

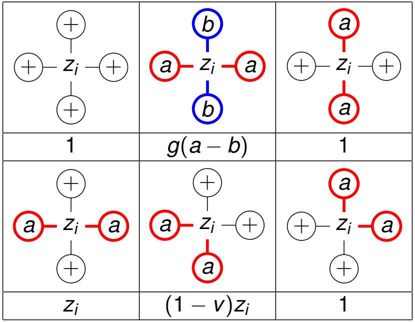
- Jimbo and Miwa were aware of relationships between 6-vertex model and vertex operators in their work on Soliton theory.
- For $\widehat{\mathfrak{gl}}(1|1)$ but without the q deformation such relations were found by Brubaker and Schultz. Similar facts were found by P. Zinn-Justin.
- The relationship between the XYZ Hamiltonian and the eight vertex model in Baxter's work is of the same nature.

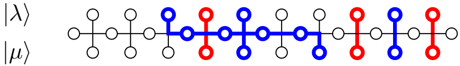
States of the model



In the model, we have a grid with r rows and infinitely many columns extending to the left and right. We have n colors identified with the integers $0 \leqslant i < n$. The allowed spins are either a color, or $+$. In a state, each edge is assigned a spin. The vertical edge in column k is only allowed two possible spins, either $+$ or the color i that is $\equiv k \bmod n$. In this example, $n = 2$. The colors are blue (0) and red (1). Only finitely many horizontal edges are colored. But every vertical edge sufficiently far to the right (column $\ll 0$) is colored and every column sufficiently far to the left (column $\gg 0$) is uncolored ($+$).

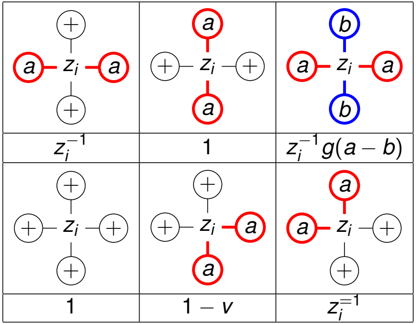
Delta Ice

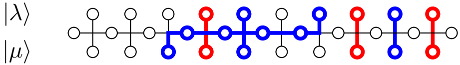




With these Boltzmann weights, let $T_\Delta(z_i)$ be the row transfer matrix. This is a “right moving” operator like $\exp(\sum k^{-1} J_k)$.

Gamma Ice





With these Boltzmann weights, let $T_\Gamma(z_i)$ be the row transfer matrix. This is a “left moving” operator like $\exp(\sum k^{-1} J_{-k})$.

Main Theorem

As with the vertex operators, the row transfer matrix $T_\Delta|\lambda\rangle$ produces a finite sum of states but $T_\Gamma|\mu\rangle$ produces an infinite sum. However $\langle\mu|T_\Gamma|\lambda\rangle$ is a finite sum. We recall

$$H_\pm(z) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - v^k) z^{\pm nk} J_{\pm k}.$$

Theorem (Main Theorem)

$$T_\Delta(z) = \exp(H_+(z)), \qquad T_\Gamma(z) = \exp(H_-(z)).$$

Proof

We introduce the fermionic operators $\psi_k^* : \mathfrak{F}_m \to \mathfrak{F}_{m+1}$:

$$\psi_k(\xi) = u_k \wedge \xi.$$

Also let $\rho_k^*(z) = \psi_k^* - z\psi_{k-n}^*$. It may be checked that

$$[J_k, \psi_j^*] = \psi_{j-nk}^*$$

and a little algebra based on the Baker-Campbell-Hausdorff formula then shows that

$$e^{H_+(z)}\rho_k^* e^{-H_+(z)} = \rho_k^*(vz^n).$$

Proof, continued

The key step is to show that $T_\Delta(z)$ satisfies the same identity

$$T_\Delta(z)\rho_k^* = \rho_k^*(vz^n)T_\Delta(z).$$

This may be translated into a relationship between partition functions. Cutting the row apart, it is sufficient to check a linear relationship between (up to) four terms which are partition functions of the finite system

$$\begin{array}{cccccc} & \color{red}{\varepsilon_k} & \varepsilon_{k-1} & \cdots & \color{red}{\varepsilon_{k-n}} & \\ a & & & & & b \\ & \color{red}{\delta_k} & \delta_{k-1} & \cdots & \color{red}{\delta_{k-n}} & \end{array}$$

Only the spins at the red-colored locations change between the terms (due to the application of ρ_k^*.) This may be checked on a case-by-case basis.