Lattice models as Poincaré pairings

(aka "everything is geometry"
aka "yet another interpretation of the lattice models for Whittaker functions")

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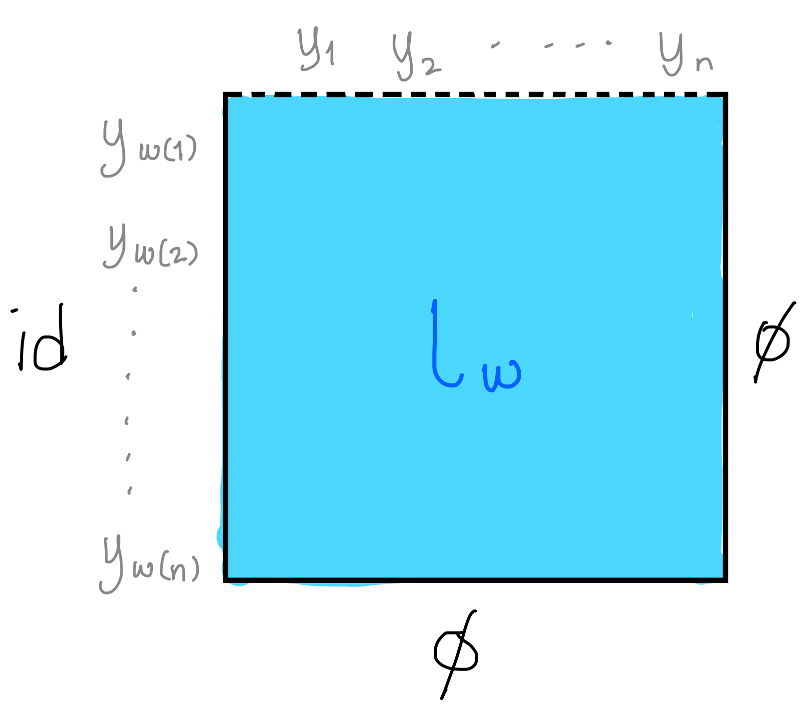
Outline

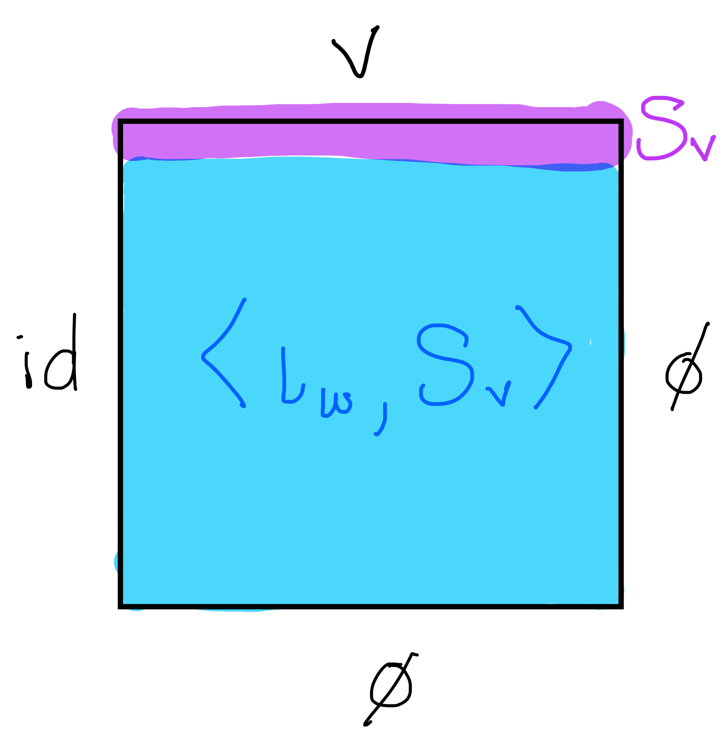
① Bird's eye view:

bases of integrable systems = bases in p-adic rep theory = bases in equivariant cohomology

② Some equivariant geometry

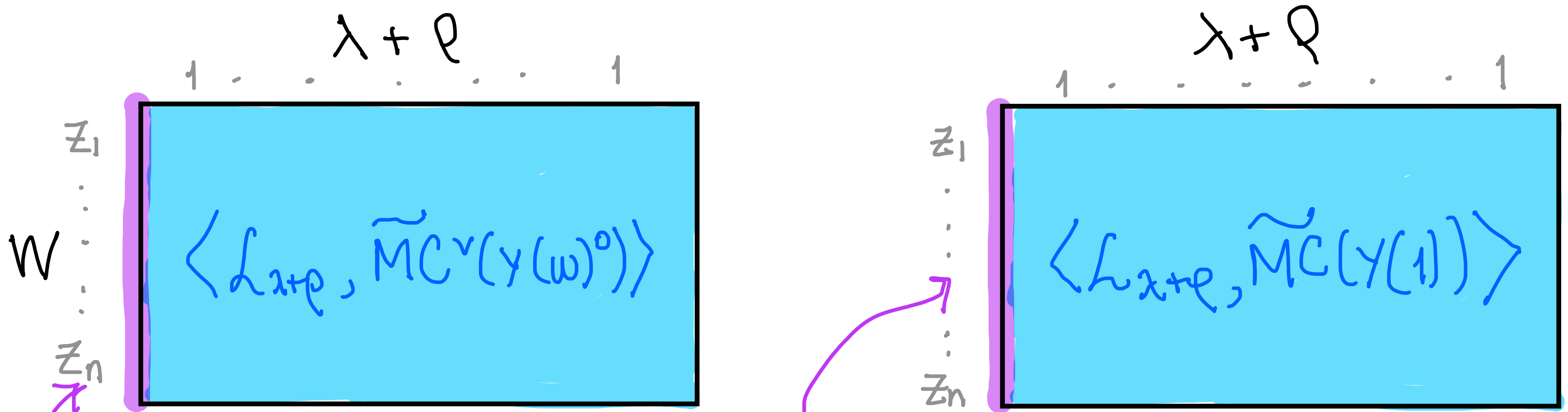
③ Example: Frozen pipes model for Schubert polynomials





④ Example: Colored & uncolored Whittaker lattice models

<u>Iwahori</u> | <u>Spherical</u>



this vector $\leftrightarrow$ Poincaré dual of $\widetilde{MC}^\vee(Y(\omega)^\circ)$ / $\widetilde{MC}(Y(1))$

⑤ Future directions?

① Bird's eye view of

Schubert Calculus ⟿ integrable systems

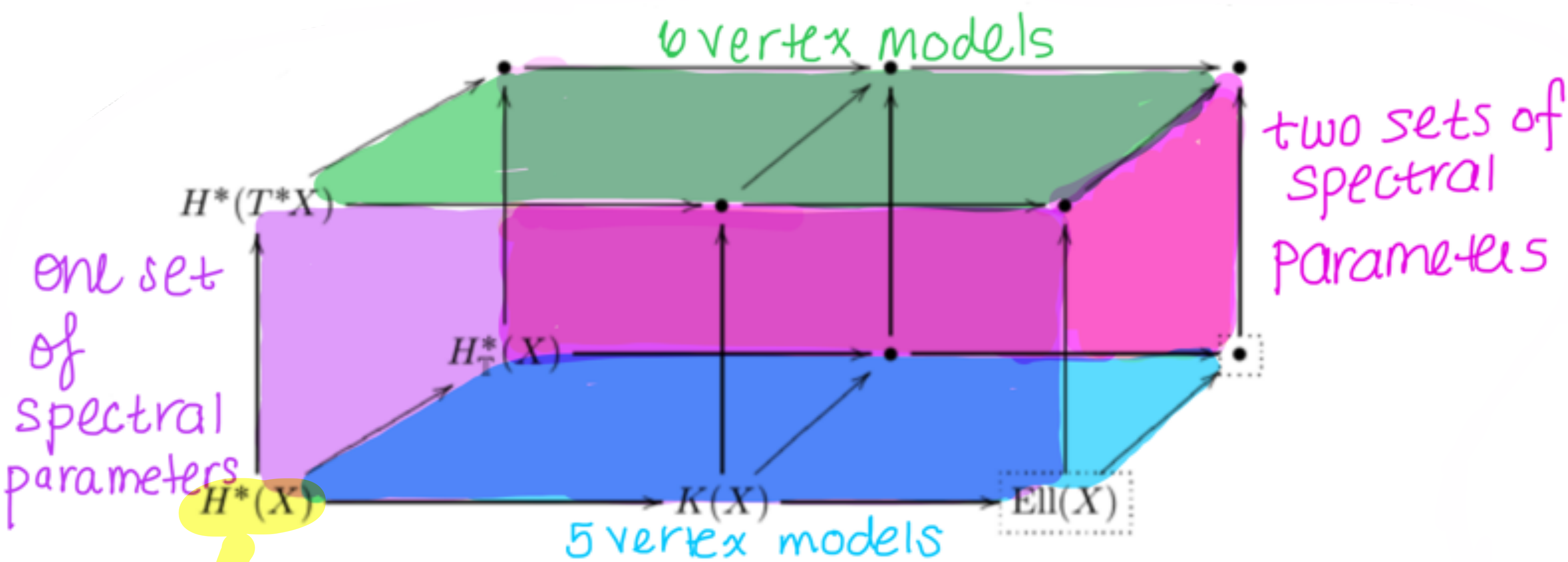


FIGURE 1. Three "orthogonal" directions to generalize classical Schubert calculus

(not pictured: quantum Schubert calculus)

[Source: Rimanyi, "h-deformed Schubert Calculus in equivariant cohomology, K-theory, & elliptic cohomology"]

Generalized equivariant cohomology (H_T^*, K_T, Ell_T)	$\longleftrightarrow$	Bethe algebra of quantum integrable systems

- fixed point basis (easy) $\longleftrightarrow$ • Bethe basis (hard)

- geometric basis $\longleftrightarrow$ • spin basis (easy)

(Schubert classes / $\boxed{\text{stable envelopes}}$) (hard)

defined by Maulik & Okounkov for general Nakajima quiver varieties — in particular, for $X = T^*(G/B)$ (cotangent bundle)

2 Some equivariant geometry

$$X \circlearrowleft \Gamma$$

topological space — group action

Def (Γ-equivariant cohomology)

If Γ acts freely on X,

$$H^*_\Gamma(X) := H^*(X/\Gamma)$$

Otherwise, let $E\Gamma$ be a contractible space w/ free Γ-action. Then

$$H^*_\Gamma(X) := H^*((X \times E\Gamma)/\Gamma).$$

Note that $H^*_\Gamma(pt) = H^*(\underbrace{E\Gamma/\Gamma}_{})$ is not necessarily trivial! (= $B\Gamma$, classifying space of Γ)

For this talk, $\Gamma = T = (\mathbb{C}^\times)^n$, max'l torus of $G := GL_n(\mathbb{C})$.

First let $n=1$. Then we can take

$$ET = \left\{ (z_i)_{i>0} \;\middle|\; \begin{array}{l} z_i \in \mathbb{C}, \text{finitely many} \\ z_i \neq 0 \end{array} \right\}$$

so

$$ET/T = \mathbb{CP}^\infty \quad \& \quad H_T(pt) = \mathbb{Z}[y]$$

For general n,

$$H_T(pt) = \mathbb{Z}[y_1, \ldots, y_n].$$

Consider

$$\pi : X \longrightarrow \{pt\}$$

This induces

$$\pi^* : H_\Gamma^*(pt) \longrightarrow H_\Gamma^*(X)$$

(so $H_\Gamma^*(X)$ is a $H_\Gamma^*(pt)$-module)

& (by Poincaré duality)

$$\pi_* : H_\Gamma^*(X) \longrightarrow H_\Gamma^*(pt)$$

which we use to define the

Poincaré / intersection pairing:

$$\langle a, b \rangle = \pi_* (a \underset{\text{cup prod}}{\cup} b)$$

Equivariant K-theory

similar, but now:

- $K_\Gamma(X)$ consists of equivariant vector bundles $E \to X$ (or, equivalently if X smooth, equivariant sheaves)

- $K_\Gamma(pt) \cong \mathrm{Rep}(\Gamma)$

$$\left(\text{so } K_T(pt) = \mathbb{Z}[\underbrace{e^{\pm t_1}, \ldots, e^{\pm t_r}}_{\text{characters} \leftrightarrow \text{basis of } \mathrm{Lie}(T)}]\right)$$

Advantage of equivariant-ness:

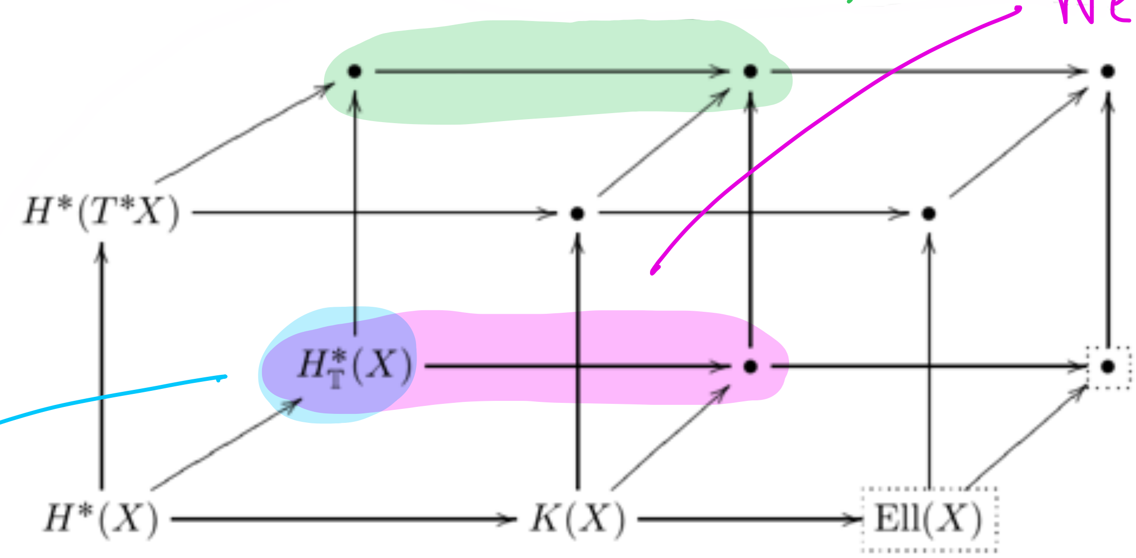
$\boxed{\text{Often, all the information about } H^*_\Gamma(X) \text{ is captured by the cohomology of the fixed point locus } X^\Gamma.}$

③ Example: Frozen pipes model

(exposition heavily influenced by Zinn-Justin, "Lectures on Geometry, Quantum Integrability, & Symmetric functions")

(can extend to)

we are here (+ connective K-theory)



focus today (simplest case)

FIGURE 1. Three "orthogonal" directions to generalize classical Schubert calculus

Let $G = GL_n$ $\quad B = \begin{pmatrix} * & \\ 0 & \end{pmatrix}$ (upper triangular)

$T \cong (\mathbb{C}^\times)^n$ $\quad X = G/B$ flag variety

Then

$$X = \bigsqcup_{w \in S_n} C_w \quad \text{(Schubert cell / Bruhat decomp)}$$

$C_w = BwB/B$

$$X_w := \overline{C_w} = \bigsqcup_{v \leq w} C_v$$ is a Schubert variety

The Schubert classes $\mathfrak{S}_w := [X_w]$ form an additive basis of $H_T^*(X)$

&

$$H_T^*(X) \cong \frac{\mathbb{C}[x_1, \ldots, x_n, \overbrace{y_1, \ldots, y_n}^{\text{"equivariant parameters"}}]}{\langle f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) \mid f \in H_G(pt) \rangle}$$

In this $\cong$, $[X_w] \longrightarrow$ Schubert polynomial

(K-theory: Grothendieck poly

connective K-theory: β-Grothendieck poly)

In the equivariant theory, also have basis of <u>twisted Schubert classes</u> for every $v \in S_n$:

$$\mathfrak{S}_w^{(v)} := [\, v X_w v^{-1} \,]$$

T- fixed points: coordinate flags

$$F_w := wB/B \quad \left(\leftrightarrow 0 \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \dots \subset \mathbb{C}^n \right)$$

Localization

$$\tilde{H}^*_T(X) := \text{localize } H^*_T(X) \text{ at } H_T(pt)$$

Thm The classes $l_w := [F_w]$ form a basis of $\tilde{H}^*_T(X)$ as a vector space over $\tilde{H}^*_T(pt)$.

How to actually decompose in this basis?

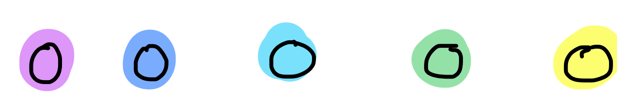
Thm

$$[X] = \sum_{w \in S_n} \frac{[F_w]}{\prod \text{weights of } T \text{ acting on } \boxed{T_{F_w} X}}$$

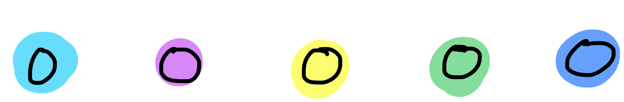
(tangent space @ F_w)

Now, how are we going to get a lattice model out of this??

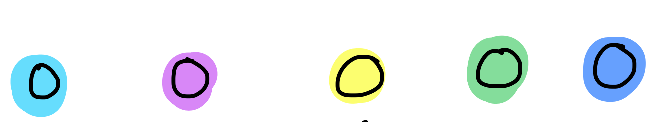
Fix an n-dim'l vector ("ket") of colors

$\vec{c} = |$  $\rangle \in V^{\otimes n}$

Then

$|$  $\rangle \longleftrightarrow S_w$

$w\vec{c}$

$\langle$  $| \longleftrightarrow$ Poincaré dual to S_w $\left(= S_w^{(w_0)}\right)$

$w\vec{c}$

& the Boltzmann weights will come from the change of basis matrix, R_v, between $\{S_w\}$ & $\{S_w^{(v)}\}$:

$$S_w = \sum_{u \in S_n} (R_v)_{uw} \, S_u^{(v)}$$

Calculating R_{s_i}, $s_i = (i, i+1)$:

Let
$$P_i = \left\{ \begin{array}{cc} & i \\ i+1 & \begin{pmatrix} \ddots & & * \\ & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \\ & & \ddots \end{pmatrix} \end{array} \right\}$$

Consider

$$\begin{tikzcd} & P_i \times_B X_w \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_w & & P_i/B \cong \mathbb{P}^1 \end{tikzcd}$$

two fixed pts, $[1] \longleftrightarrow$ & $[s_i] \longleftrightarrow$

Then by the localization formula:

$$[\mathbb{P}^1] = \frac{[1]}{y_{i+1}-y_i} + \frac{[s_i]}{y_i - y_{i+1}}$$

*divided difference operator! (in general, this op is dual to the one usually defined on the x-variables to define Schub/Grot polynomials)

So since

$$f_* g^* [1] = [X_w]$$

$$f_* g^* [s_i] = [s_i \cdot X_w]$$

$$f_* g^* [\mathbb{P}^1] = \begin{cases} [P_i \cdot X_w] & \text{if } \dim P_i \cdot X_w = \dim X_w + 1 \\ 0 & \text{else} \end{cases}$$

we have

$$[X_w] = \underbrace{[s_i \cdot X_w]}_{= \mathfrak{S}_w^{(s_i)}} + (y_{i+1} - y_i) \underbrace{[P_i \cdot X_w]}_{= \mathfrak{S}_{s_i(w)}^{(s_i)}} \delta_{\dim P_i X,\, \dim X + 1}$$

if $wc_i < wc_{i+1}$

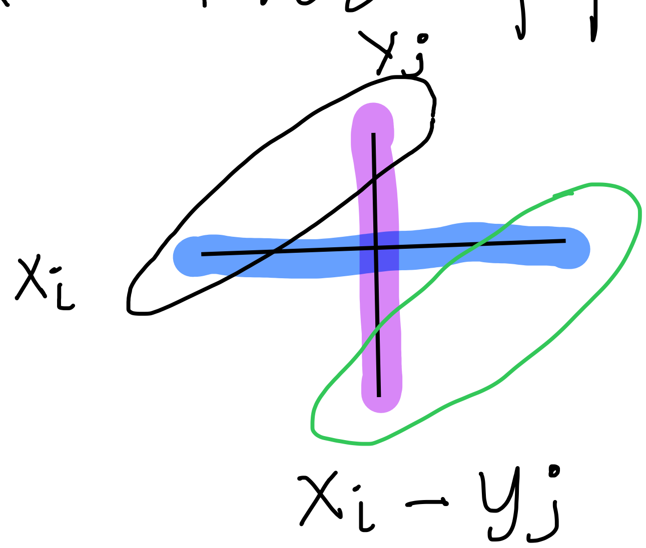
So R_{s_i} acts nontrivially only on the $i, i+1$ vector entries. If ● < ● are the colors appearing in these spots, this nontrivial piece of R_{s_i} has the form:

$$\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & y_{i+1}-y_i & \\ & & & 1 \end{pmatrix}$$

spectral parameters

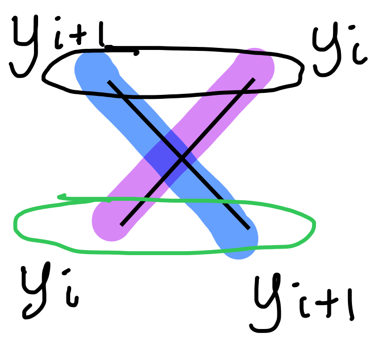
$\longleftrightarrow$ Frozen pipes Boltzmann weights, e.g.

(with $\beta = 0$, $y_j \mapsto -y_j$)



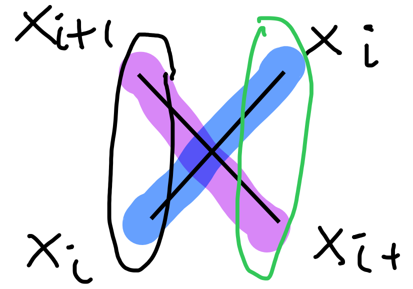
& the row/column R-matrix weights.

column



$y_{i+1} - y_i$

row

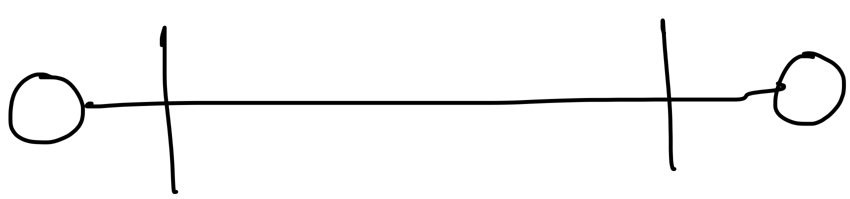


$x_{i+1} - x_i$

So... one way to analyze this lattice model is by applying operators to Schubert classes.

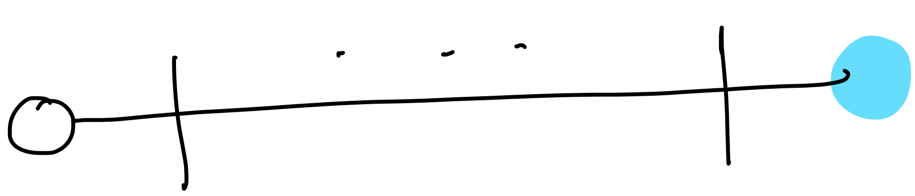
Another way: algebraic Bethe ansatz

Think of one row of a lattice as an operator



"A" operator

"B" operator



"C" operator

"D" operator

Yang Baxter eqn $\Longrightarrow$ commutation relations between these.

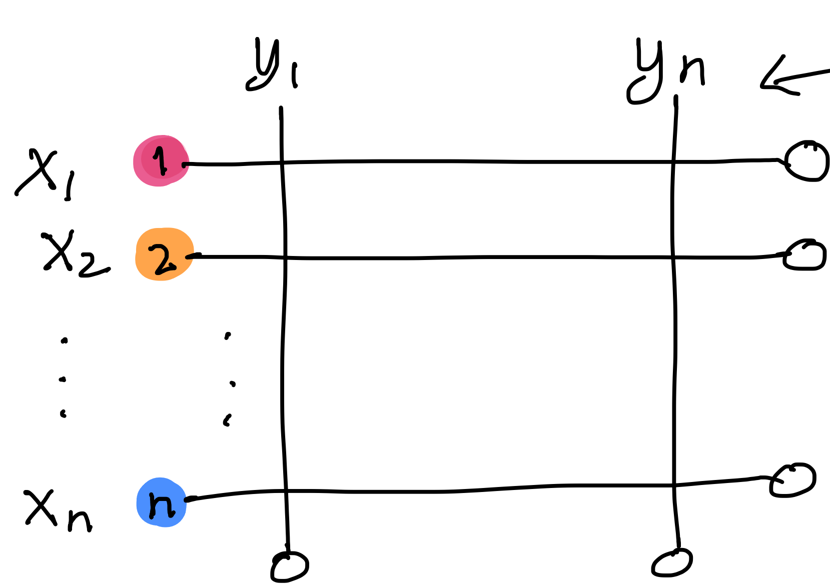
The operators + their relations generate a "Yang Baxter algebra"

(~ degeneration of Yangian / quantum gp; special case of Maulik Okounkov construction)

<u>Bethe ansatz</u>: Look @

$$B_1(x_1)\cdots B_n(x_n)|\phi\rangle$$

$=$

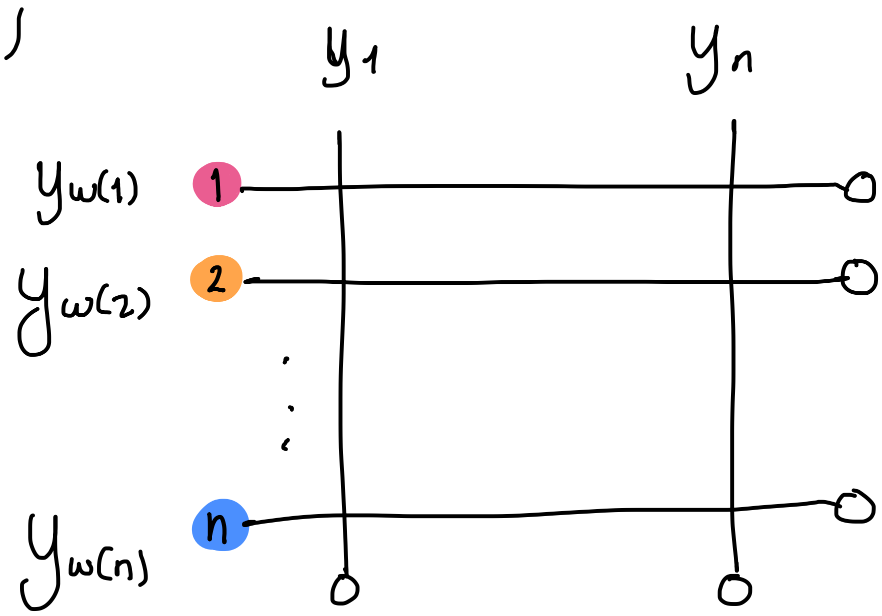


top boundary not specified yet

Use commutations to find conditions on $x_1, \dots, x_n$ so that this is an eigenvector of the transfer matrix $T(u) = \sum_{i=1}^{n} D_{ii}(u)$

$\rightsquigarrow$ need $\{x_1, \dots, x_n\}$ to be a permutation of $\{y_1, \dots, y_n\}$

Further,



$\longleftrightarrow$ ℓ_w fixed pt class

& the partition function is the expansion of ℓ_w into S_v's: $\sum_{v \in S_n} \langle \ell_w, S_v \rangle S_v$

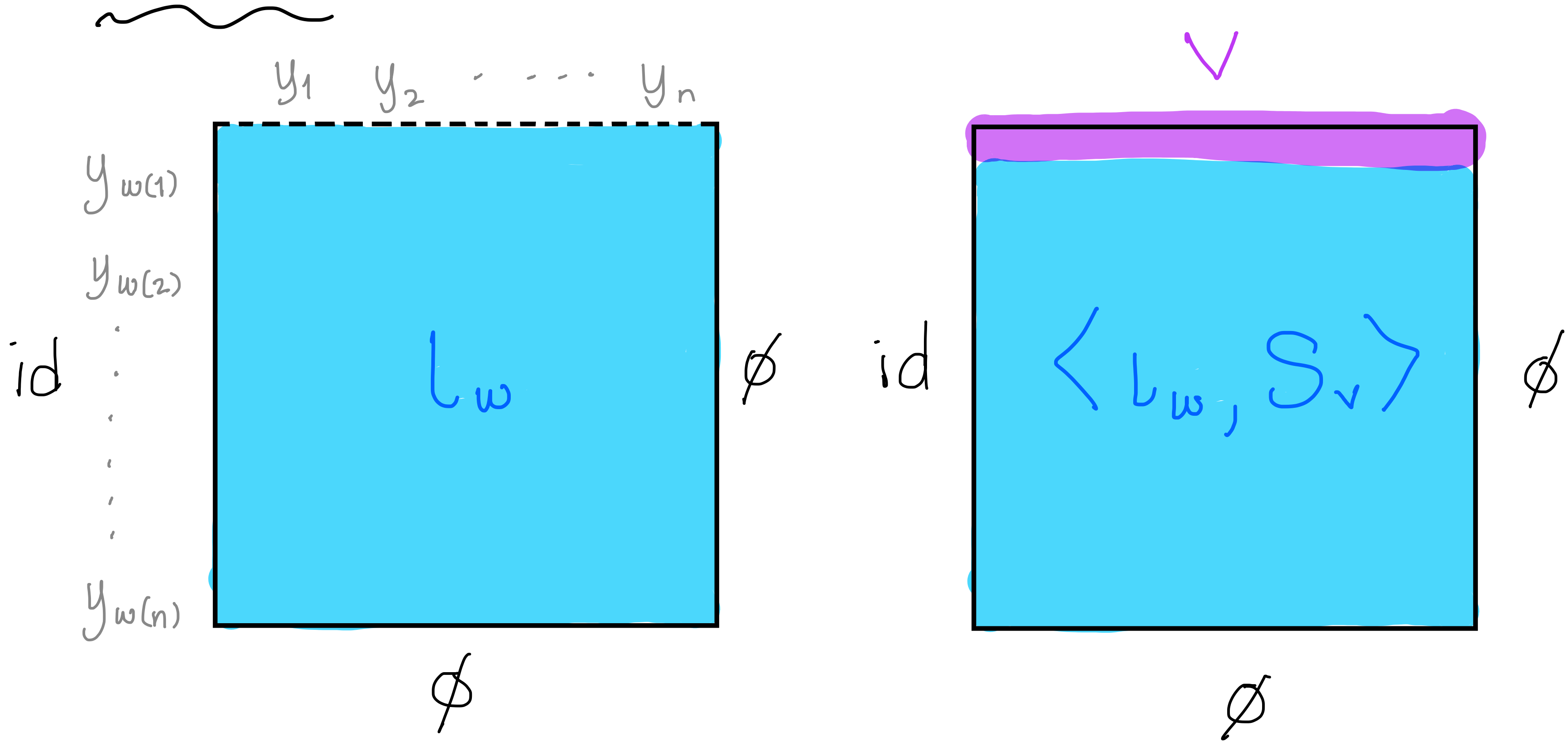
So picking a top boundary $v\vec{C}$

$\longleftrightarrow$ calculating the coefficient

$$\langle L_w, S_v \rangle \in H_T(pt) \cong \mathbb{Z}[y_1, \dots, y_n]$$

aka the double Schubert polynomial evaluated at $\{x_i \longmapsto y_{w(i)}\}$.

Cartoons:



4 Example: Colored & uncolored Whittaker function lattice models

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In lattice model world, we will be here:

$K_T(T^*X)$ (or really, $K_T(pt) \cong R(T)$)

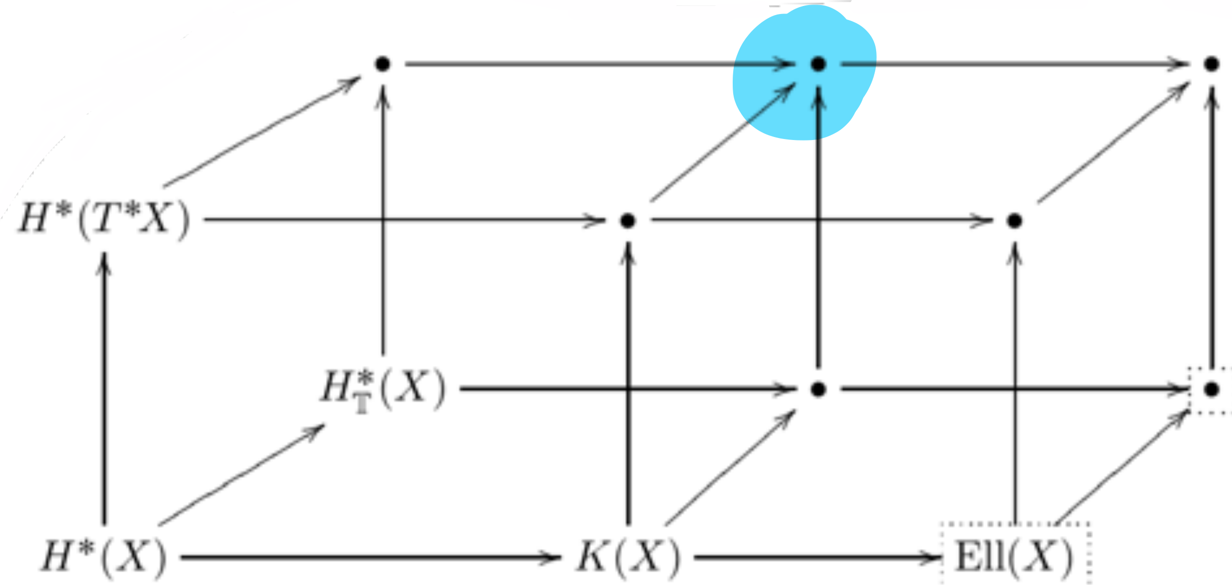


FIGURE 1. Three "orthogonal" directions to generalize classical Schubert calculus

Quick review of Whittaker functions

Take $G = GL_n(F)$, F nonarchimedean field, $\mathcal{O}$ ring of integers of F, $\mathfrak{p} = \langle \varpi \rangle$ max'l ideal of $\mathcal{O}$, $\mathcal{O}/\mathfrak{p} = \mathbb{F}_q$ the residue field, $\left\{ \begin{pmatrix} \mathcal{O} & & \mathfrak{p} \\ & \ddots & \\ \mathcal{O} & & \mathcal{O} \end{pmatrix} \right\} = J$ Iwahori subgp.

$$\tilde{J}_z : \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \longmapsto z^{\lambda} = \prod_{i=1}^{n} z_i^{\lambda_i}$$

unramified character of torus $T(F)$

The principal series representation $(\pi, I(z))$ is:

$$I(z) := \mathrm{Ind}_{B(F)}^{G(F)} \left(\delta^{1/2} \tilde{J}_z \right)$$

w/ π the right regular action of G

The space $I(z)^J$ of Iwahori - fixed vectors generates $I(z)$ as a G-module, has size $|S_n|$, & has two well-known bases:

① Standard basis $\{\Phi_w^z\}_{w \in S_n}$ of characteristic functions on the orbits of

$$G = \bigsqcup_w BwJ$$

② Casselman's basis $\{f_w\}$ defined as dual to intertwining operators $\mathcal{A}_w : I(z) \to I(wz)$:

$$\mathcal{A}_w(f_v)(1) = \delta_{w,v}$$

Aluffi, Mihalcea, Schürmann, & Su define an isomorphism Ψ between

$$\widetilde{K_T}(G/B)[q] \otimes_{K_T(pt)} \mathbb{C}_{\mathcal{T}_Z}$$

&

$$I(z)^{J}$$

such that

(opposite) Schubert cell

$$\Psi(MC^{\vee}(Y(w)^{\circ}) \otimes 1) = \Phi^{z}_{w}$$

&

$$\Psi(b_w) = f_w$$

dual motivic chern class

multiple of fixed pt ι_w

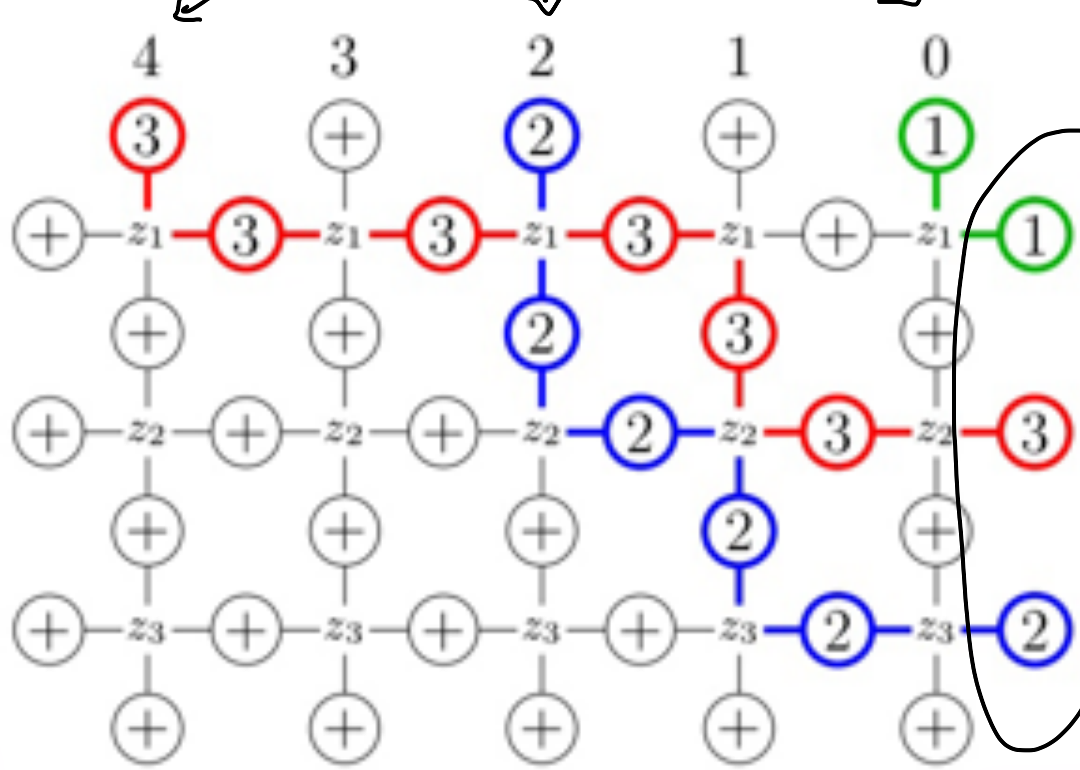
$\sim$ stable envelope basis in $K_T(T^*G/B)$

Brubaker, Bump, Buciumas, & Gustafsson (2020) define a colored lattice model that computes the Iwahori Whittaker functions

$$z^{\rho} \cdot \phi_{w,\lambda}(z) := z^{\rho} \cdot \underbrace{\Omega}_{\text{Whittaker functional}}{}_{z^{-1}}\left(\pi(\varpi^{-\lambda})\,\Phi_w^{z^{-1}}\right)$$

$(\rho = (n-1, n-2, \ldots, 2, 1, 0))$

$\lambda + \rho$



$w\vec{C}$

& the sum

$$\sum_{w} \phi_{w,\lambda}(z)$$

computes the spherical Whittaker function $\longleftrightarrow$ uncolored "Tokuyama" lattice model.

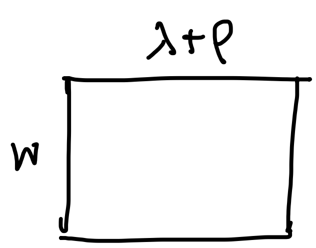
Like with the Frozen Pipes model, we have two ways of analyzing the Tokuyama model, one of which $\longleftrightarrow$ Schubert-like basis & the other $\longleftrightarrow$ fixed point basis.

Two ways of calculating:

similar to Borodin–Petrov 2016

① As sum of colored Iwahori partition functions

"Macro YBE" proof

$\sum_w$ 

Calculate using Demazure-Whittaker operators via train argument

In geometry:

$$\sum_w \left\langle \mathcal{L}_{\lambda+\rho}, (-q)^{\ell(w_0 w)} MC^\vee(Y(w)^\circ) \right\rangle$$

$$= \langle \mathcal{L}_{\lambda+\rho}, \underbrace{\widetilde{MC}^\vee(Y(1))} \rangle$$

line bundle

modified spherical vector Φ_-

② Using algebraic Bethe ansatz inspired method

"Micro YBE" proof

get Weyl character-like formula

$$\prod_{i<j} \frac{z_i - q^{-1} z_j}{z_i - z_j} \sum_w (-1)^{\ell(w)} z^{w(\lambda+\rho)}$$

In geometry:

$$\sum_w (-q)^{\ell(w_0 w)} \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{1-q^{-1}e^{\alpha}}{1-e^{\alpha}} \langle \mathcal{L}_{\lambda+\rho}, b_w \rangle$$

Casselman basis vector f_w

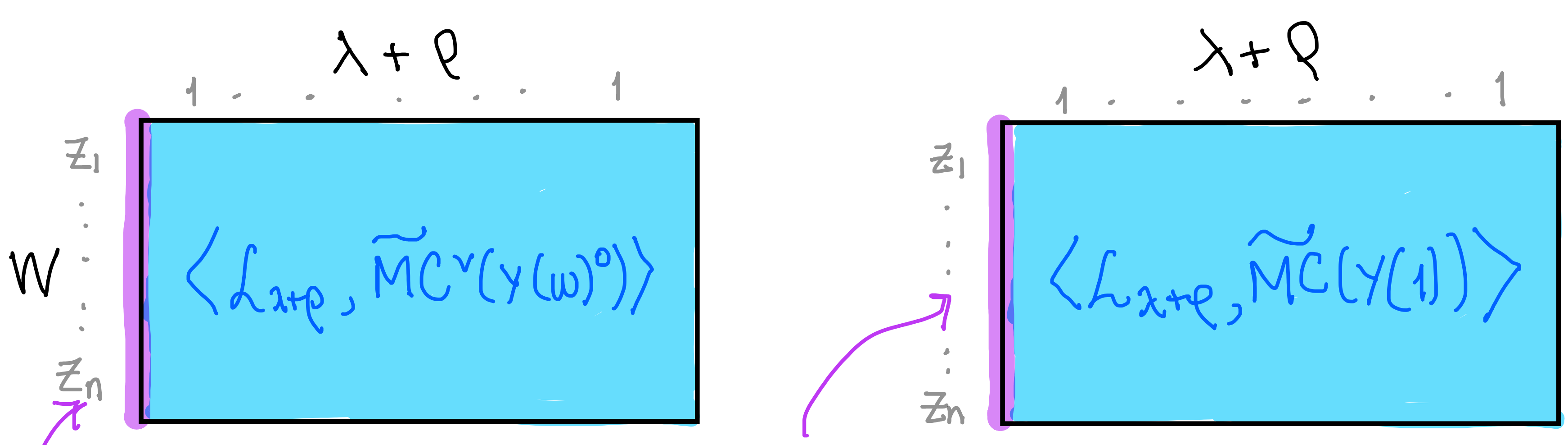
(proof of these geometric interpretations: very careful formal geometric manipulations & convention matching ☺)

- Can extract from this a (modified version of) the Langlands-Gindikin-Karpelevich formula expanding Φ_- into the f_w's.
  (& its geometric version, expanding $\widetilde{MC}(Y(1))$ into fixed pts.)

- also have these cartoons now:

Iwahori

Spherical

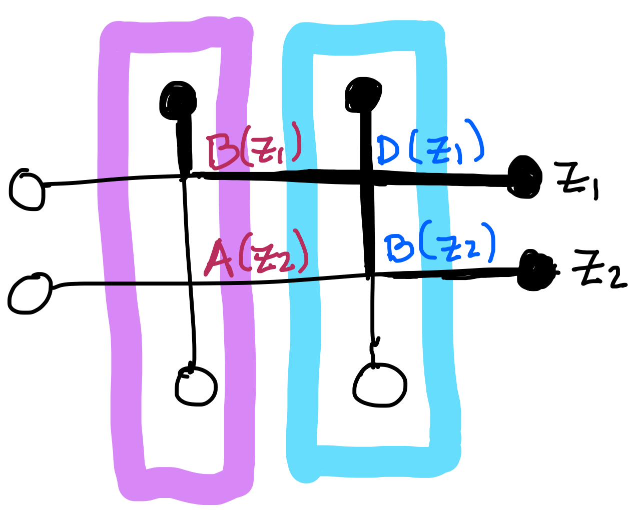


this vector $\leftrightarrow$ Poincaré dual of $\widetilde{MC}^\vee(Y(w)^\circ)$ / $\widetilde{MC}(Y(1))$

A picture to give the idea of the algebraic Bethe ansatz - like method in the simplest case:

take each state of the model:

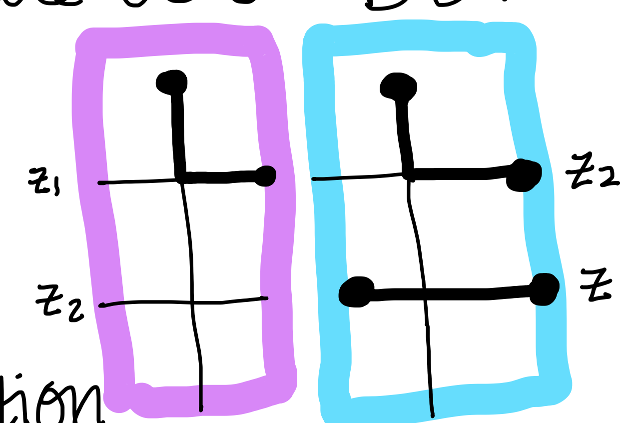
& commute columns separately until all A's & D's are below B's:



= factor from commutation

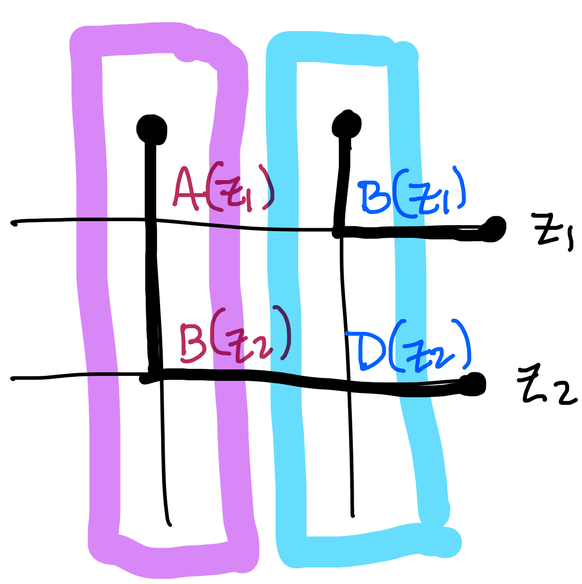
$B \leftrightarrow D$

×



weight $z_1 = z^{\mathrm{id}(\lambda+\rho)}$

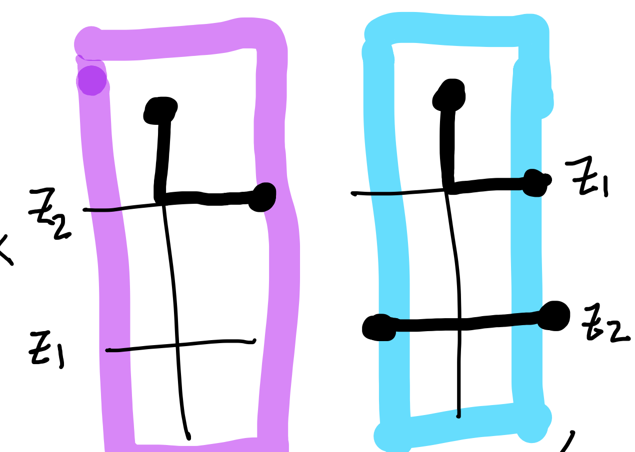
+



= factor from commutation

$B \leftrightarrow A$

×



weight

$z_2 = z^{w_0(\lambda+\rho)}$

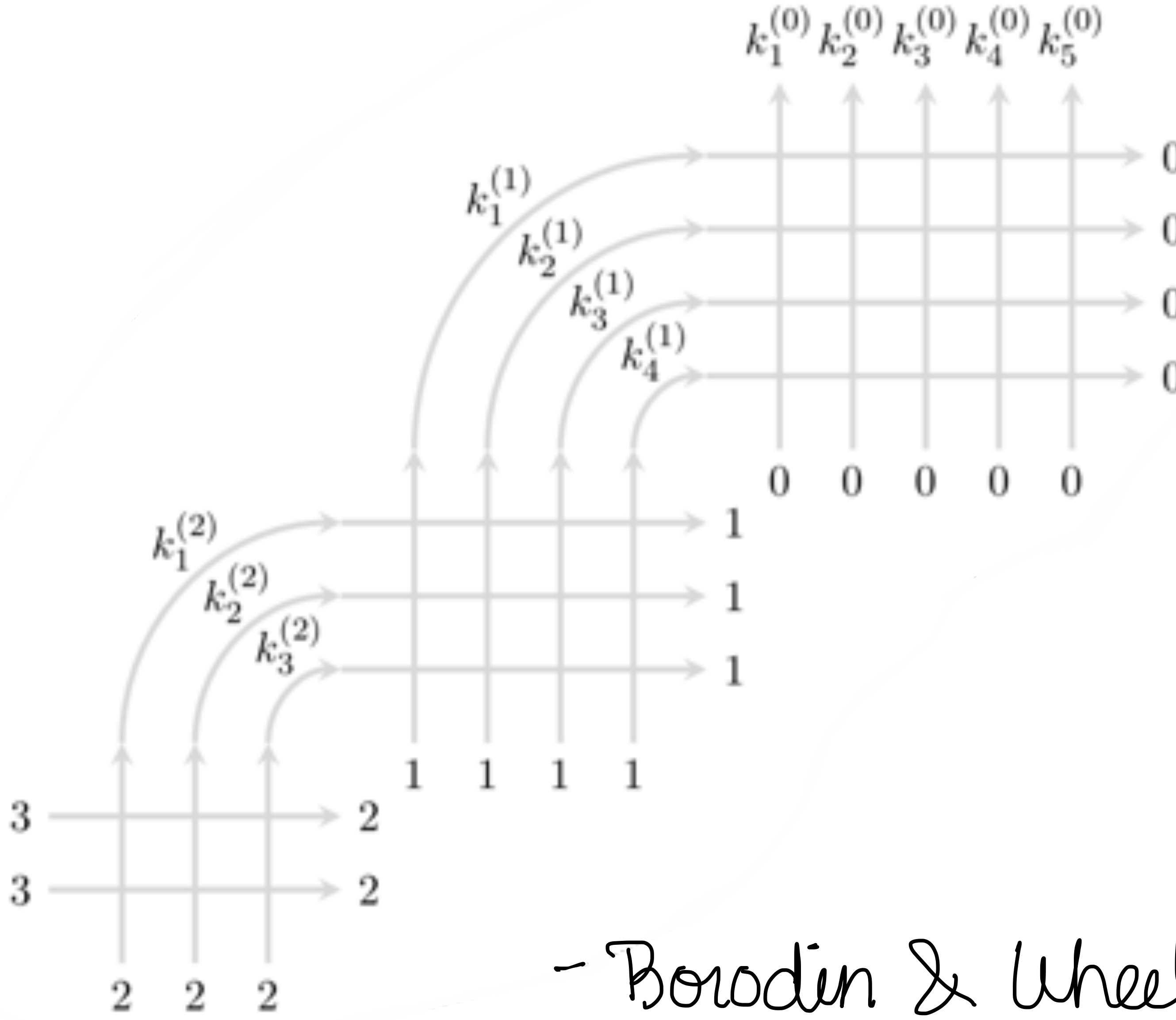
⑤ Future directions?

Initially, I hoped to find a lattice proof of the Bump-Naruse-Nakasuji conjecture describing the expansion

$$\Phi_u = \sum_w m_{u,w} f_w .$$

$$\left(\longleftrightarrow \quad MC^\vee(Y(w)^\circ) = \sum_w m_{u,w} b_w \right)$$

It is potentially extractable from a formula of Borodin & Wheeler obtained via the nested Bethe ansatz:



– Borodin & Wheeler 2018, Ch.7

would need to collapse this inner sum; maybe not ideal

$$f_\mu(x_n,\dots,x_1) = \sum_{\sigma^{(1)}\in\mathfrak{S}_n} \boxed{\cdots \sum_{\sigma^{(n-1)}\in\mathfrak{S}_2}} f_\delta\left(x_{\sigma^{(1)}(1)},\dots,x_{\sigma^{(1)}(n)}\right) \prod_{1\leqslant i<j\leqslant n} \frac{q x_{\sigma^{(1)}(i)} - x_{\sigma^{(1)}(j)}}{x_{\sigma^{(1)}(i)} - x_{\sigma^{(1)}(j)}}$$

sum over fixed pts?

$$\times \prod_{b=2}^{n} \psi_{\{a_1^{(b)},\dots,a_{n-b+1}^{(b)}\}}\left(\sigma^{(b)}\cdot(x_1,\dots,x_{n-b+1}); \sigma^{(b-1)}\cdot(x_1,\dots,x_{n-b+2})\right) \prod_{1\leqslant i<j\leqslant n-b+1} \frac{q x_{\sigma^{(b)}(i)} - x_{\sigma^{(b)}(j)}}{x_{\sigma^{(b)}(i)} - x_{\sigma^{(b)}(j)}}, \tag{7.5.1}$$

nonsymmetric Hall Littlewood poly, specializes to [factor of (-q)'s] × Iwahori

Other ideas...? ☺