Refined Dual Grothendieck Polynomials From 3 Perspectives

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Geometric Motivation:

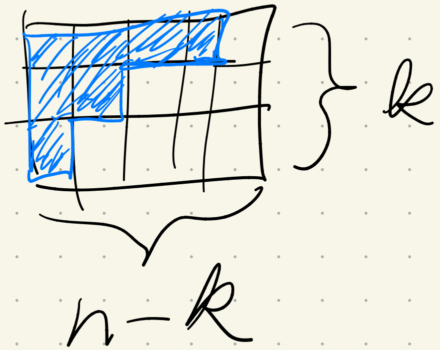
$Gr(k,n)$ = Grassmannian

k-dim subspaces of $\mathbb{C}^n$

We want to study its K-theory ring.

Use Schubert varieties: $\overline{BwB/P}$

Indexed by Grassmannian permutations, which have 1 descent and in bijection with partitions $\lambda \subseteq$



Can give a basis by Grothendieck polys

$$G_\lambda = s_\lambda + HOT$$

$\uparrow$ Schur func.

Combinatorial formula [Buch, '02]

5-vertex model [Motegi–Sakai, '13]

Since they are symmetric functions, can compute the dual basis w.r.t. Hall inner prod.

$$\langle G_\lambda, \underbrace{g_\mu}_{} \rangle = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle$$

Dual Grothendieck polynomials

1st Perspective: Combinatorics [Lam–Pylyavskyy '07]

$$g_\lambda = \sum_{T \in RPP_\lambda} \beta^{|\lambda|-|T|} x^T \qquad \prod_i x_i^{\#\text{ columns containing } i}$$

Reverse Plane Partitions

Fillings of λ s.t. rows & columns weakly increase

β counts the degree diff.

Ex/

1	1	1	2
1	2	3	
1	3	3	

$$\beta^3 x_1^3 x_2^2 x_3^2$$

If we want to record which rows we obtain duplicate entries $\begin{array}{|c|}\hline i \\ \hline i \\ \hline\end{array}$ duplicate, we refine β to params $t_1, \ldots, t_{e-1}$. This is still a symmetric function and $\exists$ bijection [LP'07, Galashin-Grinberg-Liu'15]

$$\phi : RPP_\lambda \to \bigsqcup_{\mu \subseteq \lambda} \big(\underbrace{SSYT_\mu}_{\text{Semistd Young Tab.}}, \underbrace{ET^{\mu}_{\lambda}}_{\text{Elegant tableaux}} \big)$$

Semistd Young Tab.

RPPs with $\beta = 0$

equiv. columns strictly inc.

Elegant tableaux

SSYT shape λ/μ

flagging cond. [s.t. entries in row i are $< i$.

This bijection uses RSK and is weight preserving.

$x \to SSYT_\mu$

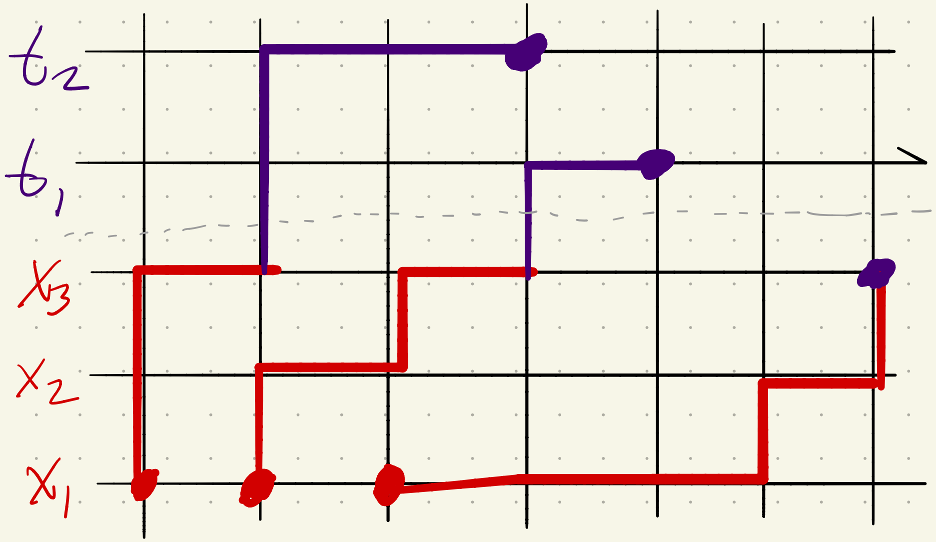
$t \to ET^{\mu}_{\lambda}$

Combining these together, we can write a single tableau with two ordered alphabets $X < T$ with a flagging condition.

Ex/ $\lambda = 433$ $\mu = 421$

1	1	1	2
2	3	t_1	
3	t_2	t_2	

$\longmapsto$



This is from LGV Lemma using the standard proof of Jacobi-Trudi formula modified for the flagging condition.

Thm [Motegi–S., 20]/ A dual Grothendieck polynomial (in finite # vars) is a sum over such lattice paths.

Cor/

homogeneous sym. func.

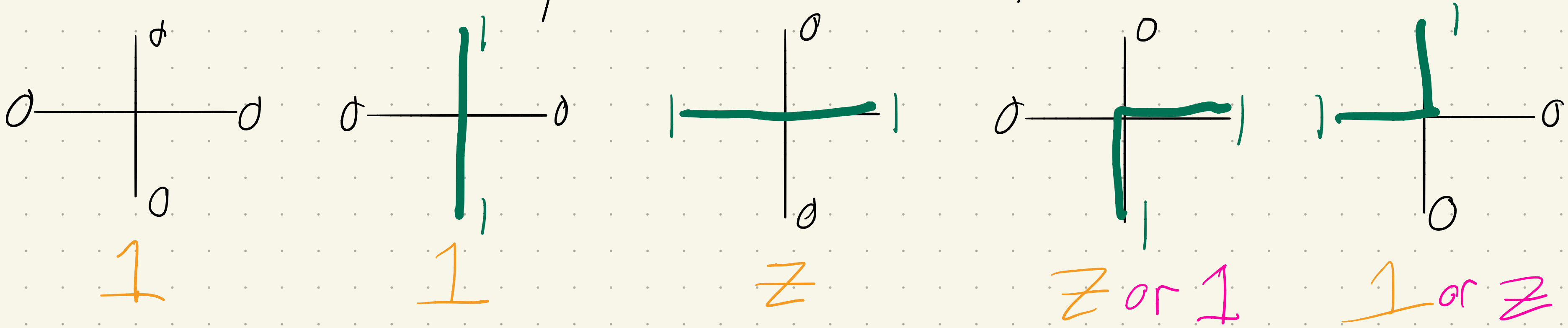
$$g_\lambda = \det\left[h_{\lambda_i+j-i}(x, t_1, \ldots, t_{i-1})\right]_{i,j=1}^{n}$$

$$= \det\left[e_{\lambda'_i+j-i}(x, t_1, \ldots, t_{\lambda'_i-1})\right]_{i,j=1}^{n}$$

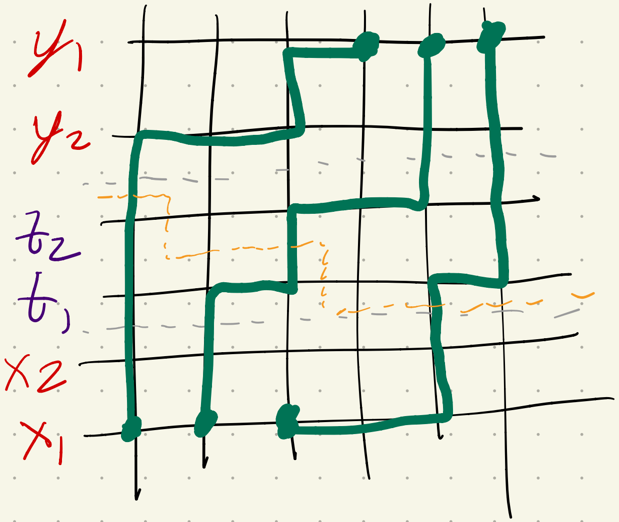
elementary sym. func.

2nd Perspective: Solvable Lattice Models

Translate lattice paths into 5-vertex lattice model



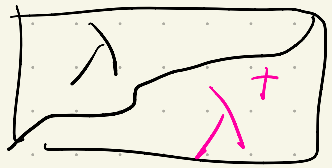
We perform Cauchy-like proof:

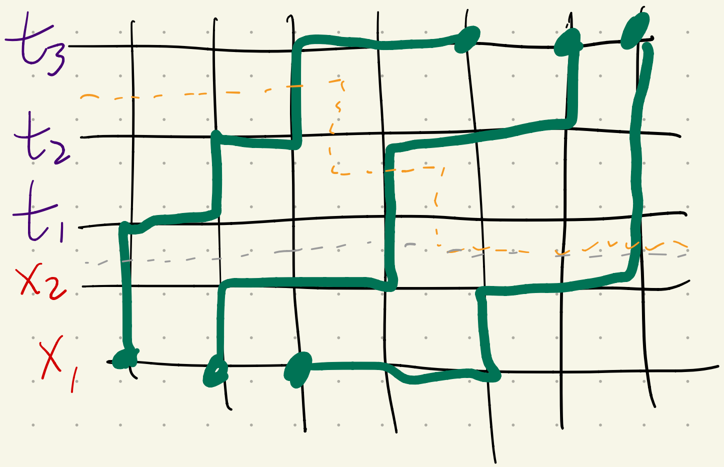


Cor/ Cauchy identity

$(t_{\ell-1},\dots,t_1)$ $=$ t^+

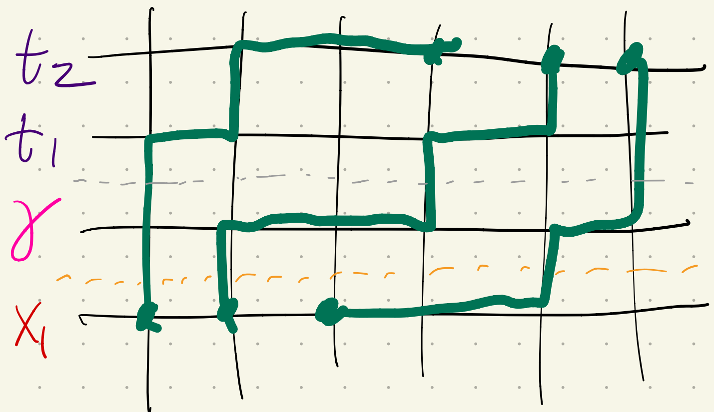
$$S_{m\ell}(x,t,y)=\sum_{\lambda\subseteq m^\ell} g_\lambda(x;t)\, g_{\lambda^+}(y;t^+)$$





Cor/ Littlewood identity

$$S_{m\ell}(x;t,t_\ell)=\sum_{\lambda\subseteq m^\ell}\prod_{i=1}^{\ell} t_i^{m-\lambda_i}\, g_\lambda(x;t)$$



Cor/ Branching rule

$$g(x,\gamma;t)=\sum_{\mu\subseteq\lambda}\gamma^{\lambda_1-\mu_1}\, t_1^{\lambda_2-\mu_2}\cdots t_{\ell-1}^{\lambda_\ell-\mu_\ell}\, g(x;\gamma,t)$$

Noting that the bottom left entries are fixed, we can transform this into the 5-vertex model used by [MS, '13], which adds an extra parameter η.

Thm/ The partition func $= \displaystyle\sum_{\mu\subseteq\lambda}\sum_{SET^{\mu}_{\lambda}} t^{T} G_{\mu}(x;\eta)$

set-valued elegant tableaux

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & \\ \cline{1-1}\end{array} \qquad \begin{array}{c} \max A \le \min B \\ \wedge \\ \min C \end{array}$$

Specializing $\eta=0$

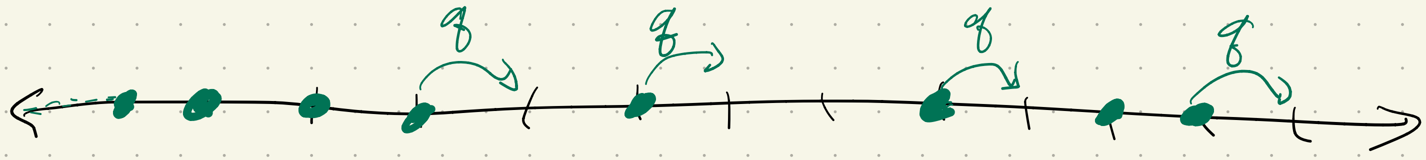
Cor/ Cauchy-Littlewood identity

$$\sum_{\lambda\subseteq m^{\ell}} \prod_{i=1}^{\ell} t_i^{m-\lambda_i}\, g_{\lambda}(x;t)$$

$$= \prod_{i=1}^{\ell} t_i^{m} \prod_{1\le i<j\le n} \frac{1}{(x_i-x_j)(t_i^{-1}-t_j^{-1})} \det\left[\frac{(x_i t_j^{-1})^{m+n}-1}{x_i t_j^{-1}-1}\right]_{i,j=1}^{n} \Bigg|_{t_{\ell+1}=\cdots=t_n=\infty}$$

3rd Perspective: Probability

Totally
Asymmetric
Simple
Exclusion
Process



Step initial condition

Record the time it takes for the nth particle to move ℓ steps.

This leads to a random matrix.

Last Passage Percolation

(geometric distribution)

Random matrix w/ entries $[w_{ij}]$ s.t. $P(w_{ij}=k) = (1-t_i x_j)(t_i x_j)^k$

Last passage time $G(\ell,n) = \max\limits_{\text{paths } (1,1)\to(\ell,n)} \sum w_{ij}$ $\quad (i,j)\to(i+1,j)$ $\quad (i,j)\to(i,j+1)$

Take $x_i = t_j = \sqrt{q}$ to recover TASEP with

$G^*(\ell,n) = G(\ell,n) + \ell + n - 1$ being the time needed.

Define $G(i) = (G(\ell,i), \dots, G(\ell,1))$

Thm/[Motegi-S,'20]   [$t=x=q$ Yeliussizov '19]

Transition probability $P(G(i)=\lambda \mid G(j)=\mu) = g_{\lambda/\mu} \prod_{i,j}(1-x_i t_j)$

Pf/ $$g_{\lambda/\mu}(x_n;t) = \prod_{j=1}^{\ell-1} t_j^{\max(\mu_j,\lambda_{j+1})-\mu_j} \prod_{j=1}^{\ell} x_n^{\lambda_j-\max(\mu_j,\lambda_{j+1})}$$

Branching rule $$\times \underbrace{H(\lambda_j-\max(\mu_j,\lambda_{j+1}))}$$

Heaviside step func. $H(x) = \begin{cases} 0 & x \le 0, \\ 1 & x > 0. \end{cases}$ $\boxtimes$

Thm/ [Motegi–S, '20; indep. J.S. Kim]

$$g_{\lambda/\mu}(x;t) = \det\left[\sum_{m=0}^{\infty} \alpha_m^{ij}(t)\, h_{\lambda_i-\mu_j-i+j-m}(x)\right]_{i,j=1}^{\ell}$$

$$\alpha_m^{ij}(t) = \begin{cases} h_m(t_j,\dots,t_{i-1}) & \text{if } i \geq j, \\ e_m(-t_i,\dots,-t_{j-1}) & \text{if } i < j. \end{cases}$$

$$g_{\lambda/\mu}(x;t) = \det\left[\sum_{m=0}^{\infty} \tilde{\alpha}_m^{ij}(t)\, e_{\lambda'_i-\mu'_j-i+j-m}(x)\right]_{i,j=1}^{\lambda_1}$$

$$\tilde{\alpha}_m^{ij}(t) = \begin{cases} e_m(t_{\mu'_j+1},\dots,t_{\lambda'_i-1}) & \text{if } \mu'_j \geq \lambda'_i - 1, \\ h_m(-t_{\lambda'_i},\dots,-t_{\mu'_j}) & \text{else.} \end{cases}$$

Pf/ Algebraic

Future Work

- Develop colored lattice model for $g_{\lambda/\mu}(x;t)$
- Understand the η-deformed version as symmetric functions and probablistically
- Determine a geometric reason for t params.
- Are there good interpretations of integral repr.

Thm [Motegi–S., '20]

$$g_\lambda(x;t) = \frac{1}{(2\pi i)^\ell} \oint \cdots \oint \frac{\prod_{i=1}^{\ell} z_i^{\lambda_i+\ell-i} \prod_{1\le i<j\le \ell} (z_i - z_j) z_j}{\prod_{i=1}^{\ell} \prod_{m=1}^{\ell} (z_i - x_m) \prod_{1\le i<j\le \ell} (z_j - t_i)} \, dz_1 \cdots dz_\ell$$