# Local relations and $q$-moments of height functions of stochastic vertex models

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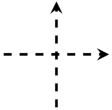
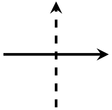
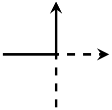
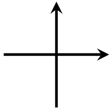
# Stochastic six-vertex model

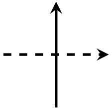
Fix $q, z$ and define the following six-vertex weights

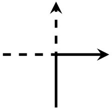

$= 1$


$= \frac{z-1}{z-q}$


$= \frac{1-q}{z-q}$


$= 1$

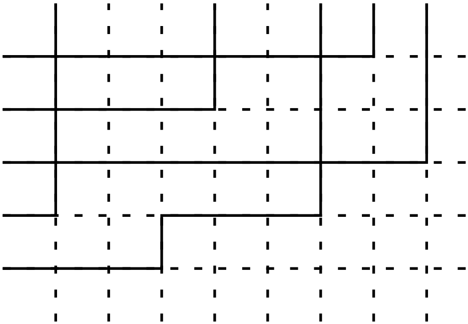

$= \frac{q(z-1)}{z-q}$


$= \frac{z(1-q)}{z-q}$

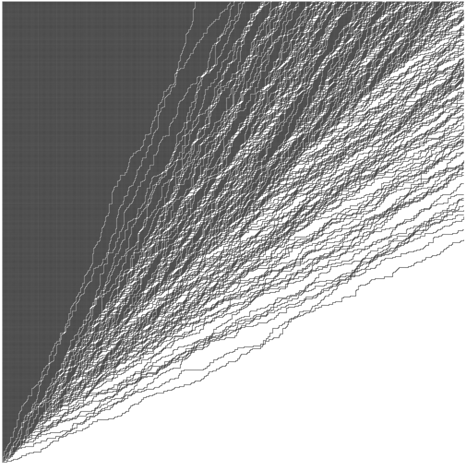
- The weights are positive for $q \in (0,1)$ and $z > 1$.
- The weights are stochastic.

Consider a model with the step boundary conditions:

- The bottom boundary is empty;
- The left boundary is completely occupied;
- Free conditions on the top and the right boundaries.

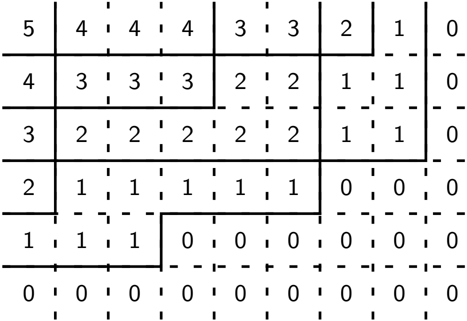


We want to describe the large scale behavior. For example, to explain the picture like this:

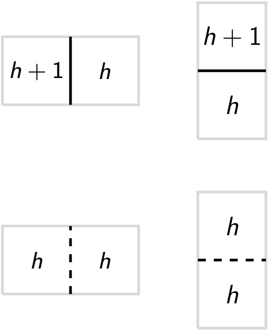


$z = 4/3$, $q = 1/3$ simulation by L. Petrov

It is useful to define the height function $h(x, y)$:




Local rules




Defined up to a global shift – we fix $h\left(\frac{1}{2}, \frac{1}{2}\right) = 0$.

## Theorem [Borodin,Corwin,Gorin-2014]

For $z^{-1} < y/x < z$ we have $\lim_{L\to\infty} \frac{h(xL,yL)}{L} = \frac{\left(\sqrt{yz}-\sqrt{x}\right)^2}{z-1}$. Moreover, the fluctuations are given by GUE Tracy-Widom distribution scaled by $\sigma_{x,y} L^{1/3}$.

The first step in the proof of this result is an expression for $q$-moments $\mathbb{E}[q^{kh(x,y)}]$:

## Theorem [Borodin,Corwin,Gorin-2014]

$$\mathbb{E}\left[q^{kh(x,y)}\right] = \frac{q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \prod_{i=1}^{k} \left(\frac{1-w_i}{1-qw_i}\right)^{x-\frac{1}{2}} \left(\frac{1-qzw_i}{1-zw_i}\right)^{y-\frac{1}{2}} \frac{dw_i}{w_i}$$

One can then collect the $q$-moments into a $q$-deformed moment generating function and perform asymptotic analysis when $x, y \to \infty$.

How to obtain this expression? One possible approach relies on special functions and Cauchy-type identities:

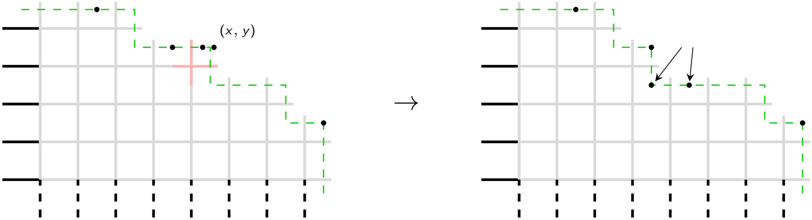
- Using higher spin vertex model introduce spin Hall-Littlewood functions.
- Using Yang-Baxter equation prove a Cauchy-type summation identity.
- Using Bethe ansatz find an explicit expression and orthogonality relations.
- Using Cauchy-type identities and orthogonally find expressions for observables, leading to evaluation of $q$-moments.

This approach is constructive and allows to obtain expression for $q$-moments starting from the vertex model.

However, it involves a lot of moving parts, and some of them are not yet understood for more general vertex models...

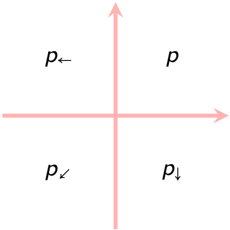
Alternative approach: show that both sides satisfy the same recurrence relation.

Consider a sum of height functions evaluated at $k$ points on an up-left path. It turns out we can describe the behavior of the $q$-moments when we "flip"a corner of this path over a single vertex:




$$\mathbb{E}\left[q^{\cdots+rh(x,y)+\ldots}\right] = \sum_{a+b+c=r} A_{a,b,c}\mathbb{E}\left[q^{\cdots+ah(x-1,y)+bh(x-1,y-1)+ch(x,y-1)+\ldots}\right]$$

For any fixed incoming configuration of a vertex the following relation on $q$-moments holds:




$$\mathbb{E}\left[ q^{rh(p)} \,\middle|\, \substack{\textit{incoming}\\ \textit{configuration}} \right] = \frac{q^r z - q}{z-q} \qquad q^{rh(p_{\downarrow})}$$
$$+ \frac{1-q^r}{1-q}\,\frac{1-qz}{z-q} \qquad q^{h(p_{\swarrow})+(r-1)h(p_{\downarrow})}$$
$$+ \frac{1-q^r}{1-q}\,\frac{z-1}{z-q} \qquad q^{h(p_{\leftarrow})+(r-1)h(p_{\downarrow})}$$

The relation can be verified by case by case analysis – four cases, one for each incoming configuration.
Note that coefficients do not depend on the incoming configuration!

Using the recurrence relation we can prove the following:

### Theorem [Bufetov,K.-20]

Let $x_1 \leq x_2 \leq \cdots \leq x_k$ and $y_1 \geq y_2 \geq \cdots \geq y_k$.

$$\mathbb{E}\left[q^{\sum_{i=1}^k h(x_i,y_i)}\right] = \frac{q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \prod_{i=1}^k \left(\frac{1-w_i}{1-qw_i}\right)^{x_i - \frac{1}{2}} \left(\frac{1-qzw_i}{1-zw_i}\right)^{y_i-\frac{1}{2}} \frac{dw_i}{w_i}$$

Proof:

- The local relation from the previous slide translates to a recurrence relation on the left-hand side;
- One can directly verify that the right-hand also satisfies the same relation;
- Finally, for the base case when either $x_i = \frac{1}{2}$ or $y_i = \frac{1}{2}$, the left-hand is deterministic, while the right-hand side can be computed taking residues.

So, using the elementary approach outlined above, we can directly prove integral expressions for $q$-moments given two main ingredients:

- A guess how the expression looks like (might come from computer simulations, Cauchy-identity arguments for simpler models).
- A local relation for the height function (the subject of this talk).

# Local relations

For now we focus on one vertex with a fixed incoming configuration.

Returning to the local relation for the six-vertex model:

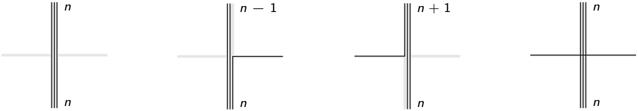
$$\mathbb{E}\left[ q^{rh(p)} \middle| \substack{\textit{incoming} \\ \textit{configuration}} \right] = \sum_{k,l} w \left( \begin{smallmatrix} & k & \\ j & + & l \\ & i & \end{smallmatrix} \right) \; q^{rh(p)}$$

$$= \frac{q^r z - q}{z - q} \; q^{rh(p_\downarrow)} + \frac{1 - q^r}{1 - q} \frac{1 - qz}{z - q} \; q^{h(p_\swarrow) + (r-1)h(p_\downarrow)} + \frac{1 - q^r}{1 - q} \frac{z - 1}{z - q} \; q^{h(p_\leftarrow) + (r-1)h(p_\downarrow)}$$

Four linear equations, three variables – linear algebra problem.

For $r > 1$ this relation is not unique: we can use arbitrary $q^{ah(p_\leftarrow) + bh(p_\swarrow) + ch(p_\downarrow)}$ with $a + b + c = r$, giving non-degenerate system of $\binom{r+2}{2}$ variables and four equations.

But there are more general weights: six-vertex model comes as coefficients of $R$-matrix of $U_q(\widehat{\mathfrak{sl}}_2)$ acting on $V_2 \otimes V_2$, where $V_2$ is the standard representation. Replacing one of the representation by a Verma module $W_\lambda$ we obtain higher-spin stochastic six-vertex weights, which can be obtained explicitly by fusion:

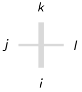



$$\frac{1-suq^n}{1-su} \qquad \frac{su(q^n-1)}{1-su} \qquad \frac{1-s^2q^n}{1-su} \qquad \frac{-su+s^2q^n}{1-su}$$

When $s = q^{-1/2}$ and $u = zq^{-1/2}$ the higher spin weights reduce to the six-vertex weights before.

Local relation still holds for the higher spin weights. Fix parameters of the vertex $u, s$ and the initial conditions, then

$$\mathbb{E}^{\mathrm{h.s.}}_{u,s}\left[q^{rh(p)}\,\middle|\,\substack{\textit{incoming}\\ \textit{configuration}}\right]=\sum_{k,l} w_{u,s}\left(\text{vertex}\right)\; q^{rh(p)}$$

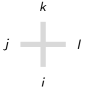


$$=\frac{1-suq^r}{1-su}q^{rh(p_\downarrow)}+\frac{1-q^r}{1-q}\,\frac{-s^2+qsu}{1-su}\;q^{h(p_\swarrow)+(r-1)h(p_\downarrow)}+\frac{1-q^r}{1-q}\,\frac{s^2-su}{1-su}\;q^{h(p_\leftarrow)+(r-1)h(p_\downarrow)}$$

Now there are still $\binom{r+2}{2}$ variables, but infinite number of possible initial conditions, leading to infinite number of linear equations with a solution listed above.

Still can be readily proved directly considering cases $j = 0, 1$.

Fusing in the other direction, we obtain fully-fused vertex weights $\mathcal{W}_{z;t,s}$:




$$\mathcal{W}_{z;t,s}\left( \cdot \right) = \mathbb{1}_{i+j=k+l} z^{l-j} t^{2k-2i} s^{2l} \sum_{p=0}^{\min(j,k)} z^{p} t^{-2p} \frac{(\frac{zs^2}{t^2};q)_{k-p}(t^2;q)_l(\frac{t^2}{z};q)_p(z;q)_{j-p}}{(zs^2;q)_{k+l-p}(t^2;q)_j} \begin{bmatrix} k+l-p \\ l \end{bmatrix}_q \begin{bmatrix} j \\ p \end{bmatrix}_q$$

The local relation still holds!

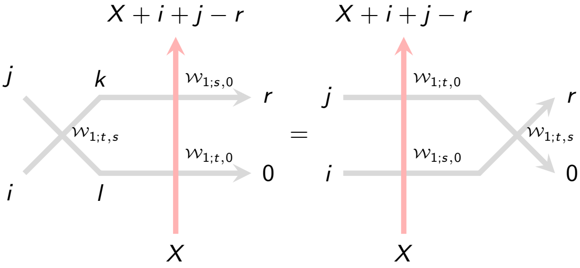
$$\mathbb{E}^{\mathrm{f.f.}}_{z,s,t}\left[ q^{rh(p)} \middle| \begin{smallmatrix} \textit{incoming} \\ \textit{configuration} \end{smallmatrix} \right] = \sum_{\substack{a,b,c \\ a+b+c=r}} \frac{z^b s^{2a+2b}}{t^{2b}} \frac{(z;q)_a (t^2/z;q)_b (zs^2/t^2;q)_c}{(zs^2;q)_r} q^{ah(p_{\leftarrow})+bh(p_{\swarrow})+ch(p_{\downarrow})}$$

Proof of the last relation: First we consider the case $z = 1$, when both weights and the relation simplify.

$$\mathcal{W}_{1;t,s}\left(\begin{array}{ccc} & k & \\ j & + & l \\ & i & \end{array}\right) = \mathbb{1}_{i+j=k+l}(s/t)^{2l}\frac{(s^2/t^2;q)_{i-l}(t^2;q)_l}{(s^2;q)_i}\begin{bmatrix} i \\ l \end{bmatrix}_q$$

$$\mathbb{E}^{\mathrm{f.f.}}_{1,s,t}\left[q^{rh(p)}\,\middle|\,\begin{array}{c}\textit{incoming}\\ \textit{configuration}\end{array}\right] = \sum_{\substack{b,c\\ b+c=r}}\left(s^2/t^2\right)^b\frac{(t^2;q)_b(s^2/t^2;q)_c}{(s^2;q)_r}q^{bh(p_{\swarrow})+ch(p_{\downarrow})}$$

This relation is a particular case of the Yang-Baxter equation for the fully fused weights, when all spectral parameters are 1, and one spin parameter is 0:



To finish the proof, one should find two expressions identical to the $z = 1$ weights inside the fully fused weights

$$\mathcal{W}_{z;t,s} = z^{l-j} t^{2k-2i} s^{2l} \sum_{p} z^{p} t^{-2p} \frac{(\frac{zs^2}{t^2};q)_{k-p}(t^2;q)_l}{(zs^2;q)_{k+l-p}} \begin{bmatrix} k+l-p \\ l \end{bmatrix}_q \frac{(\frac{t^2}{z};q)_p (z;q)_{j-p}}{(t^2;q)_j} \begin{bmatrix} j \\ p \end{bmatrix}_q$$

Using this observation, the local relation for the fully fused weights can be shown by two successive applications of $z = 1$ case above.

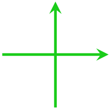
So, the local relation for $q$-moments of the height function is a general property, which can be expected from stochastic and solvable vertex weights.

Why we were interested in it? Actually, the original motivation was to verify expressions for $q$-moments of height function in colored models.

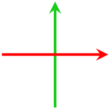
For the colored models, the methods based on Cauchy identities still partially work, but all the required steps for such methods are not yet completely understood in full generality. However, the local relations can prove the expressions for $q$-moments in the general case.

# Colored stochastic six-vertex model

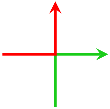
Let $u, v$ be row and column parameters. For a pair of colors $i < j$ set



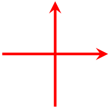
$= 1$



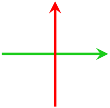
$$= \frac{u - v}{u - qv}$$



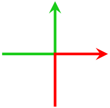
$$= \frac{v(1-q)}{u - v}$$



$= 1$



$$= \frac{q(u - v)}{u - qv}$$

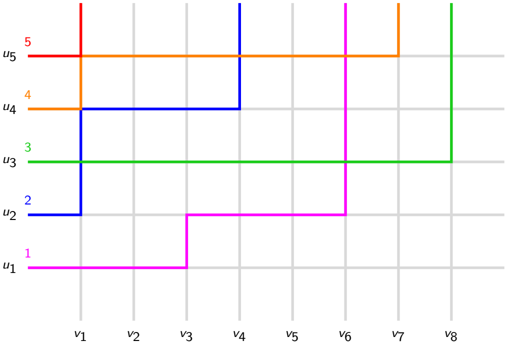


$$= \frac{u(1-q)}{u - qv}$$

Originates from $U_q(\widehat{\mathfrak{sl}}_{n+1})$.

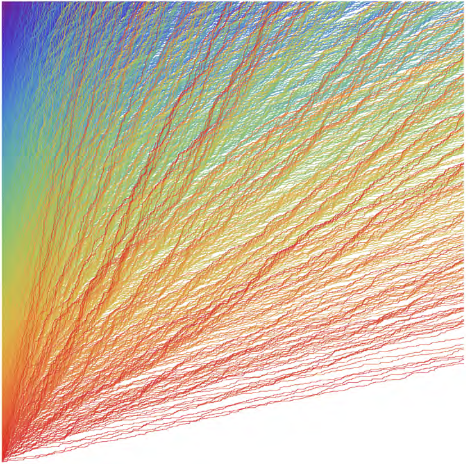
# Model

All colors enter from the left in increasing order, nothing enters from below (empty = color 0).



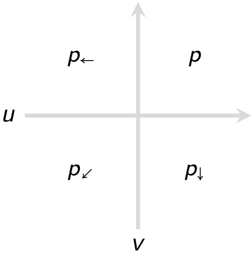

As in the one-colored case we can define colored height functions $h_{\geq c}$, which count only the paths of colors $\geq c$.

We hope to understand the large-scale pictures like the one below:



Also made by L. Petrov

Local relation still holds: fix incoming boundary conditions and a sequence of colors $c_1 \leq \cdots \leq c_r$.




$$\mathcal{Q}_{\geq \mathrm{c}}([p_1][p_2]\dots) := q^{h_{\geq c_1}(p_1)+h_{\geq c_2}(p_2)+\dots}$$

$$\begin{aligned}\mathbb{E}\left[\mathcal{Q}_{\geq \mathrm{c}}([p]^r) \,\middle|\, \begin{matrix}\text{incoming}\\ \text{colors}\end{matrix}\right] &= \frac{q^r u - qv}{u - qv}\mathcal{Q}_{\geq \mathrm{c}}([p_{\downarrow}]^r) \\ &+ \frac{v - qu}{u - qv}\sum_{i=0}^{r-1} q^i \mathcal{Q}_{\geq \mathrm{c}}([p_{\downarrow}]^i[p_{\swarrow}][p_{\downarrow}]^{r-i-1}) \\ &+ \frac{u - v}{u - qv}\sum_{i=0}^{r-1} q^i \mathcal{Q}_{\geq \mathrm{c}}([p_{\downarrow}]^i[p_{\leftarrow}][p_{\downarrow}]^{r-i-1})\end{aligned}$$

Using the local relation, we can establish the following integral expression:

### Theorem [Bufetov,K.-2020]

Let $c_1 \leq c_2 \leq \cdots \leq c_k$ be colors, $x_1 \leq \cdots \leq x_k$ and $y_1 \geq \cdots \geq y_k$ be coordinates of the computed height functions $h_{\geq c_{\pi(i)}}(x, y)$ for a permutation $\pi$. Then

$$\mathbb{E}\left[q^{h_{\geq c_{\pi(1)}}(x_1,y_1)+\cdots+h_{\geq c_{\pi(k)}}(x_k,y_k)}\right] = \frac{q^{\frac{k(k-1)}{2}-l(\pi)}}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \prod_{a=1}^{k} \frac{dw_a}{w_a}$$

$$T_{\pi^{-1}} \left( \prod_{a=1}^{k} \prod_{j=1}^{c_a - 1} \frac{1 - q u_j w_a}{1 - u_j w_a} \right) \prod_{i=1}^{k} \prod_{j=1}^{j<y_a} \frac{1 - u_j w_a}{1 - q u_j w_a} \prod_{j=1}^{i<x_a} \frac{1 - q v_i w_a}{1 - v_i w_a}$$

Here $T_\pi$ is defined by

$$T_i = q + \frac{w_{i+1} - q w_i}{w_{i+1} - w_i}(\sigma_i - 1).$$

Special cases:

- All colors $c_i$ are equal to 1, that is, one-colored model. We can set $\pi = id$.

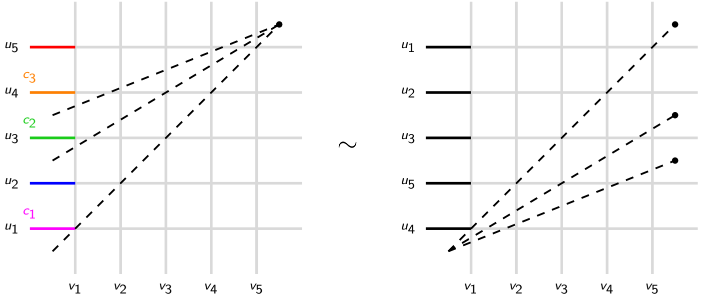
$$\mathbb{E}\left[q^{h(x_1,y_1)+\cdots+h(x_k,y_k)}\right] = \frac{q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \prod_{i=1}^{k} \prod_{j=1}^{j<y_a} \frac{1 - u_j w_a}{1 - q u_j w_a} \prod_{j=1}^{i<x_a} \frac{1 - q v_i w_a}{1 - v_i w_a} \frac{dw_a}{w_a}$$

- Height functions of different colors at a single point. All $x_i$ and $y_j$ are equal, and we can set $\pi = id$.

$$\mathbb{E}\left[q^{h_{\geq c_1}(x,y)+\cdots+h_{\geq c_k}(x,y)}\right] = \frac{q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \prod_{a=1}^{k} \prod_{j=c_a}^{j<y} \frac{1 - u_j w_a}{1 - q u_j w_a} \prod_{j=1}^{i<x} \frac{1 - q v_i w_a}{1 - v_i w_a} \frac{dw_a}{w_a}$$

The formulas on the previous slide look similar, and they can be exactly matched:

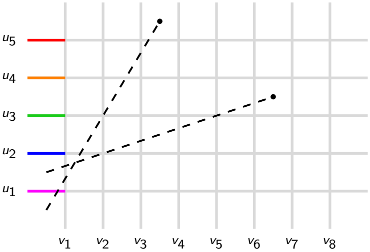
$$\mathbb{E}\left[q^{h_{\geq c_1}(x,y)+\cdots+h_{\geq c_k}(x,y)}\right] = \widetilde{\mathbb{E}}\left[q^{h(x,y-c_1)+\cdots+h(x,y-c_k)}\right]$$

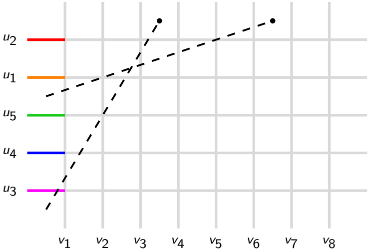


This matching is a special case of *shift-invariance*. More generally:

Given two generic stochastic colored vertex models $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, the vectors of height functions have the same finite-dimensional distributions iff the one-dimensional distributions coincide.

Visualizing height functions by "cuts", the multidimensional distributions of height functions are determined by row and column paramters interesting the cuts.



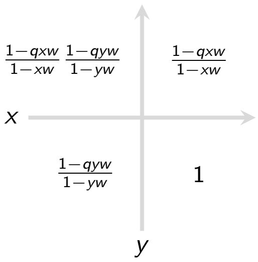



Shift-invariance:

- Was initially discovered for the case $\pi = \widetilde{\pi} = id$ (when all dashed lines intersect) in [Borodin,Gorin,Wheeler-19].
- For $\pi = \widetilde{\pi}$ proved in [Bufetov,K.-20] for more general domains by matching integral formulas for $q$-moments. Since $q$-moments uniquely determine the joint distributions, such a matching is enough.
- The most general case is proved in [Galashin-20] by considering flip symmetry.

The recurrence relation for the integral expressions:

$$\frac{q^{\frac{k(k-1)}{2}-l(\pi)}}{(2\pi i)^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \left( \prod_{i=1}^{k} F^{-(\gamma_i,\delta_i)}(w_i) \right) T_{\pi^{-1}} \left( \prod_{i=1}^{k} F^{(\alpha_i,\beta_i)}(w_i) \right) \frac{dw_a}{w_a}$$



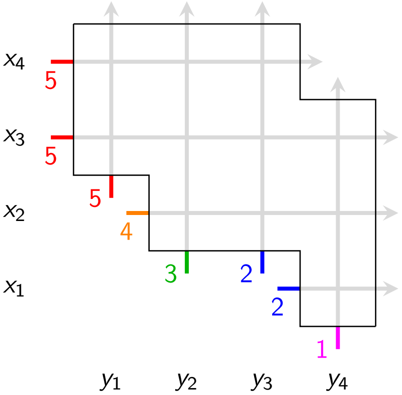

$$\prod_{i=1}^{r} \frac{1-qxw_i}{1-xw_i}$$

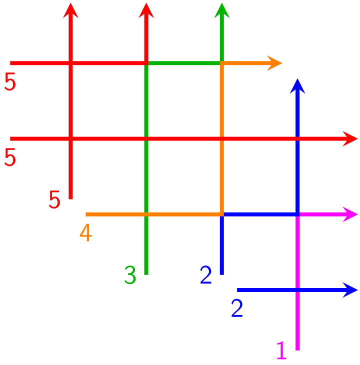
$$= \frac{qy - q^r x}{qy - x}$$

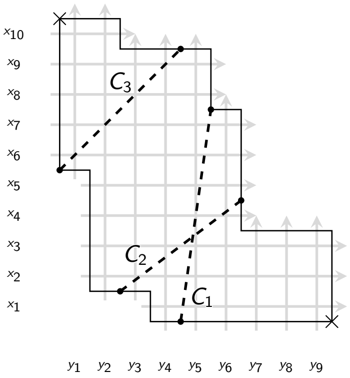
$$+ \frac{qx - y}{qy - x} \sum_{i=0}^{r-1} T_i \dots T_1 \frac{1-qyw_1}{1-yw_1}$$

$$+ \frac{y - x}{qy - x} \sum_{i=0}^{r-1} T_i \dots T_1 \frac{1-qxw_1}{1-xw_1} \frac{1-qyw_1}{1-yw_1}$$

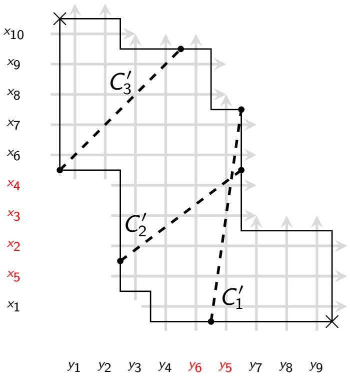
# Some additional pictures





x10
x9
x8
x7
x6
x5
x4
x3
x2
x1
y1 y2 y3 y4 y5 y6 y7 y8 y9
C3
C2
C1



x10
x9
x8
x7
x6
x4
x3
x2
x5
x1
y1 y2 y3 y4 y6 y5 y7 y8 y9
C3′
C2′
C1′