Alcove walk formula for SSV polynomials

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Solvable Lattice Models Seminar

SSV polynomials $E^{(n)}_{m}(x; q, k, G_1, \dots, G_{\lfloor \frac{n}{2} \rfloor})$

$n=1$ → Macdonald polynomials $E_m(x; q, k)$

$q \to \infty$ or $q \to 0$, G's → Gauss sums → Metaplectic Iwahori Whittaker functions $\phi_{\varepsilon, w}(x; \varpi^{-\lambda} w')$

$$\phi_{\varepsilon, w}(x; \varpi^{-\lambda}) \sim x^{-\rho} \lim_{q \to \infty} E^{(n)}_{-w(\lambda+\rho)}(x^{-1})$$

and

$$\phi_{\varepsilon, w}(x; \varpi^{-\lambda} w_0) \sim x^{-\rho} \lim_{q \to 0} E^{(n)}_{-ww_0(\lambda+\rho)}(x^{-1})$$

for λ dominant, $\rho = (r-1, r-2, \dots, 0)$, w_0 the longest element of S_r, G parameters appropriately specialized.

Outline:

- Weyl groups and alcove walks
- SSV polynomials, Ram-Yip style alcove walk formula
- Consequences of alcove walk formula
- Connection with metaplectic Iwahori-Whittaker functions

Basic Notation (Type A, GL_r)

Fix $r \in \mathbb{Z}_{>0}$. Consider $\mathbb{R}^r$, with basis $\varepsilon_1, \dots, \varepsilon_r$, inner product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

We use the following notation:

$\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq r\}$

$\Phi_+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r\}$

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$\theta = \varepsilon_1 - \varepsilon_r$, longest root

$Q = \bigoplus_{i=1}^{r-1} \mathbb{Z}\alpha_i$, root lattice

$P = \mathbb{Z}^r$, weight lattice

$W_0 =$ group generated by reflections in Φ $\cong S_r$, finite Weyl group

W_0 is a Coxeter group with presentation

$$W_0 = \langle s_1, \dots, s_{r-1} : s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } i,j \text{ non-consecutive} \rangle$$

where s_i is the reflection through the hyperplane orthogonal to α_i.

Affine Versions:

Let $\mathbb{R}^r_a = \mathbb{R}^{r+1} = \mathbb{R}^r \oplus \mathbb{R}\delta$. Extend bilinear form by $(\delta, \mathbb{R}^{r+1}_a) = 0$.

$\tilde{\Phi} = \{\alpha + s\delta : \alpha \in \Phi, s \in \mathbb{Z}\}$, (real) affine roots

$\tilde{\Phi}_+ = \Phi_+ \cup \{\alpha + s\delta : s \in \mathbb{Z}_{>0}\}$, positive affine roots

$\alpha_0 = \delta - \theta = \delta - (\varepsilon_1 - \varepsilon_r)$

We have a pairing $\langle \cdot , \cdot \rangle : \mathbb{R}^r_a \times \mathbb{R}^r \longrightarrow \mathbb{R}$ by

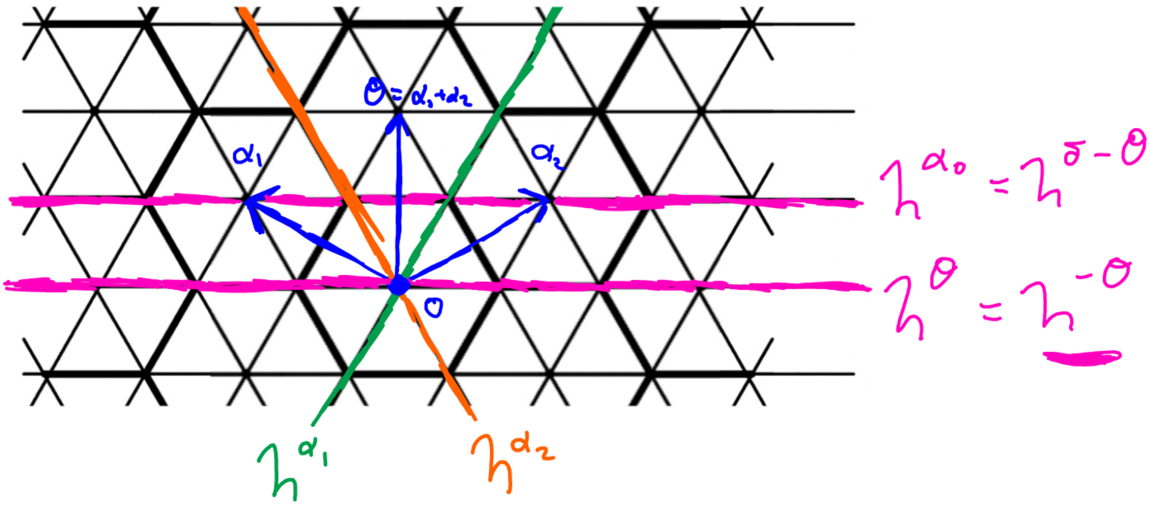
$$\langle u + s\delta , v \rangle = (u,v) + s .$$

For $\hat{\alpha} = \alpha + s\delta \in \tilde{\Phi}$, define the hyperplane

$$\mathfrak{h}^{\hat{\alpha}} = \{ v \in \mathbb{R}^r : \langle \hat{\alpha}, v \rangle = 0 \} .$$

Let $s_{\hat{\alpha}}$ be the reflection through $\mathfrak{h}^{\hat{\alpha}}$.

Example: $r = 3$. Type A_2. Picture restricted to $\mathrm{span}\{\alpha_1, \alpha_2\}$.



We have the affine Weyl group

$$W = \langle s_{\hat{\alpha}} : \hat{\alpha} \in \tilde{\Phi} \rangle = \langle s_{\alpha_i} : 0 \le i \le r-1 \rangle .$$

W is a Coxeter group with generators $s_i = s_{\alpha_i}$ $(0 \le i \le r-1)$ braid relations from

For $\mu \in Q$, let $T(\mu)$ be the translation by μ on $\mathbb{R}^r$:

$$T(\mu) v = v + \mu .$$

We have $s_0 = T(\theta) s_\theta$ and $w T(\mu) w^{-1} = T(w\mu)$ $(w \in W_0)$, leading to

$$W = W_0 \ltimes T(Q) .$$

Note W preserves $P = \mathbb{Z}^r$.

Also preserves $\{\lambda \in \mathbb{Z}^r : \sum \lambda_i = c\}$ for any c.

W also acts on $\mathbb{R}^r_a = \mathbb{R}^r \oplus \mathbb{R}\delta$ in the usual way:

For $\hat{\alpha} = \alpha + s\delta \in \tilde{\Phi}$, $\hat{v} = v + t\delta \in \mathbb{R}^r_a$,

$$s_{\hat{\alpha}} * \hat{v} = \hat{v} - \frac{2(\hat{\alpha}, \hat{v})}{(\hat{\alpha}, \hat{\alpha})}\hat{\alpha} = \hat{v} - (\alpha, v)\hat{\alpha}.$$

Versions depending on $n \in \mathbb{Z}_{>0}$:

$$\tilde{\hat{\Phi}}^{(n)} = \{ \alpha + n s \delta : \alpha \in \Phi,\ s \in \mathbb{Z} \} \subseteq \hat{\Phi}$$

$$\tilde{\hat{\Phi}}^{(n)}_+ = \Phi_+ \cup \{ \alpha + n s \delta : \alpha \in \Phi,\ s \in \mathbb{Z}_{>0} \} \subseteq \hat{\Phi}_+$$

$$\alpha_0^{(n)} = n\delta - \theta, \qquad \alpha_i^{(n)} = \alpha_i \quad (i \neq 0)$$

$$s_0^{(n)} = s_{\alpha_0^{(n)}} = \Upsilon(n\theta) s_\theta, \qquad s_i^{(n)} = s_{\alpha_i} = s_i \quad (i \neq 0)$$

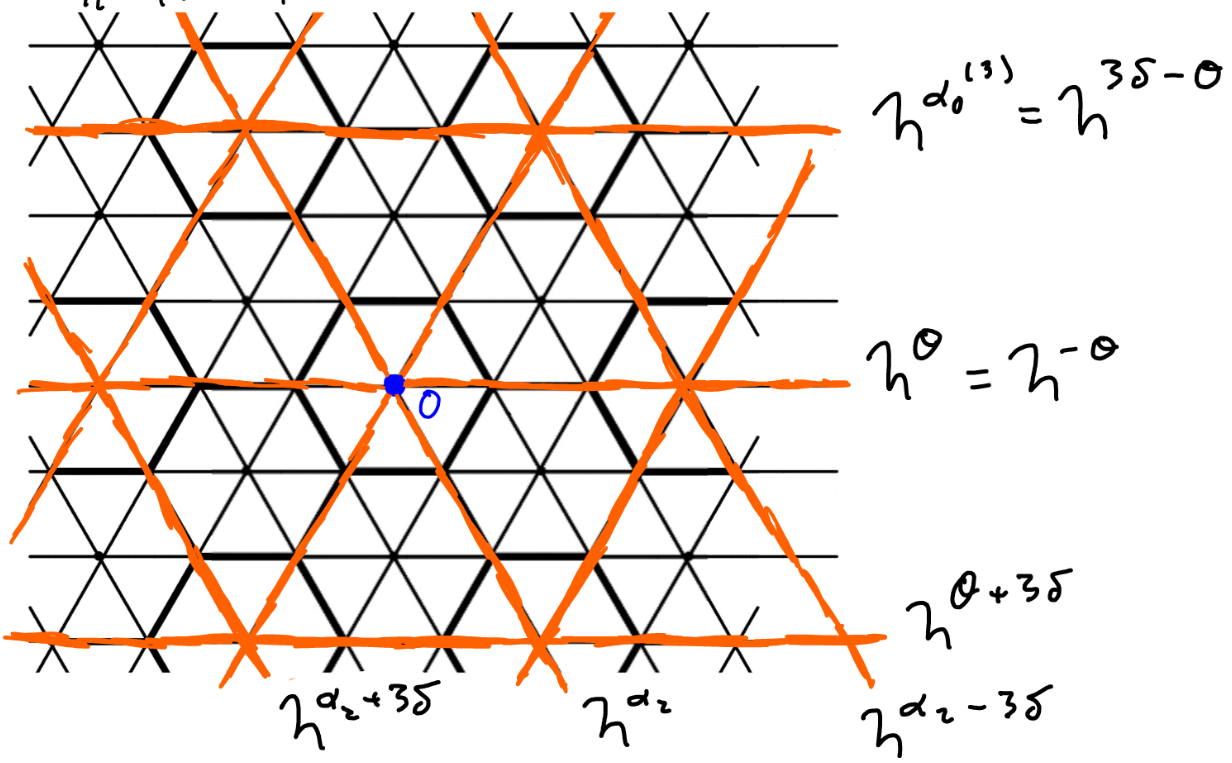
$$W^{(n)} = \langle s_{\hat{\alpha}} : \hat{\alpha} \in \tilde{\hat{\Phi}}^{(n)} \rangle = \langle s_i^{(n)} : 0 \leq i \leq r-1 \rangle$$

$$= W_0 \ltimes \Upsilon(nQ) \subseteq W.$$

We have

$$W^{(n)} \cong W \quad \text{by} \quad s_i^{(n)} \longmapsto s_i$$

Example: $W^{(3)}$ is the group of reflections over the orange hyperplanes.



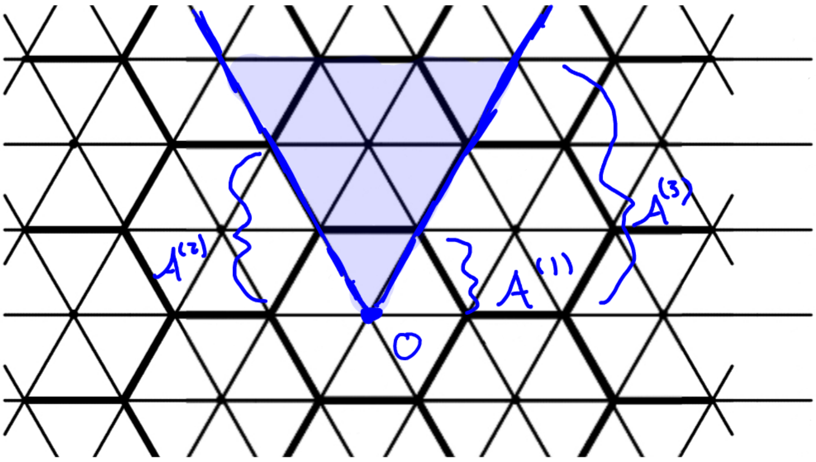
The n-alcoves are the connected components of

$$\mathbb{R}^r \setminus \bigcup_{\hat\alpha \in \tilde\Phi^{(n)}} h^{\hat\alpha}.$$

The fundamental n-alcove is

$$A^{(n)} = \{ v \in \mathbb{R}^r : \langle \alpha_i^{(n)}, v \rangle > 0 \text{ for } 0 \le i \le r-1 \}$$
$$= \{ v \in \mathbb{R}^r : v_1 > v_2 > \dots > v_r,\ v_1 - v_r < n \}$$

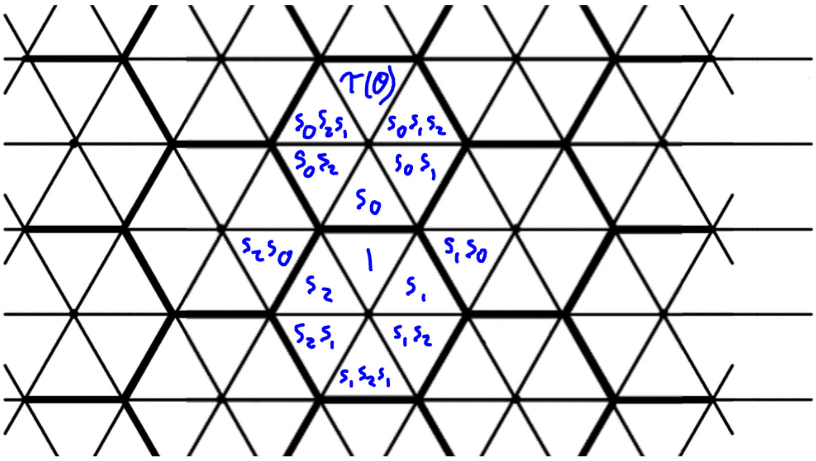
Here we see $A^{(1)}, A^{(2)}, A^{(3)}$.



(These pictures only show $\operatorname{span}\{\alpha_1, \alpha_2\}$.)

We identify elements of $W^{(n)}$ with n-alcoves by $w \longleftrightarrow wA^{(n)}$.

Here are the 1-alcoves, labeled by elements of W.



We also have the closed fundamental n-alcove

$$\overline{A^{(n)}} = \{ v \in \mathbb{R}^r : v_1 \ge \dots \ge v_r,\ v_1 - v_r \le n \}$$

and its integer points

$$\overline{A^{(n)}_{\mathbb{Z}}} = \{ \lambda \in \mathbb{Z}^r : \lambda_1 \ge \dots \ge \lambda_r,\ \lambda_1 - \lambda_r \le n \}.$$

$\overline{A}^{(n)}_{\mathbb{Z}}$ is a fundamental domain for the action of $W^{(n)}$ on $P = \mathbb{Z}^r$.

Let $w \in W^{(n)}$. Fix a reduced expression $w = s^{(n)}_{i_1} \cdots s^{(n)}_{i_\ell}$.
Let $\vec{w} = (i_1, \dots, i_\ell)$.
An n-alcove walk of type $\vec{w}$ is a

sequence of n-alcoves $p = (A_0, A_1, \dots, A_\ell)$

where $A_j = A_{j-1}$ (called a fold)

or $A_j = A_{j-1} s^{(n)}_{i_j}$ (called a crossing)

interpret A_{j-1} as an element of $W^{(n)}$

In the latter case, A_j and A_{j-1} are adjacent alcoves: if $A_{j-1} \longleftrightarrow u \in W^{(n)}$,

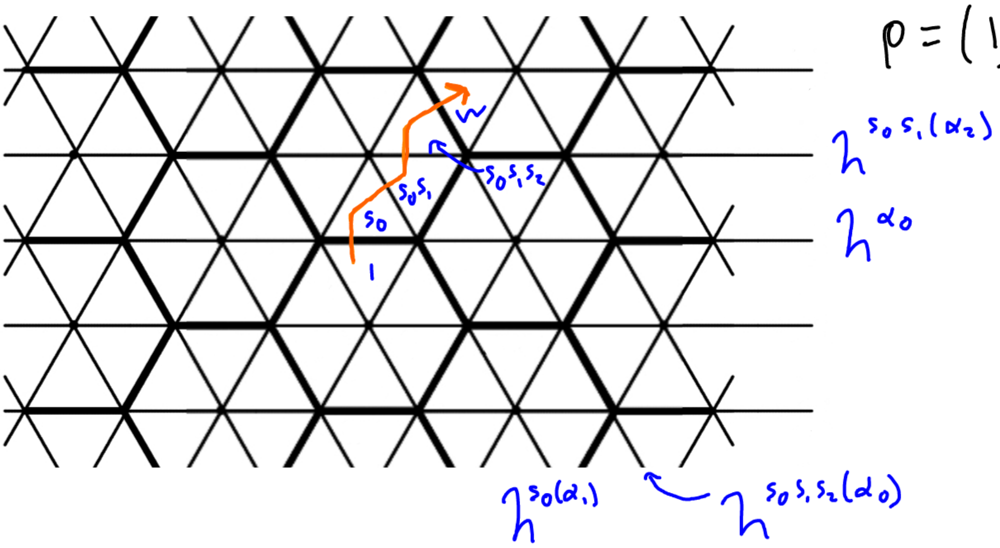
$$u s_{i_j} = u s_{i_j} u^{-1} u = s_{u * \alpha^{(n)}_{i_j}} u$$

and the hyperplane orthogonal to $u * \alpha^{(n)}_{i_j}$ is a wall of u.

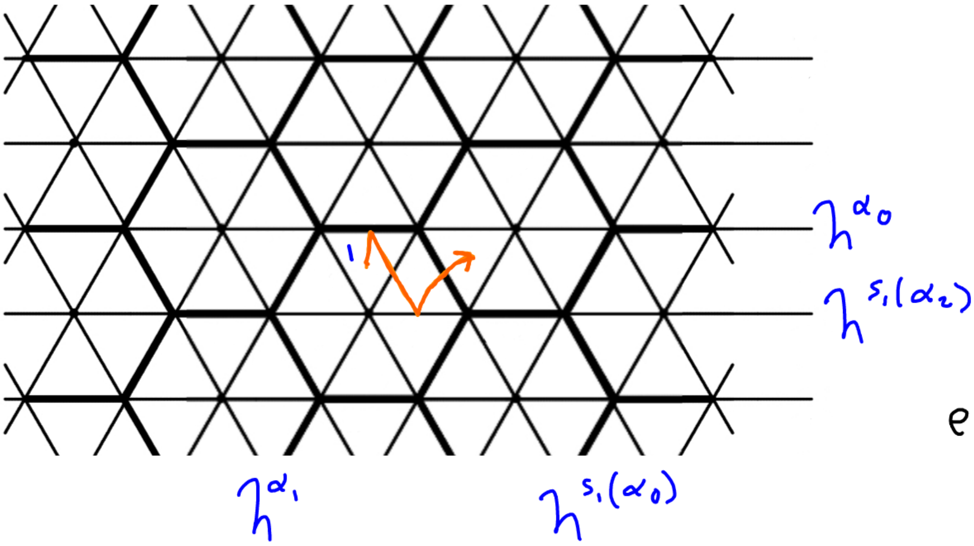
Examples: 1-alcove walks of type $(0,1,2,0)$ starting at 1:

$w = s_0 s_1 s_2 s_0$

$p = (1, s_0, s_0 s_1, s_0 s_1 s_2, s_0 s_1 s_2 s_0)$



$$\mathrm{end}(p) = s_0 s_1 s_2 s_0 = T(\alpha_1 + 2\alpha_2) \underbrace{s_1 s_2}_{\varphi(p)}$$

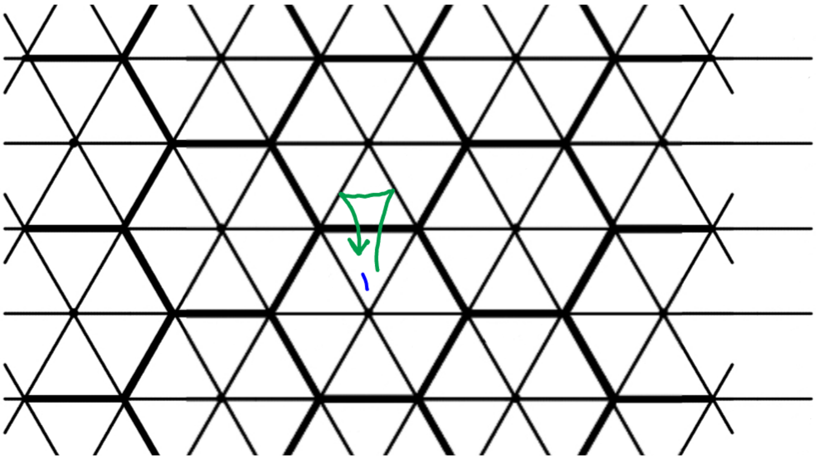


$w = s_0 s_1 s_2 s_0$

$p = (1, 1, s_1, s_1, s_1 s_0)$

fold fold

$\mathrm{end}(p) = s_1 s_0 = \tau(\alpha_2)\underbrace{s_1 s_2}_{\varphi(p)}$



$w = s_0 s_1 s_2 s_0$

$p = (1, s_0, s_0, s_0, s_0 s_0 = 1)$

folds

$\mathrm{end}(p) = 1$

We now orient the hyperplanes and assign signs to folds.

For $\alpha \in \Phi$, an alcove is on the positive side of h^{α} if it is on the same side as $A^{(n)}$. (Otherwise it's on the negative side.)

The orientation for $h^{\alpha + s\delta}$ is parallel to that of h^{α}.

A fold is positive if it stays on the positive side of the corresponding hyperplane, negative otherwise.

Let $Alc^{(n)}(\vec{w})$ be the set of all n-alcove walks of type $\vec{w}$ that start at 1.

We now introduce SSV polynomials and state the alcove walk formula.

Fix $n \in \mathbb{Z}_{>0}$; parameters q, k, and $G_s^{(n)}$, $s \in \mathbb{Z}$, satisfying:

- $G_s^{(n)} = G_t^{(n)}$ if $s \equiv t \mod n$
- $G_0^{(n)} = k$
- $G_s^{(n)} G_{-s}^{(n)} = 1$ if $s \not\equiv 0 \mod n$.

BBBG'21 notation

$G_s^{(n)} = -k^{-1} g_s$, $k^2 = v$

Let $\mathbb{F} = \mathbb{C}(q, k, G_1^{(n)}, \dots, G_{\lfloor \frac{n}{2} \rfloor}^{(n)})$.

Let $\mathbb{F}[x^{\pm 1}] = \mathrm{span}_{\mathbb{F}}\{ x^m : m \in \mathbb{Z}^r \} =$ Laurent polynomials in $x_1, \dots, x_r$

Results of Sahi, Stokman, and Venkateswaran:

- Constructed operators $Y^{\hat{v}}$, $\hat{v} \in \mathbb{Z}^r \oplus n\mathbb{Z}\delta$, on $\mathbb{F}[x^{\pm 1}]$ coming from new DAHA representation.

- The $Y^{\hat{v}}$ are commutative and diagonalizable, with eigenvalues $\gamma(\hat{v};m)$, $m \in \mathbb{Z}^r$. (Formula later.)

- There exists a unique family of Laurent polynomials $\{E_m^{(n)} : m \in \mathbb{Z}^r\}$ such that:

  1) $Y^{\hat{v}} E_m^{(n)} = \gamma(\hat{v};m) E_m^{(n)}$ for all $\hat{v} \in \mathbb{Z}^r \oplus n\mathbb{Z}\delta$, $m \in \mathbb{Z}^r$.

  2) x^m has coefficient 1 when $E_m^{(n)}$ is expanded in the monomial basis.

  We call $E_m^{(n)}$ an SSV polynomial.
  We sometimes emphasize the dependence on some or all of the parameters in our notation:
  $$E_m^{(n)} = E_m^{(n)}(x) = E_m^{(n)}(x; q, k, G_1, \ldots, G_{\lfloor \frac{n}{2} \rfloor}).$$

- $E_m^{(1)} = E_m$, the Macdonald polynomial

- $E_{nm}^{(n)}(x; q, k) = E_m(x^n; q^n, k)$.

- Connected SSV polynomials and Whittaker functions (more on that later)

- For $\lambda \in \overline{A_{\mathbb{Z}}^{(n)}}$ $(\lambda_1 \geq \ldots \geq \lambda_r,\ \lambda_1 - \lambda_r \leq n)$,
  $E_\lambda^{(n)} = x^\lambda$.

- Constructed Υ_i, $0 \leq i \leq r-1$, with $\Upsilon_i E_m^{(n)} \sim E_{s_i m}^{(n)}$

Theorem (S): Let $\mu \in \mathbb{Z}^r$. Find unique $\lambda \in \overline{A^{(n)}_{\mathbb{Z}}}$ in the $W^{(n)}$-orbit of μ. Let $w \in W^{(n)}$ be the shortest element such that $w\lambda = \mu$. Write $w = s^{(n)}_{i_1} \cdots s^{(n)}_{i_\ell}$, a reduced expression. Then

$$E^{(n)}_\mu \sim \sum_{p \in \mathrm{Alc}^{(n)}(\vec{w})} x^{\mathrm{end}(p)\lambda} \left(\prod_{a=1}^{\ell(\varphi(p))} \sigma((\lambda, \eta_{p,a})) \right) \cdot$$
$$\cdot \left(\prod_{\substack{\text{positive} \\ \text{folds } j}} \frac{k^{-1} - k}{1 - \gamma(-\beta_j ; \lambda)} \right) \cdot \left(\prod_{\substack{\text{negative} \\ \text{folds } j}} \frac{(k^{-1} - k)\,\gamma(-\beta_j ; \lambda)}{1 - \gamma(-\beta_j ; \lambda)} \right)$$

where $\varphi(p) \in W_0$ satisfies $\mathrm{end}(p) = \tau(\nu)\varphi(p)$,

$\beta_j = s^{(n)}_{i_\ell} \cdots s^{(n)}_{i_{j+1}} * \alpha^{(n)}_{i_j}$, $\eta_{p,a}$ is similar but for $\varphi(p)$,

$$\sigma(s) = \begin{cases} k^{-1} & : s = n, 2n, 3n, \ldots \\ k & : s = 0, -n, -2n, \ldots \\ G^{(n)}_s & : \text{otherwise} \end{cases}, \quad \gamma(-\beta_j ; \lambda) = q^{\langle \beta_j, \lambda \rangle} \prod_{\alpha \in \Phi^+} \sigma((\lambda, \alpha))^{(-\beta_j, \alpha)}.$$

Corollary 1: For $\lambda \in \overline{A^{(n)}_{\mathbb{Z}}}$, define a partial order $\hat{\leq}$ on $W^{(n)}\lambda$ by $u_1\lambda \,\hat{\leq}\, u_2\lambda$ iff $u_1 < u_2$ in the Bruhat order on $W^{(n)}$. Then

$$E^{(n)}_\mu = x^\mu + \sum_{\nu \,\hat{<}\, \mu} c_\nu x^\nu.$$

In particular, $\{E^{(n)}_\mu : \mu \in \mathbb{Z}^r\}$ is a basis for $\mathbb{F}[x^{\pm 1}]$.

Proof:

powers of x in formula $\longleftrightarrow \mathrm{end}(p)\lambda \longleftrightarrow u\lambda$, u a subword of w $\longleftrightarrow u\lambda \,\hat{\leq}\, w\lambda = \mu$ $\square$

Corollary 2: In Corollary 1, all $c_\nu \neq 0$.

Proof:

Specialize $0 < k < 1$, $G^{(n)}_s = 1$ $(s \neq 0)$. In the formula

$$E^{(n)}_\mu \sim \sum_{p \in \mathrm{Alc}^{(n)}(\vec{w})} x^{\mathrm{end}(p)\lambda} \left(\prod_{a=1}^{\ell(\varphi(p))} \sigma((\lambda, \eta_{p,a})) \right) \cdot$$
$$\cdot \left(\prod_{\substack{\text{positive} \\ \text{folds } j}} \frac{k^{-1} - k}{1 - \gamma(-\beta_j ; \lambda)} \right) \cdot \left(\prod_{\substack{\text{negative} \\ \text{folds } j}} \frac{(k^{-1} - k)\,\gamma(-\beta_j ; \lambda)}{1 - \gamma(-\beta_j ; \lambda)} \right),$$

all coefficients are positive as long as $0 < \gamma(-\beta_j; \lambda) < 1$.

$\gamma(-\beta_j; \lambda) = q^{\text{positive}} \cdot (\text{product of } k\text{'s and } G\text{'s})$,

so choose $0 < q < 1$ sufficiently small.

All coefficients are > 0. $\square$

Corollary 3:

If $m \mid n$, then

$$\left\{ \begin{matrix} \text{powers of } x \\ \text{in } E_\mu^{(n)} \end{matrix} \right\} \subseteq \left\{ \begin{matrix} \text{powers of } x \\ \text{in } E_\mu^{(m)} \end{matrix} \right\} \subseteq \left\{ \begin{matrix} \text{powers of } x \\ \text{in } E_\mu \end{matrix} \right\}.$$

<u>Proof sketch:</u>

Show $\hat{\leq}$ is weaker than $\overset{m}{\leq}$ (as orderings on $\mathbb{Z}^r$).

Relies on the characterization that, for $\hat{\alpha} \in \tilde{\Phi}_+^{(n)}$,

$$S_{\hat{\alpha}}\, \mu \;\hat{<}\; \mu \quad \text{iff} \quad \langle \hat{\alpha}, \mu \rangle < 0. \qquad \square$$

Corollary 4:

Can symmetrize to get alcove walk formula for $P_\mu^{(n)}$, "symmetric" analog of $E_\mu^{(n)}$.

(Symmetric with respect to Chinta-Gunnells Weyl group action.)

Since $\gamma(-\beta_j; \lambda) = q^{\text{positive}} \cdot (\text{product of } k\text{'s and } G\text{'s})$, and nothing else depends on q, can easily take q-limits:

(negative folds)

$$\frac{\gamma(-\beta_j;\lambda)}{1-\gamma(-\beta_j;\lambda)} \quad \xrightarrow{q\to 0} 0, \qquad \xrightarrow{q\to\infty} -1$$

(positive folds)

$$\frac{1}{1-\gamma(-\beta_j;\lambda)} \quad \xrightarrow{q\to 0} 1, \qquad \xrightarrow{q\to\infty} 0$$

Corollary 5:

$$\lim_{q\to 0} E_\mu^{(n)} \sim \sum_{p\in \mathrm{Alc}_+^{(n)}(\vec{w})} x^{\mathrm{end}(p)\lambda} \left(\prod_{a=1}^{\ell(\varphi(p))} \sigma((\lambda, \eta_{p,a}))\right) (k^{-1}-k)^{\#\text{ of Folds}}$$

$$\lim_{q\to \infty} E_\mu^{(n)} \sim \sum_{p\in \mathrm{Alc}_-^{(n)}(\vec{w})} x^{\mathrm{end}(p)\lambda} \left(\prod_{a=1}^{\ell(\varphi(p))} \sigma((\lambda, \eta_{p,a}))\right) (k-k^{-1})^{\#\text{ of Folds}}$$

where $\mathrm{Alc}_+^{(n)}(\vec{w}) \subseteq \mathrm{Alc}^{(n)}(\vec{w})$ is the set of walks with only positive folds, same for $\mathrm{Alc}_-^{(n)}(\vec{w})$.

Corollary 6:

If μ is dominant $(\mu_1 \geq \dots \geq \mu_r)$, then

$$\lim_{q\to 0} E_\mu^{(n)} = x^\mu.$$

If μ is antidominant $(\mu_1 \leq \dots \leq \mu_r)$, then

$$\lim_{q\to \infty} E_\mu^{(n)} = x^\mu.$$

Proof sketch: Show the first fold in an alcove walk to dominant μ must be negative. $\square$

SSV's and (averaged) metaplectic Iwahori Whittaker functions

(Notation as in Brubaker, Buciumas, Bump, Gustafsson '21.)

Denote the averaged n-fold metaplectic Iwahori–Whittaker function by $\phi_{\varepsilon,w}(x;\varpi^{-\lambda}w')$.

We are interested in the value $\phi_{\varepsilon,w}(x;\varpi^{-\lambda})$ where λ is dominant.

<u>Thm (SSV):</u> $$\boxed{\phi_{\varepsilon,w}(x;\varpi^{-\lambda}) \sim x^{-\rho} \lim_{q\to\infty} E^{(n)}_{-w(\lambda+\rho)}(x^{-1})}$$

where $\rho=(r-1,r-2,\dots,0)$ and λ is dominant.

Whittaker	SSV
$\phi_{\varepsilon,1}(x;\varpi^{-\lambda}) = x^{\lambda}$	$x^{-\rho} \lim_{q\to\infty} E^{(n)}_{-(\lambda+\rho)}(x^{-1};q,k) \overset{\text{Corollary 6}}{=} x^{-\rho}\left(x^{-(\lambda+\rho)}\right)\Big\vert_{x\mapsto x^{-1}} = x^{-\rho}x^{\lambda+\rho} = x^{\lambda}.$
depending on k and G but not q	comes from DAHA rep of SSV, depends on q
For some operator T_i, whenever $s_i w > w$, $\phi_{\varepsilon,s_iw}(x;\varpi^{-\lambda}) \sim T_i\,\phi_{\varepsilon,w}(x;\varpi^{-\lambda})$.	For some operator Υ_i, whenever $s_i\mu \neq \mu$, $E_{s_i\mu}(x) \sim \Upsilon_i\, E_\mu(x)$.

Can show that, applied to suitable $f(x) \in \mathbb{F}[x^{\pm1}]$,

$$T_i \lim_{q\to\infty} f(x) \sim \lim_{q\to\infty} x^{-\rho}\,\iota\,\Upsilon_i\,\iota\,x^{\rho} f(x)$$

where ι is the map $x\mapsto x^{-1}$. Then

$$T_i \lim_{q\to\infty} x^{-\rho} E^{(n)}_{-w(\lambda+\rho)}(x^{-1}) \sim \lim_{q\to\infty} x^{-\rho}\,\iota\,\Upsilon_i\,\iota\,x^{\rho}x^{-\rho} E^{(n)}_{-w(\lambda+\rho)}(x^{-1})$$

$$= \lim_{q\to\infty} x^{-\rho} \iota\, T_i\, E^{(n)}_{-w(\lambda+\rho)}(x)$$

$$\sim \lim_{q\to\infty} x^{-\rho} \iota\, E^{(n)}_{-s_i w(\lambda+\rho)}(x)$$

$$= \lim_{q\to\infty} x^{-\rho} E^{(n)}_{-s_i w(\lambda+\rho)}(x^{-1}).$$

So they satisfy the same recursion! □

Similar for

$$\phi_{\varepsilon,w}(x; \varpi^{-\lambda} w_0) \sim x^{-\rho} \lim_{q\to 0} E^{(n)}_{-ww_0(\lambda+\rho)}(x^{-1}).$$

Can possibly recover other $\phi_{\varepsilon,w}(x;\varpi^{-\lambda}w')$, $w' \in W_0$, using similar methods.

Questions:

1) Do the alcove walk formulas for $\lim_{q\to 0} E^{(n)}_\mu$, $\lim_{q\to\infty} E^{(n)}_\mu$, say something new about Whittaker functions?

2) Solvable lattice model for SSV polynomials?

3) Other representation-theoretic interpretations of SSV polynomials?