Solvable lattice models & double Grothendieck pol's

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lattice models $\rightsquigarrow$ special functions

1. combinatorics: Schur pol's
   Hall-Littlewood
   Macdonald
   LLT
2. p-adic RT: spherical / Iwahori } Whittaker fcts
3. algebraic geometry: Schur
   Schubert
   Grothendieck

this talk

Motivation: prove combinatorial formulas for special fcts (& more ?)

Geometry:

Q: How many lines intersect 4 given lines in general position in $\mathbb{R}^3$?

To solve this, use Schubert calculus.

The Grassmannian:

$G_r(k,n) = \{ k\text{-dim subspaces of } \mathbb{C}^n \}$ (or $\mathbb{R}^n$, $\mathbb{R}_p^n$ ..)

↳ smooth algebraic variety

$G_r(k,n) = \bigsqcup_{\lambda \subset \boxed{}_{k}^{n-k}} C_\lambda$, C_λ a Schubert cell (set of $k \times n$ matrices with RREF $\lambda + S$)

Schubert variety $X_\lambda = \overline{C_\lambda}$ (Zariski)

Schubert variety $X_\lambda \longrightarrow$ Schubert classes $[X_\lambda]$ in the cohomology ring $H^*(G(k,n))$

$$H^*(G(k,n)) \simeq \mathbb{C}[x_1,\dots,x_n]^{S_n} / I$$

$$[X_\lambda] \longmapsto s_\lambda(\bar{x})$$

$\hookrightarrow$ Schur pol.

multiplication in $H^*(G(k,n)) \longleftrightarrow$ intersecting Schubert varieties w.r.t. generically chosen bases.

Q: How many lines intersect 4 given lines in general position in $\mathbb{R}^3$?

L = line in $\mathbb{R}^3$, $\{$lines in $\mathbb{R}^3$ intersecting $L\} \rightsquigarrow$ Schubert variety in $G(2,4)$ $\rightsquigarrow s_{(1)}$

A: (vague!) 2, because $s_{(1)}^4 = 2\, s_{(2,2)} + s_{(3,1)} + s_{(2,1,1)}$.

for details, see S. Billey's ICERM slides.

$\Rightarrow$ study special functions (and LR coeff.) combinatorially
(lattice models)

$*$ Grassmannian:

Schur pol's $\underline{s_\lambda(\bar{x}')}$ $\rightsquigarrow$ X_λ in cohomology ring $H^*(G(k,n))$

(Gr.) β-Grothendieck pol $\underline{G^\beta_\lambda(\bar{x})}$ $\rightsquigarrow$ X_λ in connective K-theory ring of $G(k,n)$.

↳ lattice models: Motegi+Sakai

Wheeler+Zinn-Justin (double version + LR coeff.)

* The flag variety Fl_n

Fl_n is the set of complete flags $\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \dots \subsetneq V_n = \mathbb{C}^n$
$\dim V_i = i$.

Differently, $Fl_n \simeq GL_n(\mathbb{C})/B$, B = Borel = upper Δ matrices

Bruhat decomposition: $GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} BwB$.

$\rightarrow GL_n(\mathbb{C})/B = \bigsqcup_{w} BwB/B \rightsquigarrow C_w = BwB/B$ is a Schubert cell

$X_w = \overline{C_w}$; $X_w = \bigsqcup_{y \leq w} C_y = \bigcup_{y \leq w} X_y$.

<u>Remark:</u> Both $Gr(k,n)$ and Fl_n special cases of

d-step flag variety $Fl(n_1, .., n_d; n) := \{(V_1, .., V_d) : V_i \subset V_{i+1} \subset \mathbb{C}^n$
$\dim V_i = n_i \}$

Schubert varieties X_w :

Schubert pol's $\mathfrak{S}_w(\vec{x})$ $\rightsquigarrow$ X_w in cohomology of Fl_n
$w \in S_n$

↳ Bernstein-Gelfand-Gelfand
Demazure
Lascoux-Schützenberger

double β-Grothendieck pol's $G_w^\beta(\vec{x}, \vec{y})$ $\rightsquigarrow$ T-equivariant connective K-theory of Fl_n
↳ Hudson

↓ recover $\mathfrak{S}_w(\vec{x})$ for $\beta = 0$, $\vec{y} = 0$
$G_\lambda^\beta(\vec{x})$ for $\vec{y} = 0$, $\lambda = \lambda_w$, w = Gr permutation,
i.e. w has unique descent
$w(i) > w(i+1)$

↳ lattice models in this talk

Recursive def. of G_w^β (geometrically due to pull back + push forward maps in Bott-Samelson res. of sing. in Fl_n)

$x \oplus y := x + y + \beta x y$

s_i simple reflection, w_0 longest word in S_n.

$f \in \mathbb{C}[x_1, \dots, x_n]$, $s_i f$: swap $x_i \leftrightarrow x_{i+1}$.

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}} \qquad \partial_i^\beta f = \partial_i\big((1 + \beta x_{i+1}) f\big)$$

Def: $$G_{w_0}^\beta(\vec{x}, \vec{y}) = \prod_{i+j \le n} (x_i \oplus y_j)$$

$$G_w^\beta(\vec{x}, \vec{y}) = \partial_i^\beta \, G_{w s_i}^\beta(\vec{x}, \vec{y}) \quad \text{for } \ell(w s_i) = \ell(w) + 1.$$

lattice models for $G_w^\beta(\vec{x}, \vec{y})$:

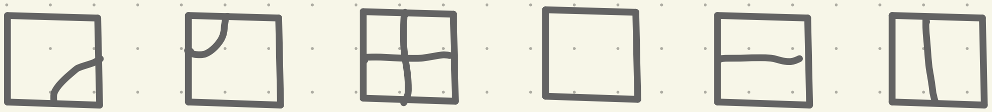
* Brubaker - Frechette - Hardt - Tibor - Weber (BFHTW) $\to$ 2X
* Buciumas - Scrimshaw (BS)

different!

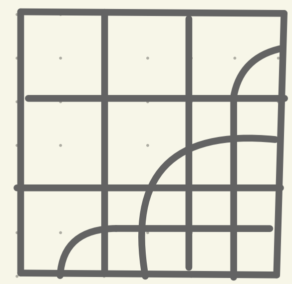
__Want to__ : prove combinatorial properties of G^B_w.

Focus on : G^B_w is a weighted sum over (bumpless) pipe dreams

__Bumpless pipe dreams__ (BPD):

tiles : 

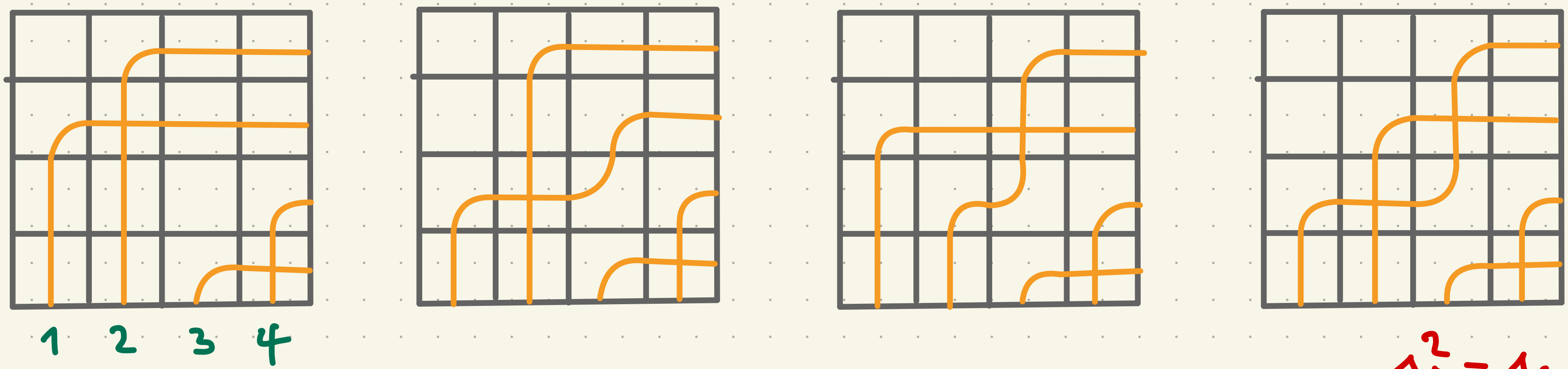
$n \times n$ BPD : tilling of $n \times n$ grid with n pipes starting bottom edge ending right edge.



$\mathcal{J}$, Key($\mathcal{J}$) $= w_0$

Key($\mathcal{J}$) $\in S_n$: trace the paths, record intersections using the Demazure product ($s_i^2 = s_i$)

Examples: $w = s_1 s_3 \in S_4$, all P with $\partial(P) = w$:
$n = 4$



$s_i^2 = s_i$.

Thm (Weigandt): $G^{\beta}_{w}(x^{-1}, y^{-1}) = \sum_{\partial(P)=w} \mathrm{wt}(P)$

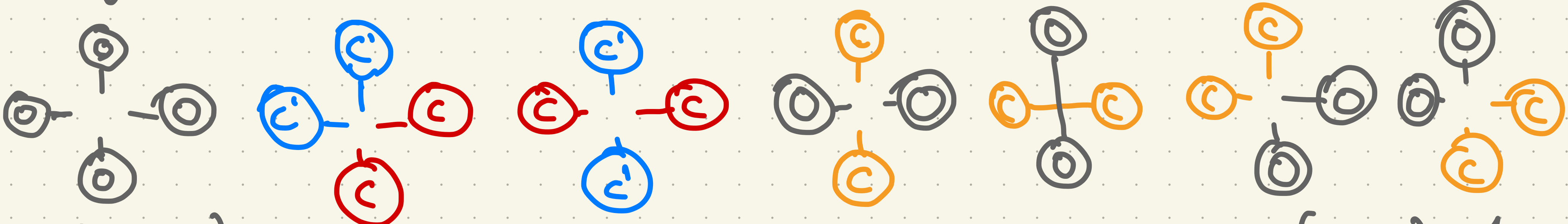
$$\mathrm{wt}(P) = \prod_{\substack{\text{tile } \square \\ \text{at } (i\ j)}} \beta(x_i \oplus y_j) \prod_{\substack{\text{tile } \boxtimes \\ \text{at } (i\ j)}} \left(1+\beta(x_i \oplus y_j)\right).$$

Proof: combinatorial, based on work of Lascoux

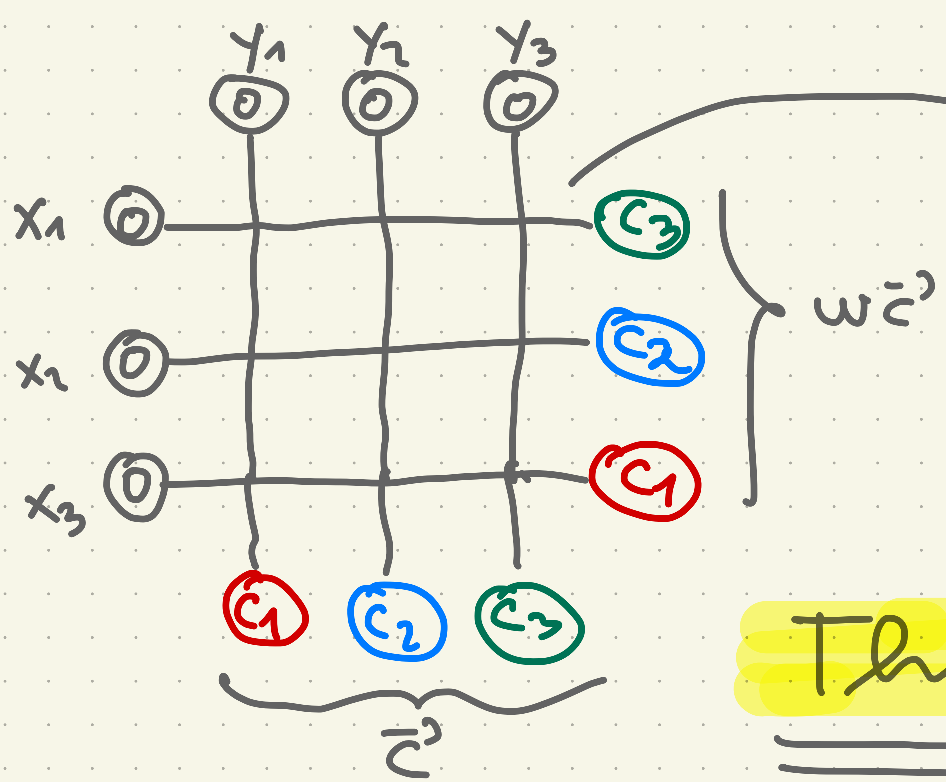
Colored lattice model for $G^{\beta}_{w}(\vec{x};\vec{y})$ (BS):

colors $c_1 > c_2 > \dots > c_n$, 0: uncolored.

weights: c any color, $c > c'$



$\beta(x_i \oplus y_j)$ $\quad 1 \quad 1 \quad 1 \quad 1 \quad 1+\beta(x_i \oplus y_j) \quad 1$



$\longrightarrow S^{\beta}_{w}(\vec{x};\vec{y})$

Thm (BS): $G^{\beta}_{w}(\vec{x};\vec{y}) = \beta^{\ell(w)} Z\left(S^{\beta}_{w}(\vec{x};\vec{y})\right)$

<u>Idea of proof</u> : once you have the weights, standard :

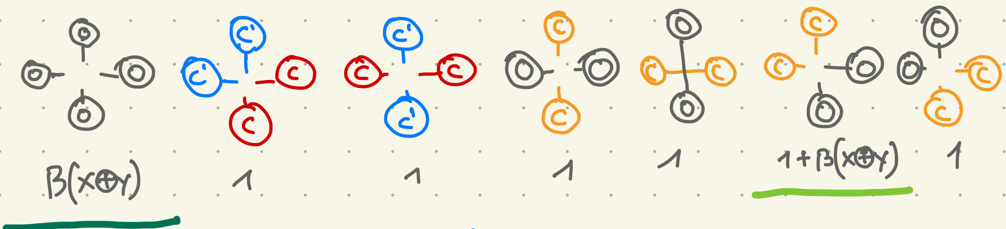
* find solution of YBE (SAGE computation + guessing)
* prove functional equation using the train argument
  compare to functional eq. for G_w^B
* compute $Z(S_{w_0}^B)$ combinatorially (one state)
  compare to $G_{w_0}^B$

Thm (Weigandt) $G^{\beta}_{w}(\bar{x}^{?}, \bar{y}^{-1}) = \sum_{\partial(P)=w} wt(P)$

B. Scrimshaw: straightforward proof, once you have model (cheating?)

there is weight preserving bijection between:

states in S_w $\longleftrightarrow$ P with $\partial P = w$



$c > c'$

$$wt(P) = \prod_{\substack{\text{tile } \boxplus \\ \text{at } (i,j)}} \beta(x_i \oplus y_j) \prod_{\substack{\text{tile } \boxtimes \\ \text{at } (i,j)}} \left(1+\beta(x_i \oplus y_j)\right).$$

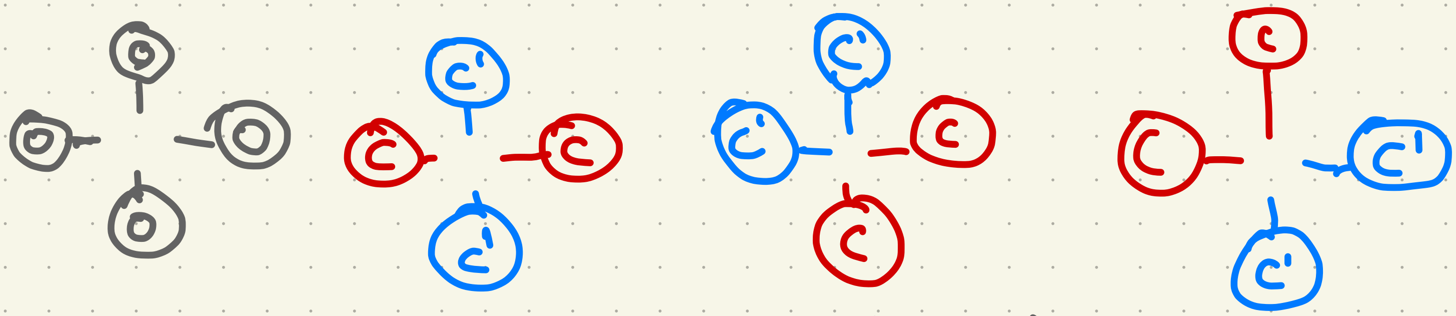
Example:

$w = s_1 s_3$

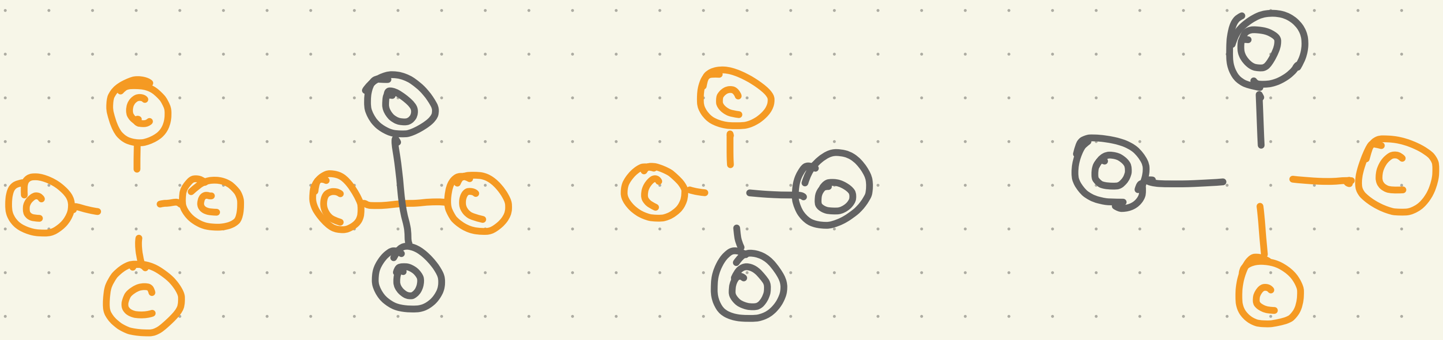
The (P) model in BFHTW

different conventions

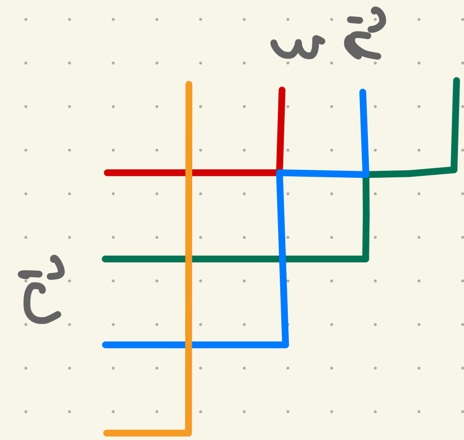
weights: c any color, c < c'



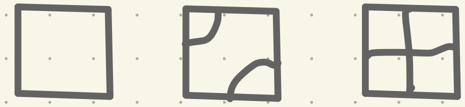
1 $\qquad x_i \oplus y_j \qquad 1+\beta(x_i \oplus y_j) \qquad 1$

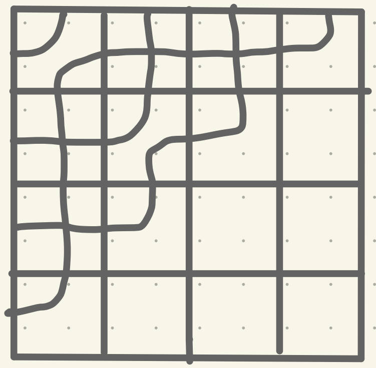


1 $\qquad x_i \oplus y_j \qquad$ 1 $\qquad 1+\beta(x \oplus y)$



Thm (BFHTW) $Z(S_w) = G^{\beta}_{w}(\vec{x}, \vec{y})$.

Pipe dreams same style of tillings, with 



<u>Thm:</u>

$$G^{\beta}_{w}(\bar{x}, \bar{y}) = \sum_{\partial P = w} wt(P)\, \beta^{ex(P)}, \quad wt(P) = \prod_{\boxplus \text{ at } (i,j)} (x_i \oplus y_j)$$

BFHTW proof: given $\mathcal{P} \in PD \to \mathrm{red}(\mathcal{P}) \in PD$

match states in the model to sets of $\mathcal{P}$ with the same $\mathrm{red}(\mathcal{P})$.

* $\partial \mathcal{P}$ not always easy to compute.

Fact: there is no weight preserving bijection between BPD and PD.

$\Rightarrow$ the two models BS and BFHTW are different.

Many other applications (most reproofs)

* BFHTW: Cauchy identity

* BS: $G^{\beta}_{w}(\bar{x};\bar{y}) = G^{\beta}_{w^{-1}}(\bar{y};\bar{x})$

diagonal flip

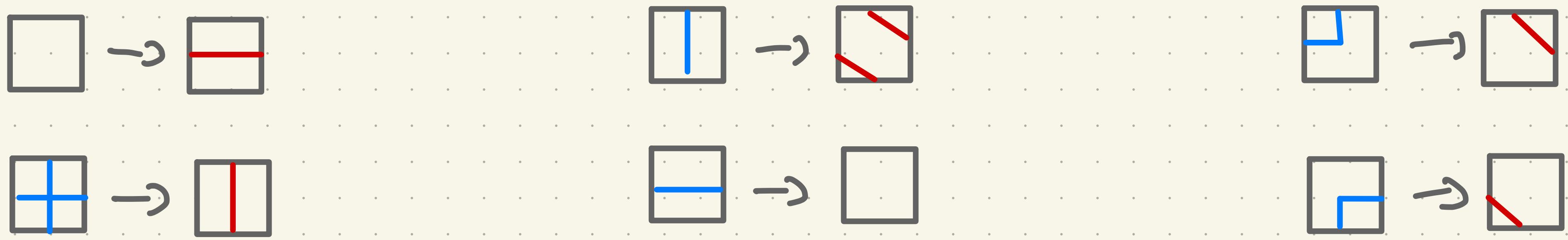
* BS: w vexillary (avoids 2143) $\rightsquigarrow$ no

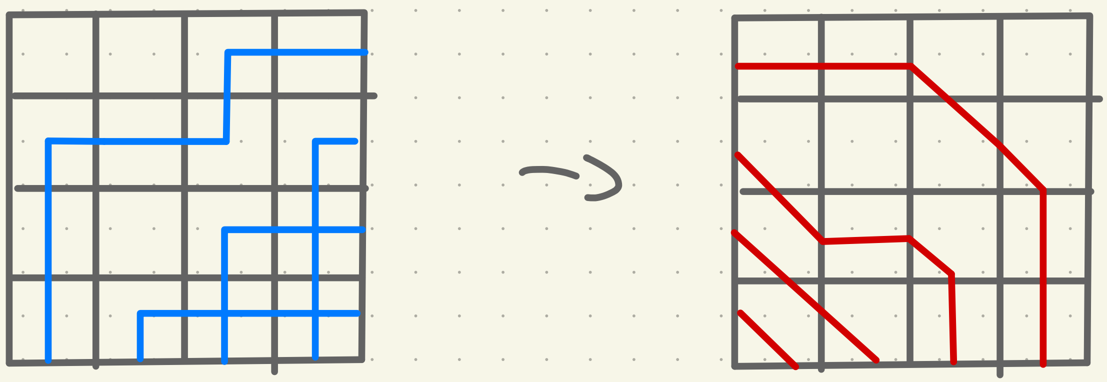
$\Rightarrow$ semidual model for flagged double Grothendieck pol's of Knutson-Miller-Yong

* BS: stable limit: factorial Grothendieck polynomial as sum over set-valued tableaux

More detail on the semidual model:

w is vexillary $\to$ no  $\to$ we may uncolor our model





given w vexillary $\to$ λ_w partition associated to w

F_w flaggin associated to w

Thm (BS) Semidual model is integrable, represents $G^{\beta}_{\lambda_w, F_w}$ $(= G^{\beta}_{w})$

semidual states $\rightsquigarrow$ marked GT patterns $\leftrightarrow$ set valued tableaux SVT

$\rightsquigarrow G^{\beta}_{\lambda_w, F_w}$ is a sum over SVT_{λ_w, F_w}

Structure coefficients:

s_λ is the character of $V_\lambda \in \mathrm{Rep}(\mathfrak{gl}_n(\mathbb{C}))$ = semisimple.

$$V_\lambda \otimes V_\mu = \bigoplus_\nu \underbrace{c_{\lambda\mu}^{\nu}}_{} V_\nu .$$

$\hookrightarrow$ non-negative integers (Littlewood Richardson coefficients)

$$\Leftrightarrow \quad s_\lambda \cdot s_\mu = \bigoplus_\nu c_{\lambda\mu}^{\nu} s_\nu .$$

Problem: give a combinatorial algorithm for $c_{\lambda\mu}^{\nu}$.

Littlewood Richardson: tableaux of shape ν/λ and weight μ.

- first few proofs wrong

- correct proofs: Schützenberger and Thomas

Other formulas: * Yamanouchi words

* Remmel-Whitney rule

* Knutson-Tao-Woodward puzzles

* etc.

Zinn-Justin: reinterpret puzzles, pov integrable systems

$c_{\lambda\mu}^{\nu} \rightsquigarrow$ partition function

Wheeler + Zinn-Justin: $c_{\lambda\mu}^{\nu}$ for G_λ^β (Grassmannian Grothendieck) in terms of Δ integrable systems.

$c_{\lambda,\mu}^{\nu}$ for G_w^β open?

Structure coefficients for other special functions?

W_λ spherical Whittaker function for $GL_n / \mathbb{Q}_p$.

$W_\mu^{(2)}$ for $GL_n^{(2)}$ ⇝ metaplectic 2-cover

Observation: (Chinta, Ruskás)

$$W_\lambda \cdot W_\mu^{(2)} = \sum \underbrace{c_{\lambda\mu}^{\nu}}_{\to \text{ nice properties}} W_\lambda^{(2)}$$

* formulas for $c_{\lambda\mu}^{\nu}$?

* representation theoretic meaning?