Izergin-Korepin analysis on wavefunctions

Solvable Lattice Models Seminar

Kohei Motegi (Tokyo University of Marine Science and Technology)

Izergin-Korepin method

Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

Izergin-Korepin method

Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

in school examinations

arithmetic sequences

$$a_{n+1} = 3a_n,\ a_0 = 1 \longrightarrow a_n = 3^n$$

Izergin-Korepin method

Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

in school examinations

arithmetic sequences

$$a_{n+1} = 3a_n,\ a_0 = 1 \Longrightarrow a_n = 3^n$$

Roughly speaking lattice models

we have to first find out what these are

$$a_{n+1} = f(a_n),\ a_0$$

Izergin-Korepin method

Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

lattice models

we have to first find out what these are

$$a_{n+1} = f(a_n),\ a_0$$

Izergin-Korepin method

Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

lattice models

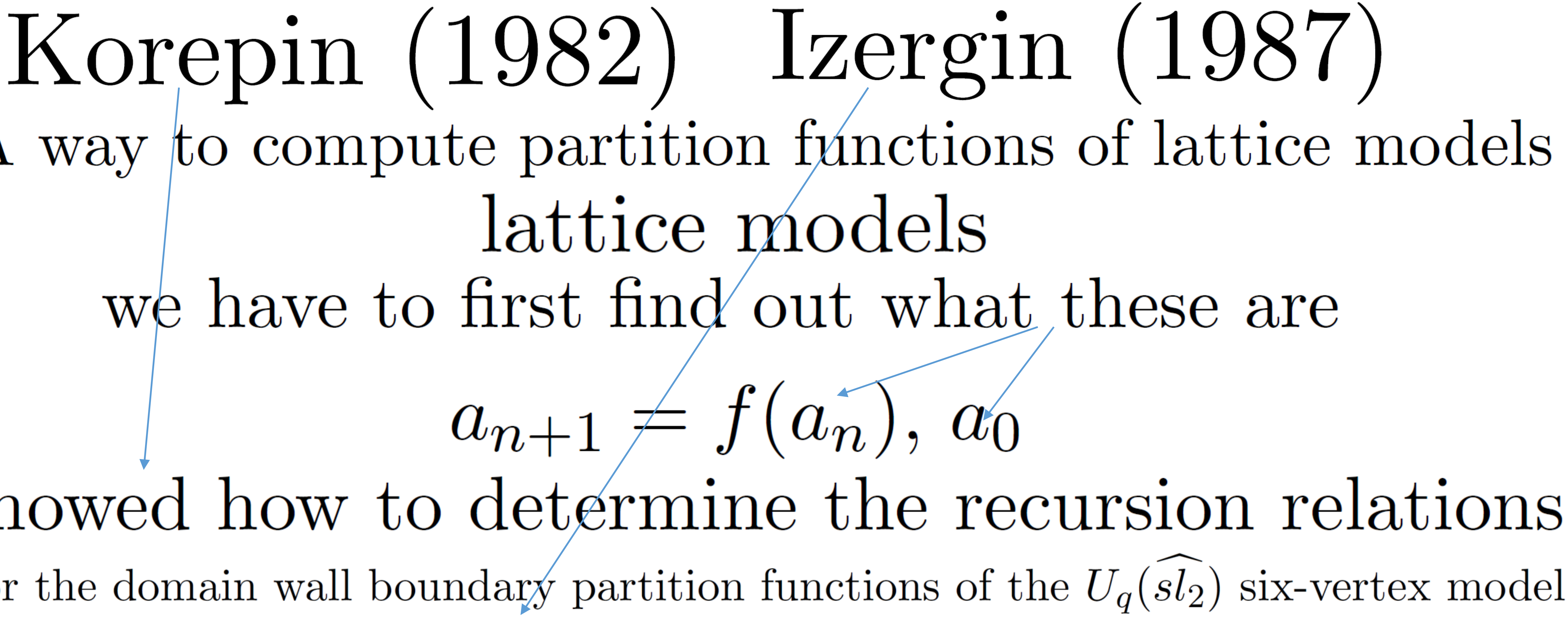
we have to first find out what these are

$$a_{n+1} = f(a_n),\ a_0$$

showed how to determine the recursion relations

for the domain wall boundary partition functions of the $U_q(\widehat{sl_2})$ six-vertex model

Izergin-Korepin method



Korepin (1982) Izergin (1987)

A way to compute partition functions of lattice models

lattice models

we have to first find out what these are

$$a_{n+1} = f(a_n),\ a_0$$

showed how to determine the recursion relations

for the domain wall boundary partition functions of the $U_q(\widehat{sl_2})$ six-vertex model

presented one form of the solution to the recursion relation

Izergin-Korepin determinant applications to

alternating sign matrices Kuperberg

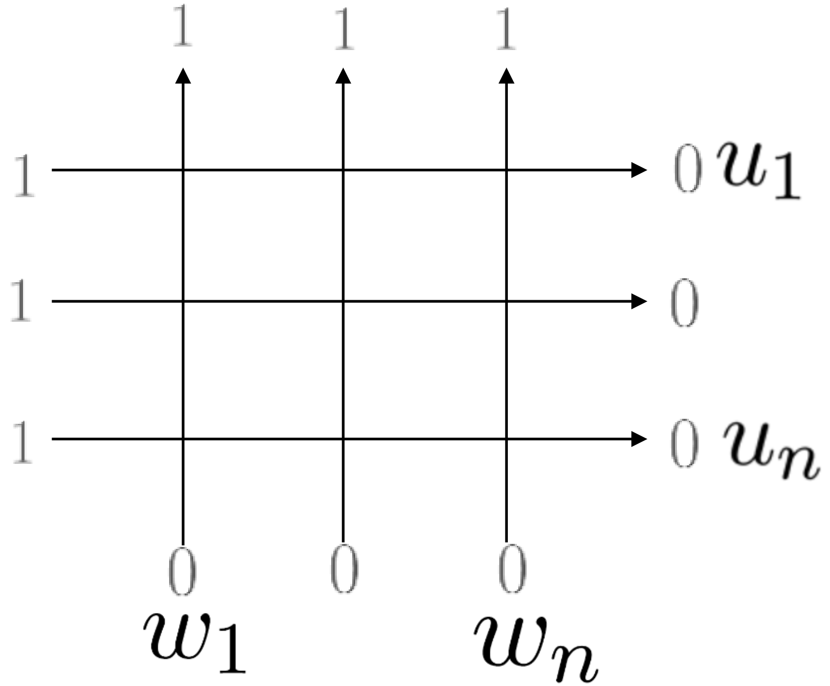
thermodynamic limit Korepin-Zinn-Justin

Izergin-Korepin method

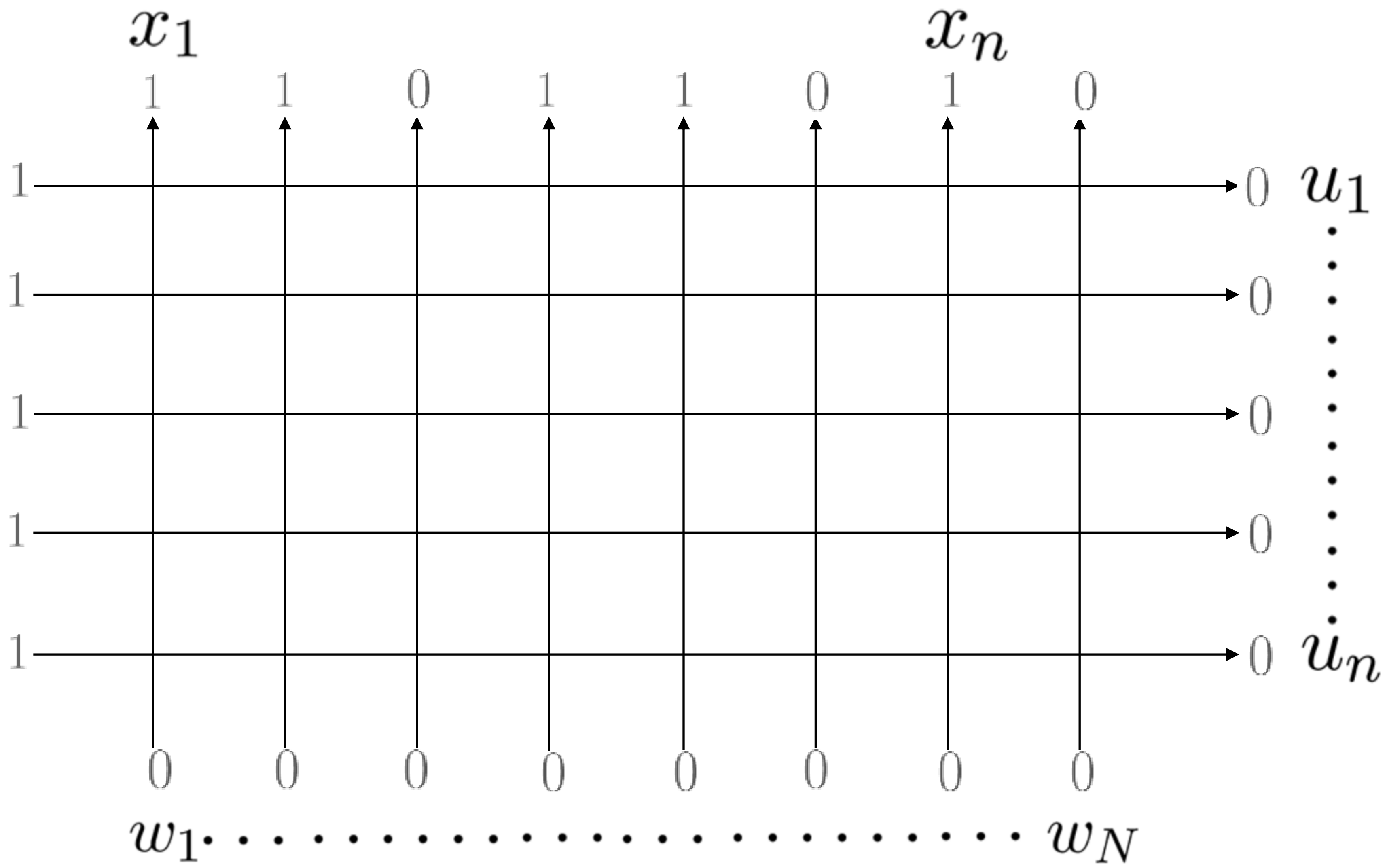
Korepin (1982) Izergin (1987)

showed how to determine the recursion relations

for the domain wall boundary partition functions of the $U_q(\widehat{sl_2})$ six-vertex model



can be regarded as the simplest case of the (off-shell Bethe) wavefunctions in many cases



wavefunctions

I will explain the Izergin-Korepin method to wavefunctions

prototype

reflecting boundary

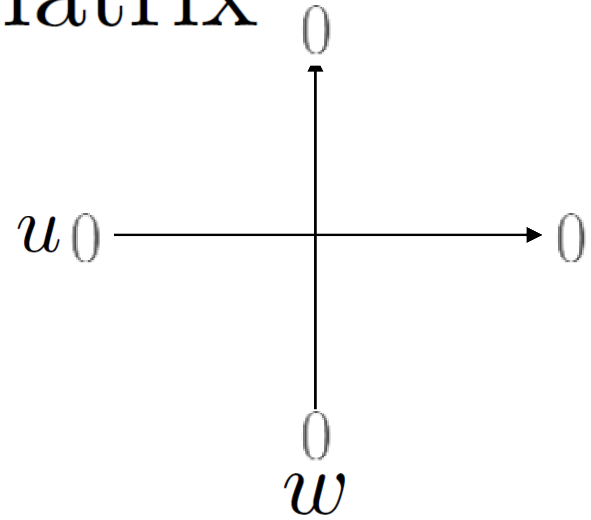
other results

Izergin-Korepin method

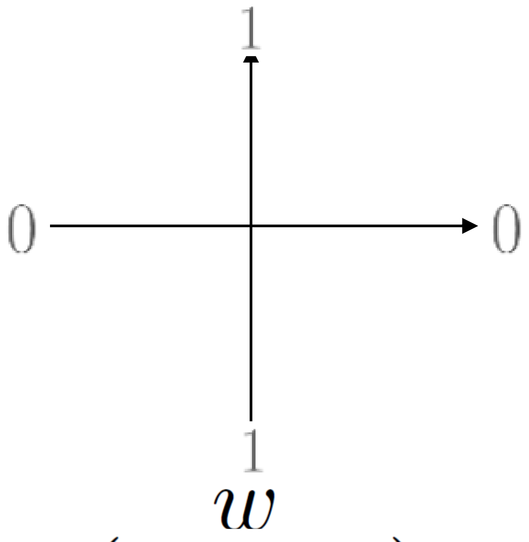
variations of the domain wall boundary partition functions Tsuchiya, Kuperberg

Slavnov's scalar product formula for the XXZ chain Wheeler

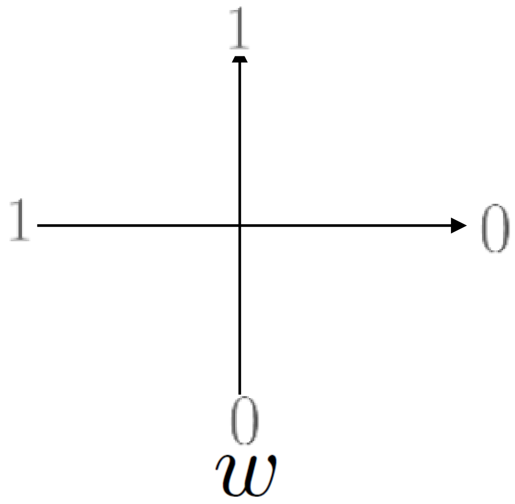
R-matrix



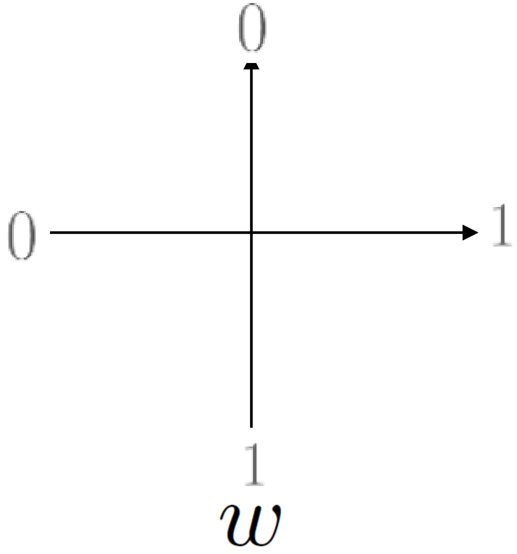
$$u - qw$$



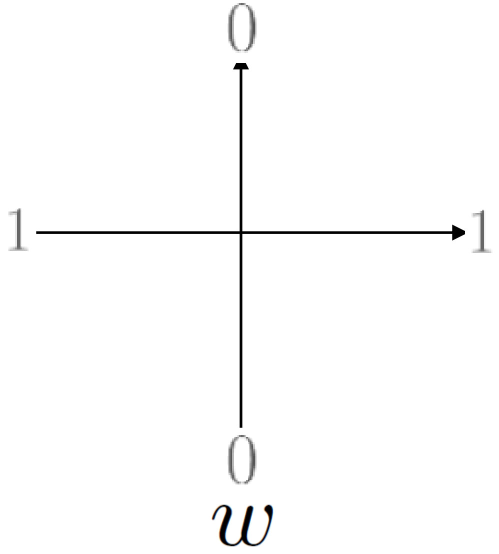
$$q(u - w)$$



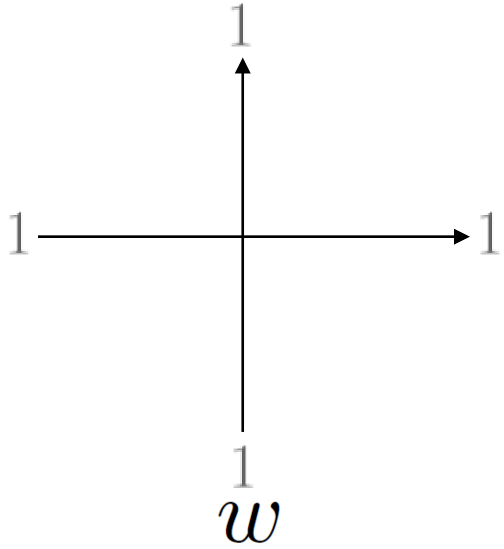
$$(1 - q)u$$



$$(1 - q)u$$



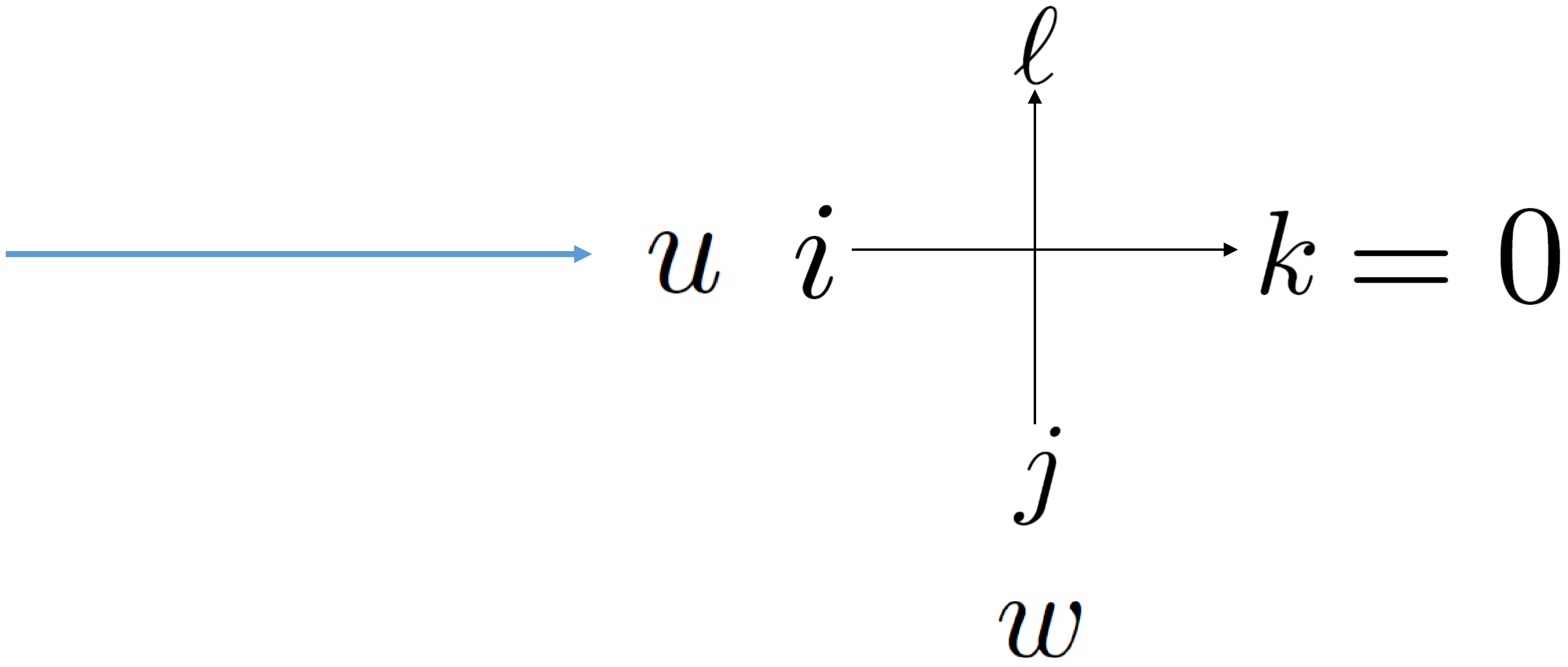
$$u - w$$



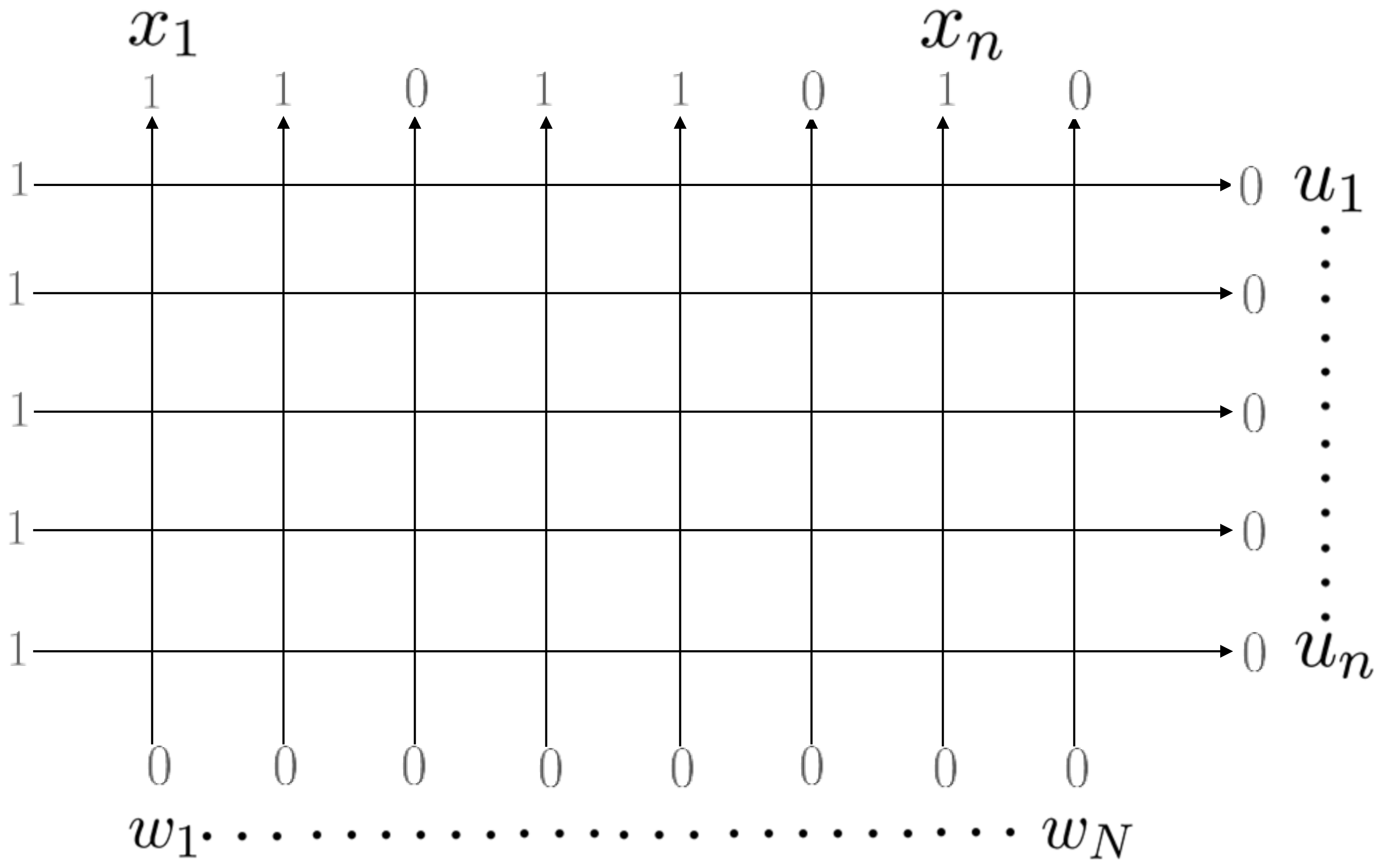
$$u - qw$$

ice-rule

$$i + j \neq k + \ell$$



wavefunctions



wavefunctions

$N = 8,\ n = 5$

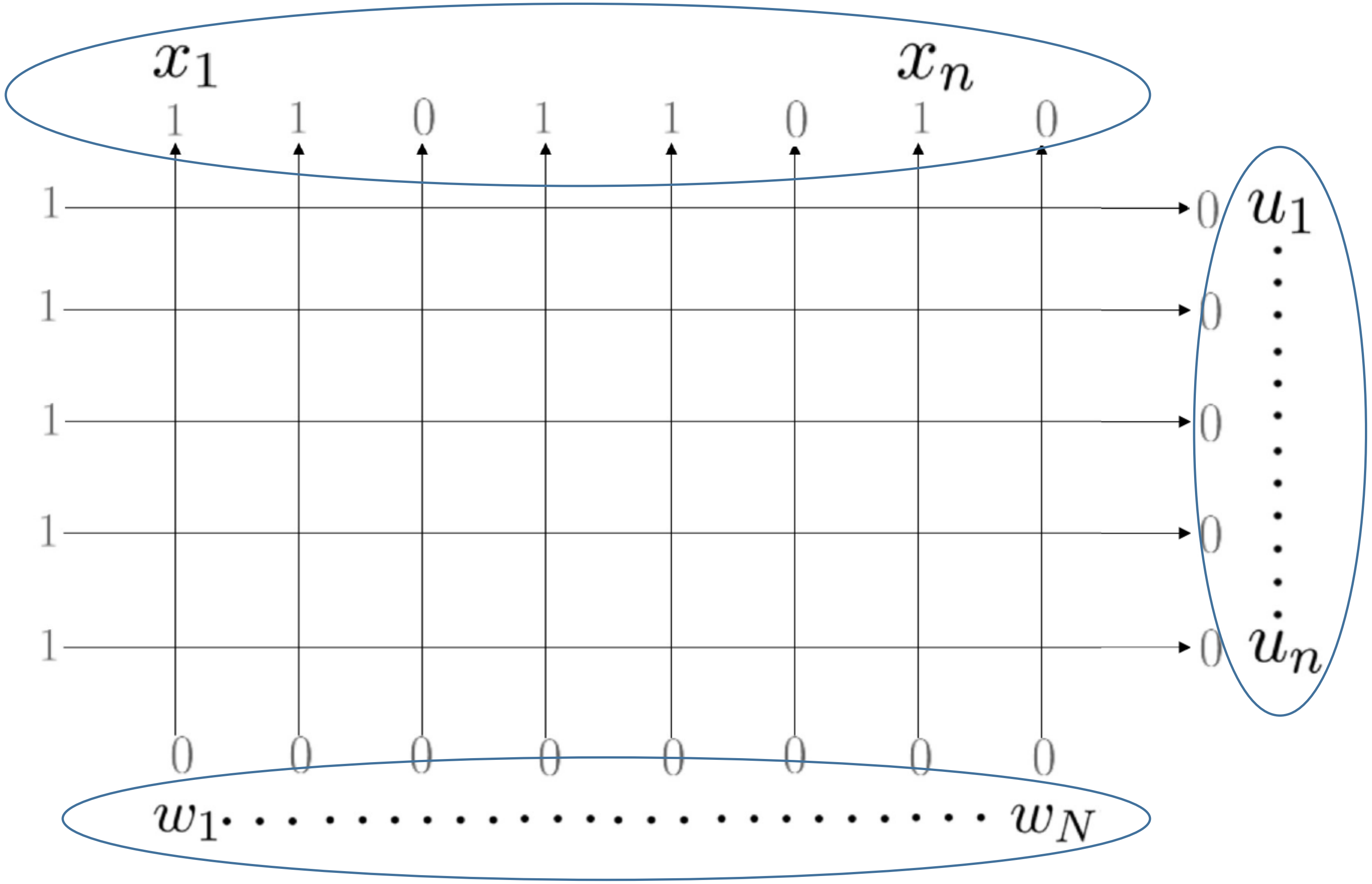
$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 6,\ x_5 = 7$

multivariable polynomials depending on

$$w_1, \dots, w_N$$

$$u_1, \dots, u_n$$

$$x_1, \dots, x_n$$



wavefunctions

$N=8,\ n=5$

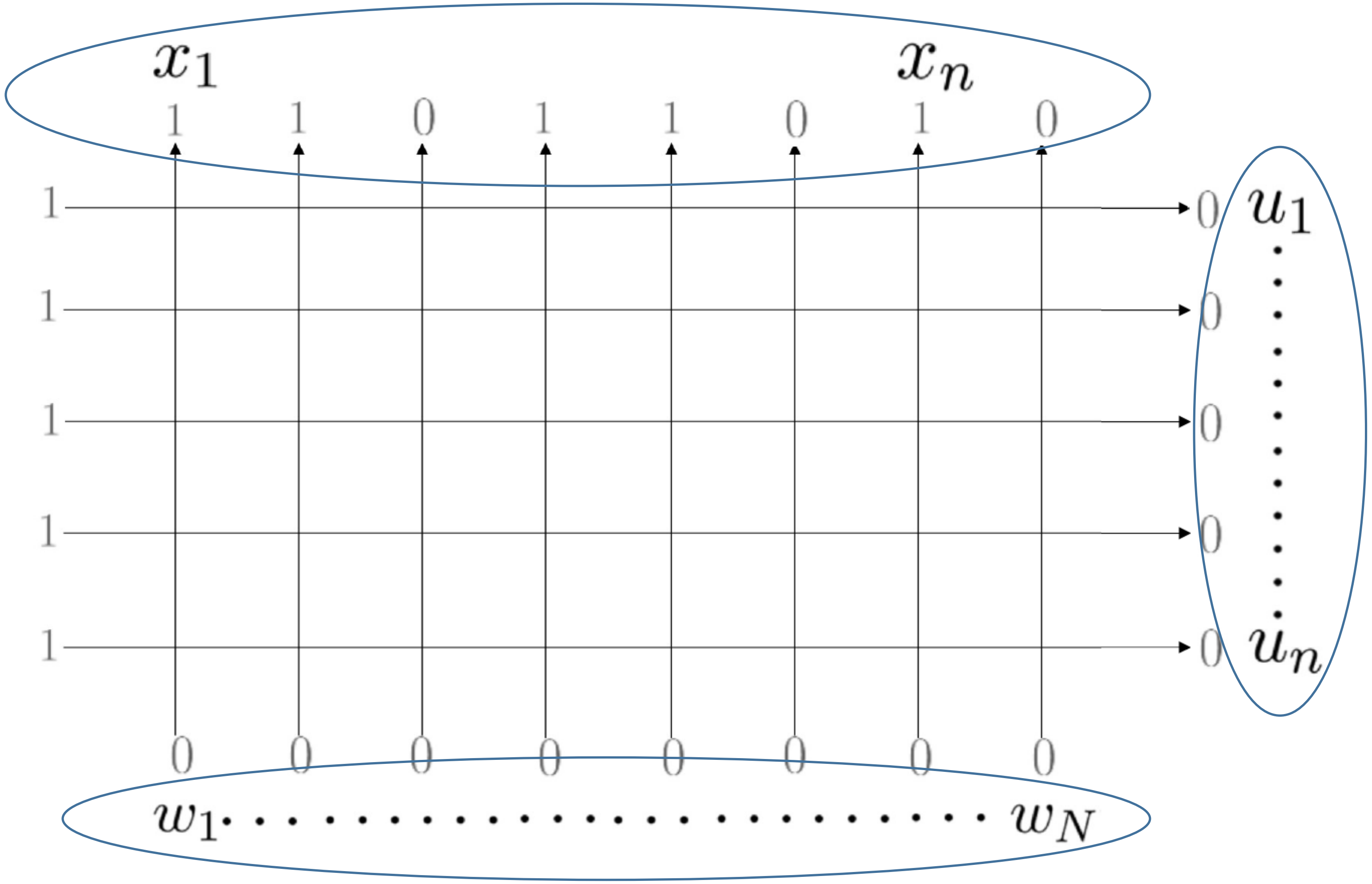
$x_1=1,\ x_2=2,\ x_3=4,\ x_4=6,\ x_5=7$

multivariable polynomials depending on

$$w_1,\dots,w_N$$

$$u_1,\dots,u_n$$

$$x_1,\dots,x_n$$



$$\parallel$$

$$W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

Steps of Izergin-Korepin analysis

Steps of Izergin-Korepin analysis

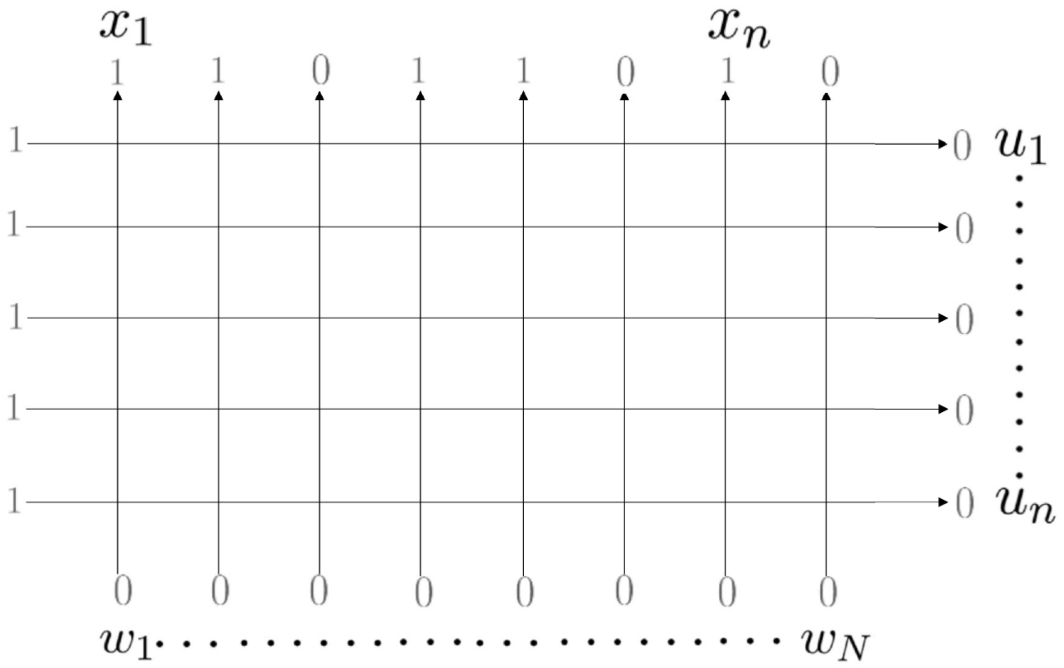
- construct Korepin's lemma

Steps of Izergin-Korepin analysis

- construct Korepin's lemma
    - construct recursive relations between

$$W_{N,n} \quad \text{and} \quad W_{N-1,n} \quad x_n \neq N$$

$$W_{N-1,n-1} \quad x_n = N$$

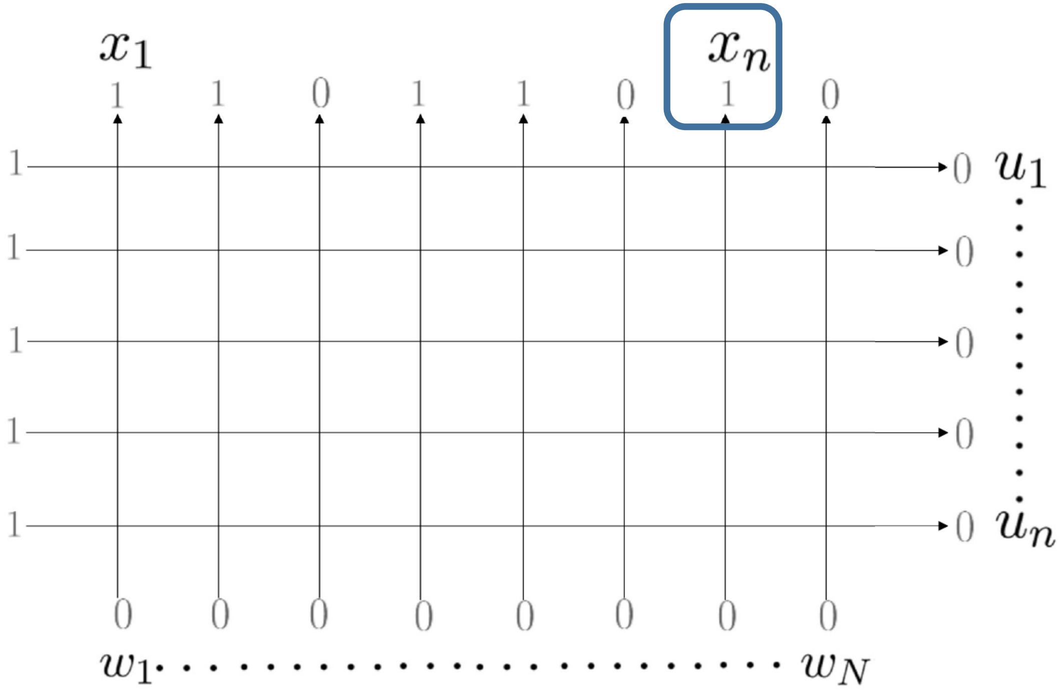


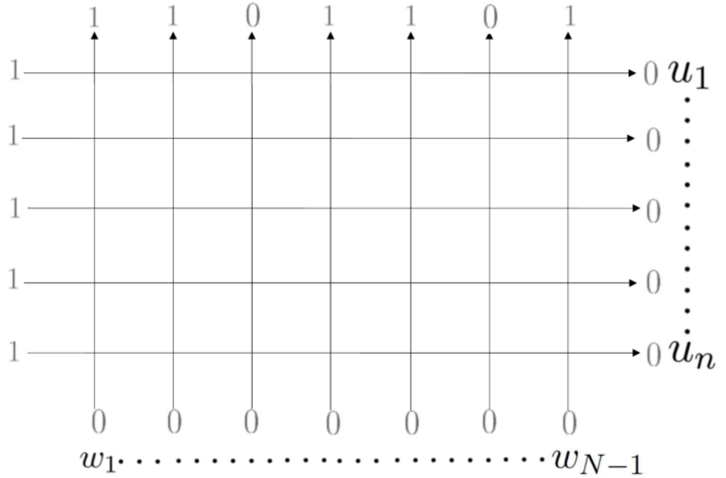
Steps of Izergin-Korepin analysis

- construct Korepin's lemma
  - construct recursive relations between

$W_{N,n}$ and $W_{N-1,n}$ $\quad x_n \neq N$

$W_{N-1,n-1}$ $\quad x_n = N$



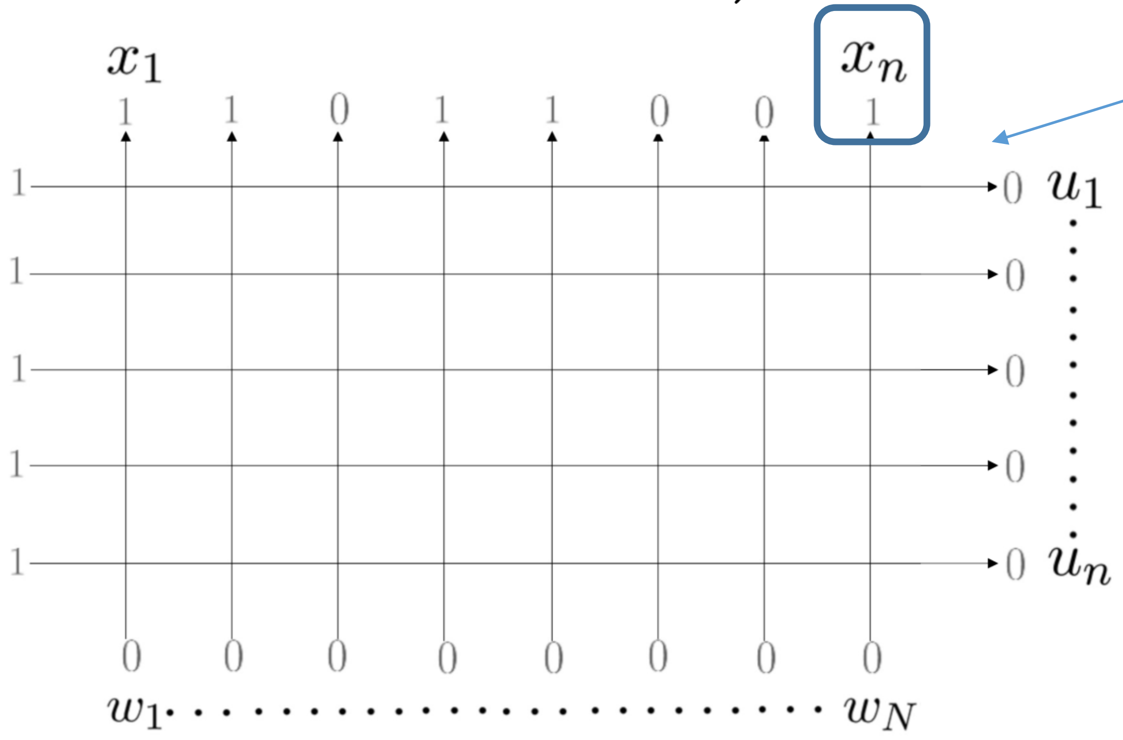


Steps of Izergin-Korepin analysis

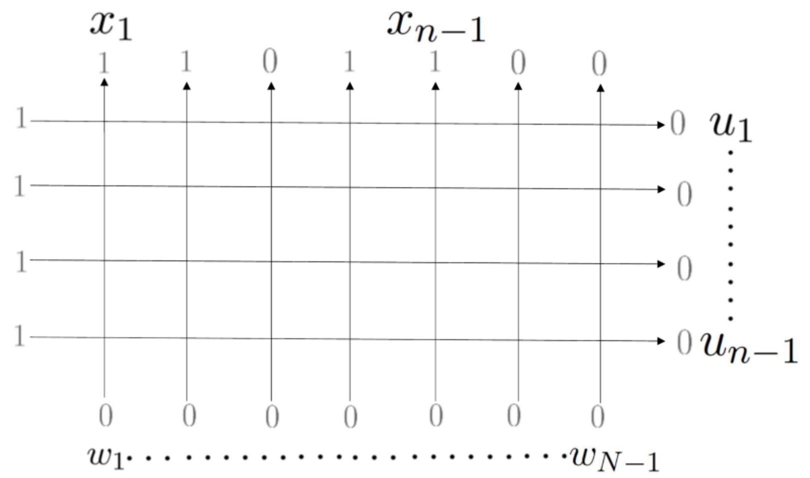
- construct Korepin's lemma
  - construct recursive relations between

$W_{N,n}$ and $W_{N-1,n}$ $x_n \neq N$

view as polynomials in w_N



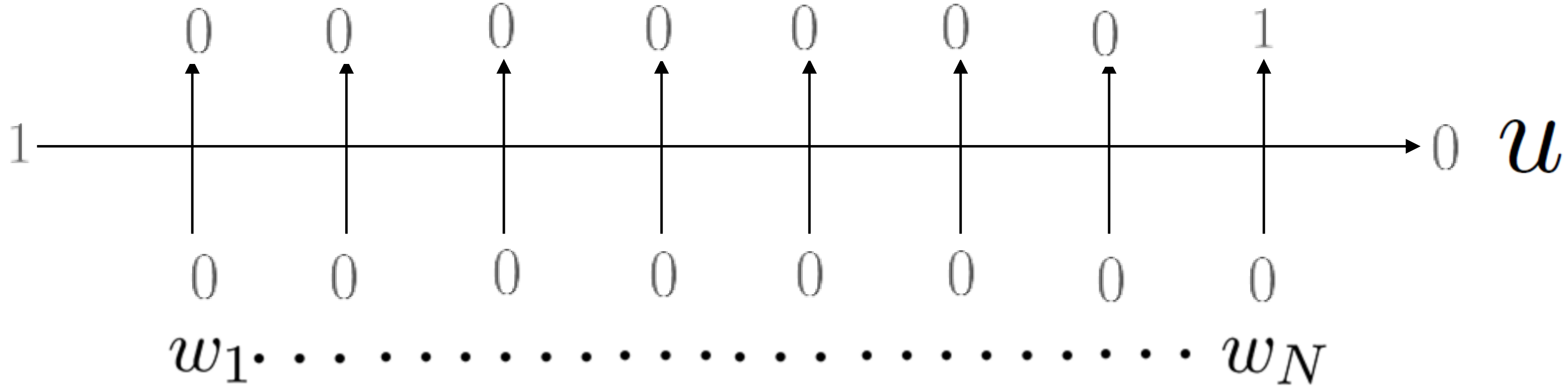
$W_{N-1,n-1}$ $x_n = N$



Steps of Izergin-Korepin analysis

- construct Korepin's lemma
    - evaluate the initial condition

$$W_{N,1}(u|w_1,\ldots,w_N|N)$$



Steps of Izergin-Korepin analysis

- construct Korepin's lemma
- show the multivariable polynomials satisfy the recursive relations and the initial condition

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n \neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$.

(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.

(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$
\begin{aligned}
&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n} \\
=&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j) \\
&\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).
\end{aligned}
$$

essentially the same with the case for the domain wall boundary partition functions

these properties mean that

$W_{N,n}$ is uniquely defined by $W_{N-1,n-1}$

$x_n = N$

recursive relations between $W_{N,n}$ and $W_{N-1,n-1}$

due to a fundamental property for polynomials

$P(w)$ polynomial in w of degree at most $n-1$
$P(x_1), \ldots, P(x_n)$ evaluations at n distinct points $w = x_1, \ldots, x_n$

Lagrange interpolation formula

$$P(w) = \sum_{j=1}^{n} P(x_j) \prod_{\substack{1 \le m \le n \\ m \ne j}} \frac{w - x_m}{x_j - x_m}$$

$P(w)$ can be constructed from $P(x_1), \ldots, P(x_n)$

$P(w)$ polynomial in w of degree at most $n-1$

$P(x_1),\ldots,P(x_n)$ evaluations at n distinct points $w = x_1,\ldots,x_n$

Lagrange interpolation formula

$$P(w) = \sum_{j=1}^{n} P(x_j) \prod_{\substack{1\le m\le n\\ m\ne j}} \frac{w - x_m}{x_j - x_m}$$

$P(w)$ can be constructed from $P(x_1),\ldots,P(x_n)$

$$W_{N,n}(w_N) = \sum_{j=1}^{n} W_{N,n}(w_N)\big|_{w_N=q^{-1}u_j} \prod_{\substack{1\le m\le n\\ m\ne j}} \frac{w - q^{-1}u_m}{q^{-1}u_j - q^{-1}u_m}$$

$P(w)$ polynomial in w of degree at most $n-1$

$P(x_1),\ldots,P(x_n)$ evaluations at n distinct points $w=x_1,\ldots,x_n$

Lagrange interpolation formula

$$P(w)=\sum_{j=1}^{n}P(x_j)\prod_{\substack{1\le m\le n\\ m\ne j}}\frac{w-x_m}{x_j-x_m}$$

$P(w)$ can be constructed from $P(x_1),\ldots,P(x_n)$

$$W_{N,n}(w_N)=\sum_{j=1}^{n}\boxed{\begin{array}{c}W_{N,n}(w_N)|_{w_N=q^{-1}u_j}\\ =*W_{N-1,n-1}\end{array}}\prod_{\substack{1\le m\le n\\ m\ne j}}\frac{w-q^{-1}u_m}{q^{-1}u_j-q^{-1}u_m}$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$*.*
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$*.*
(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\&=(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\&\quad\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

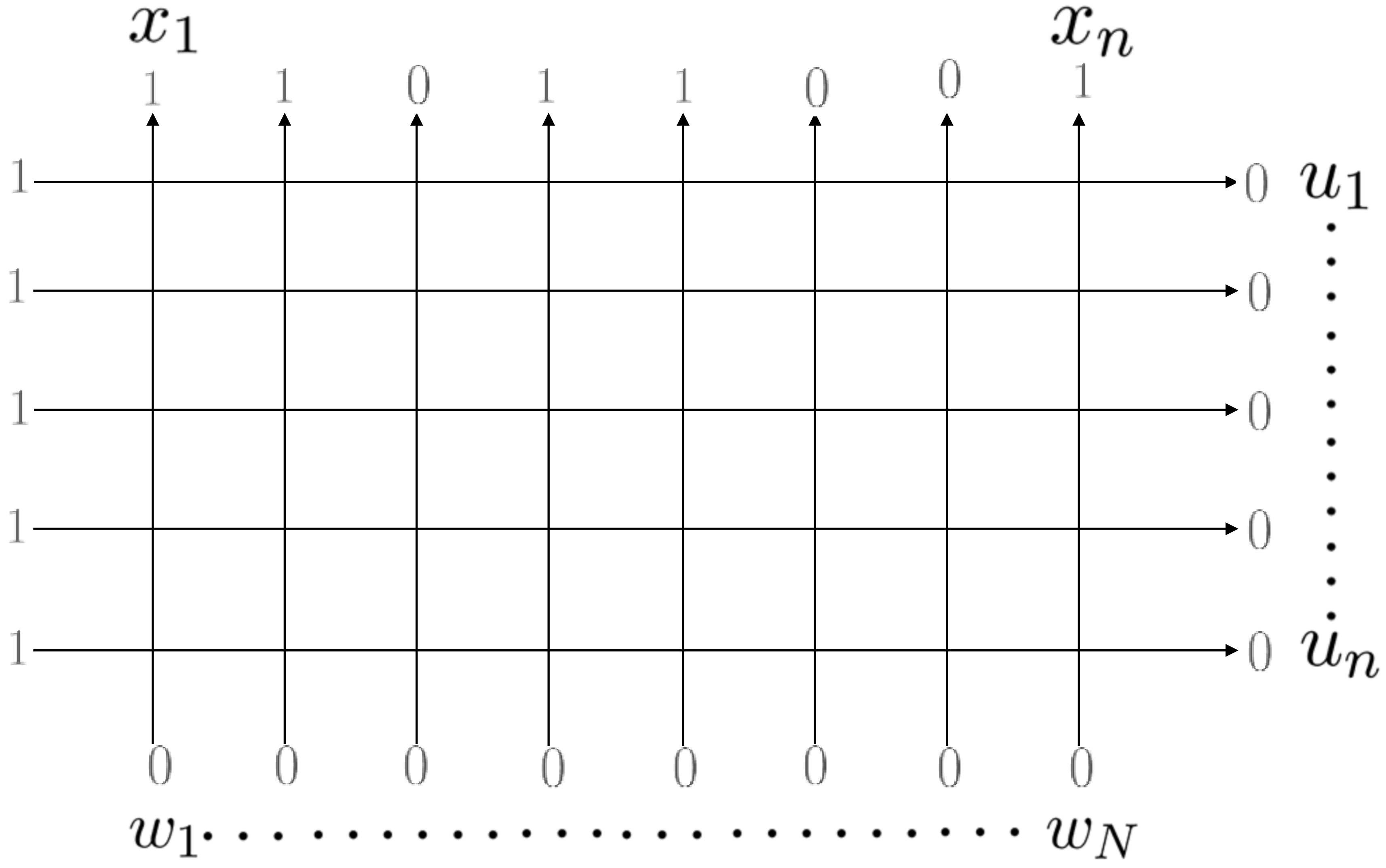
If $x_n\neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\&=\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

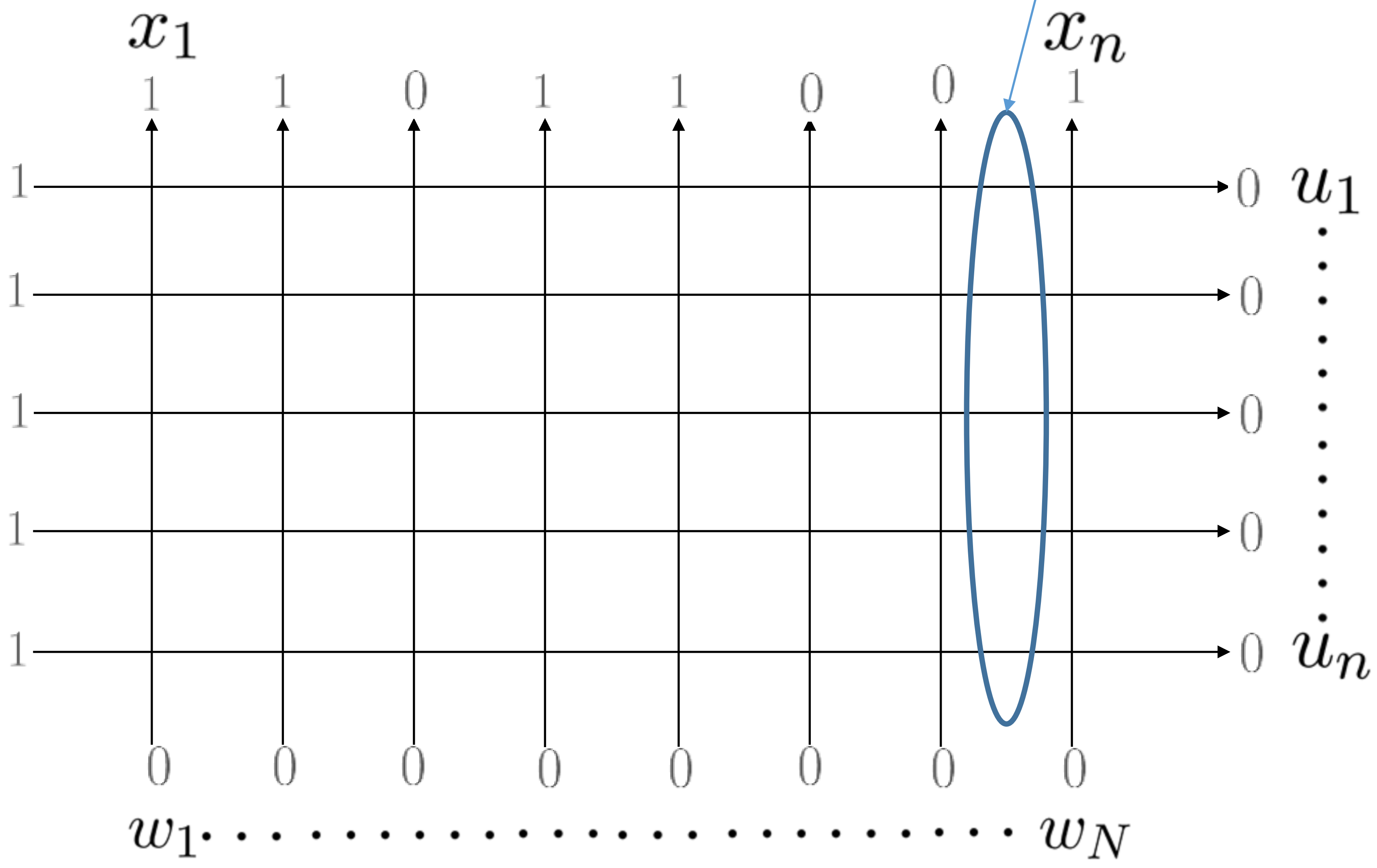
$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

wavefunctions



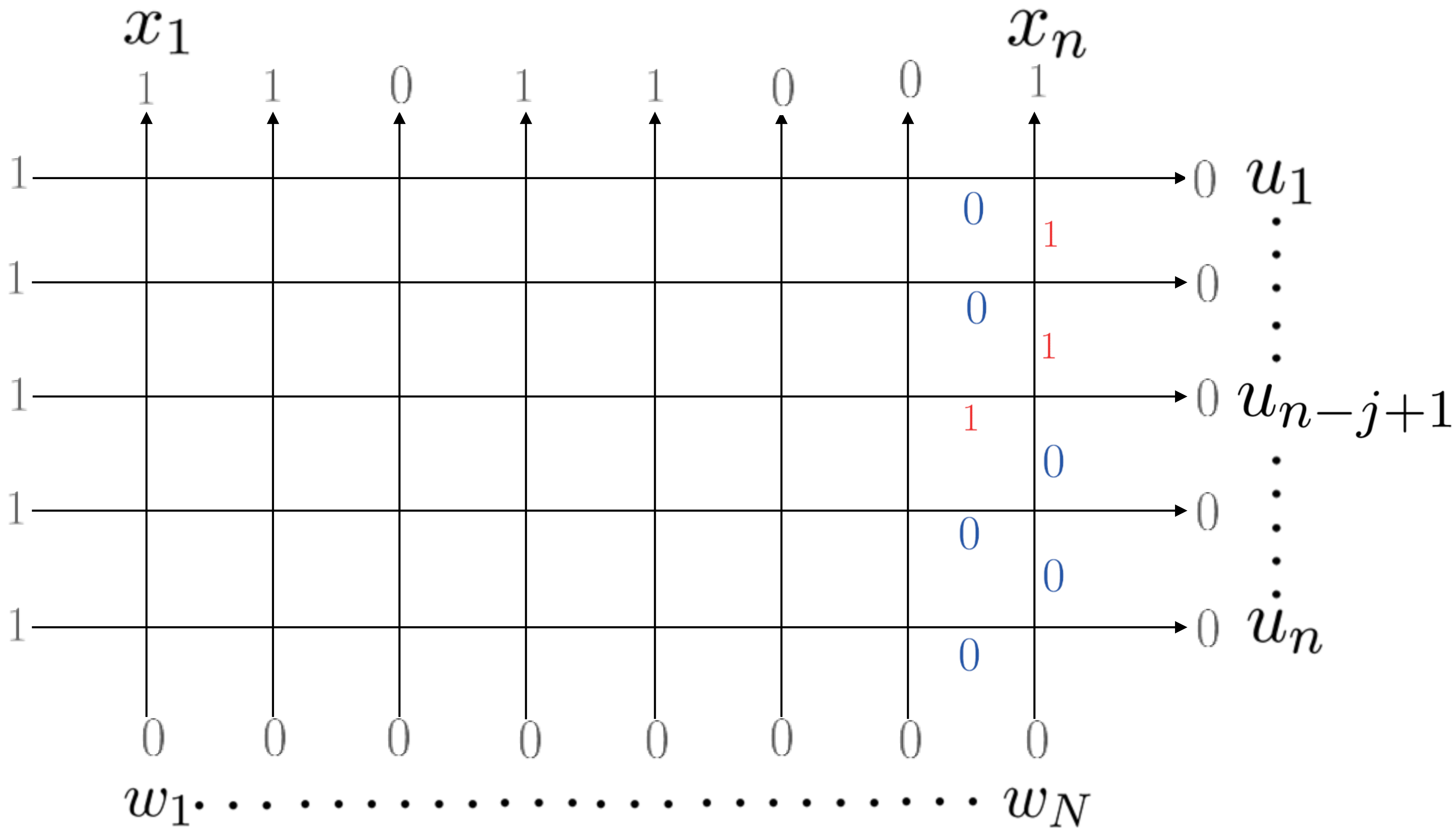
wavefunctions

ice-rule
only one 1 here, rest are 0s



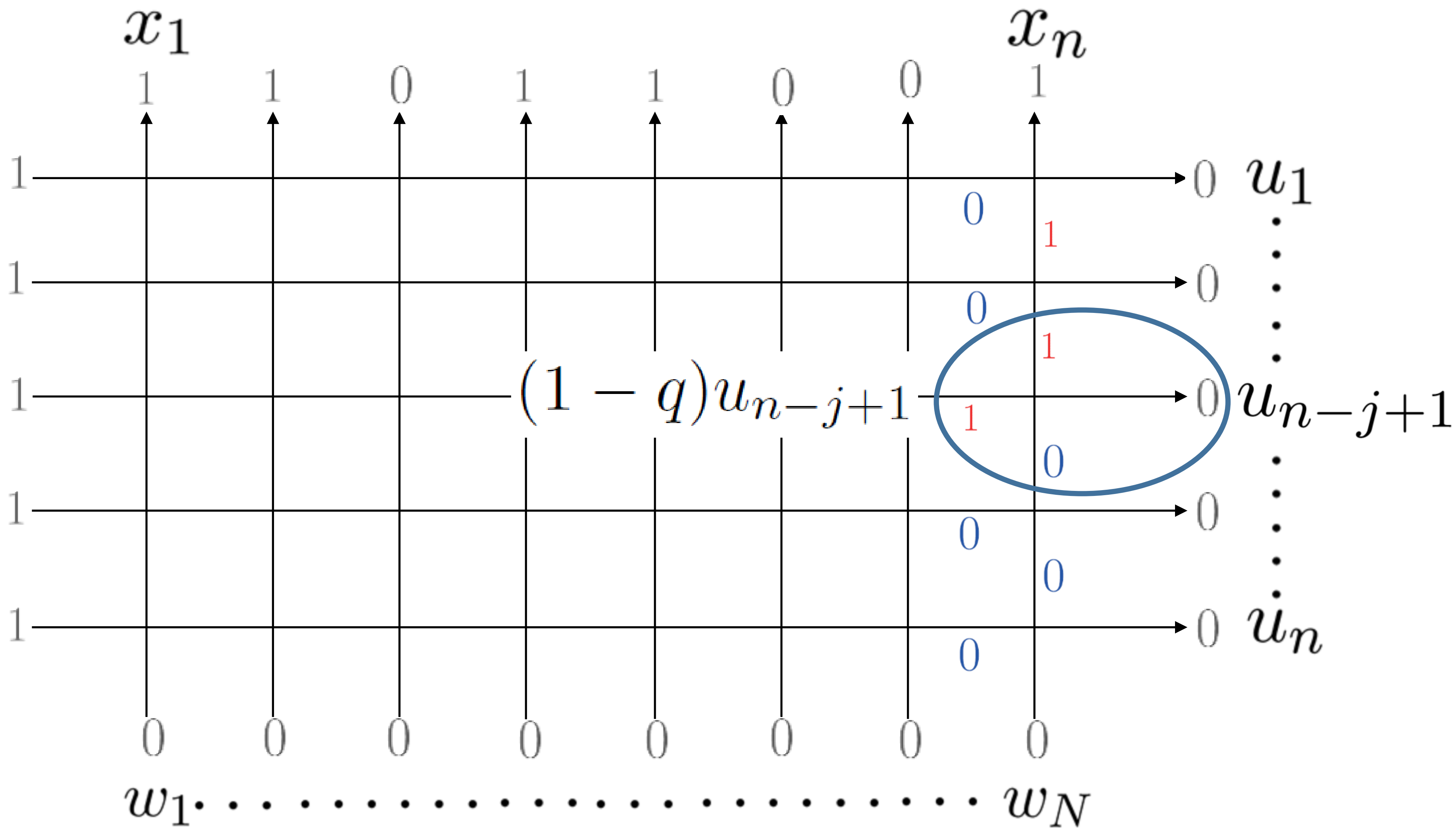
wavefunctions

$$= \sum_{j=1}^{n}$$

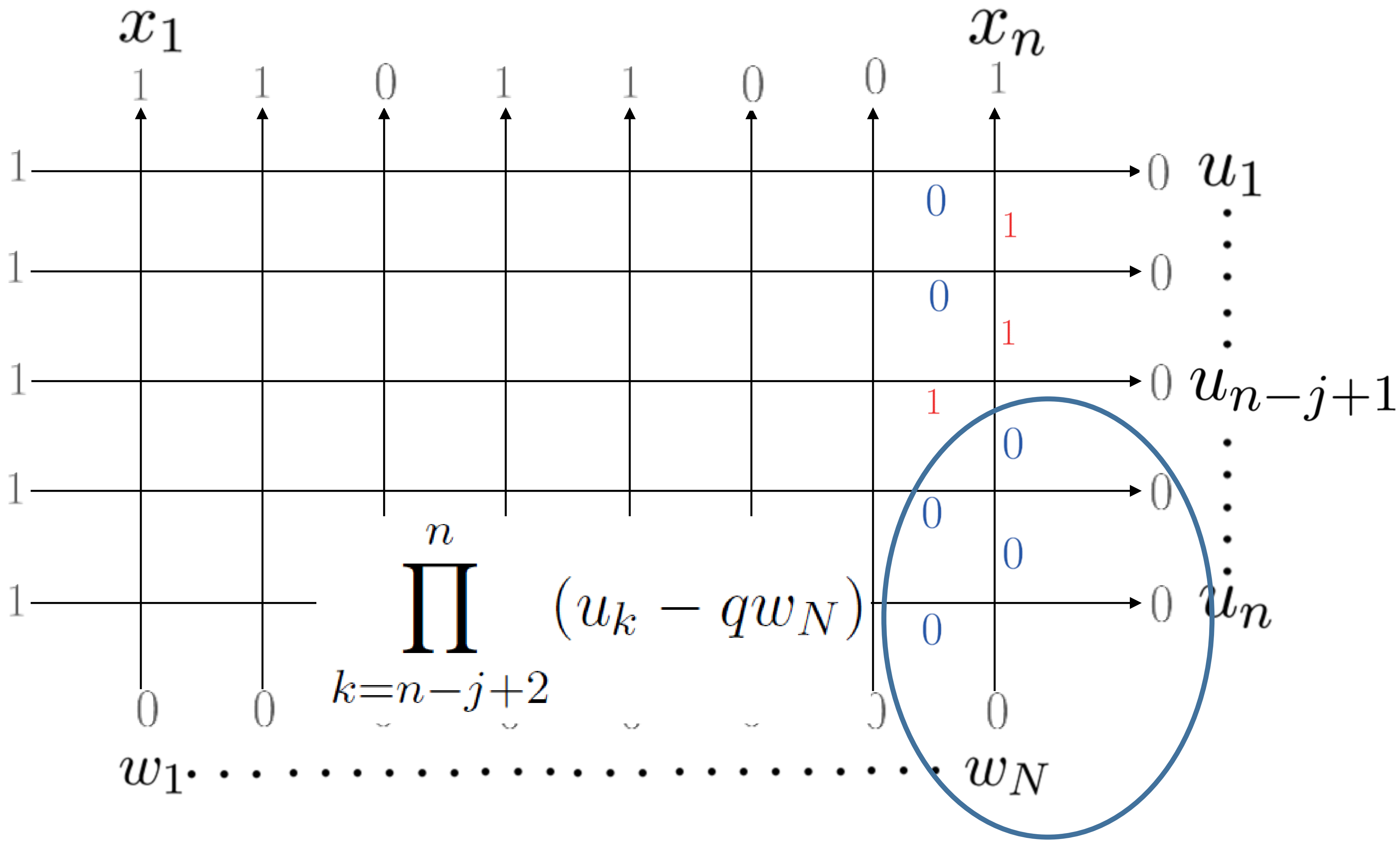


wavefunctions

$$= \sum_{j=1}^{n}$$



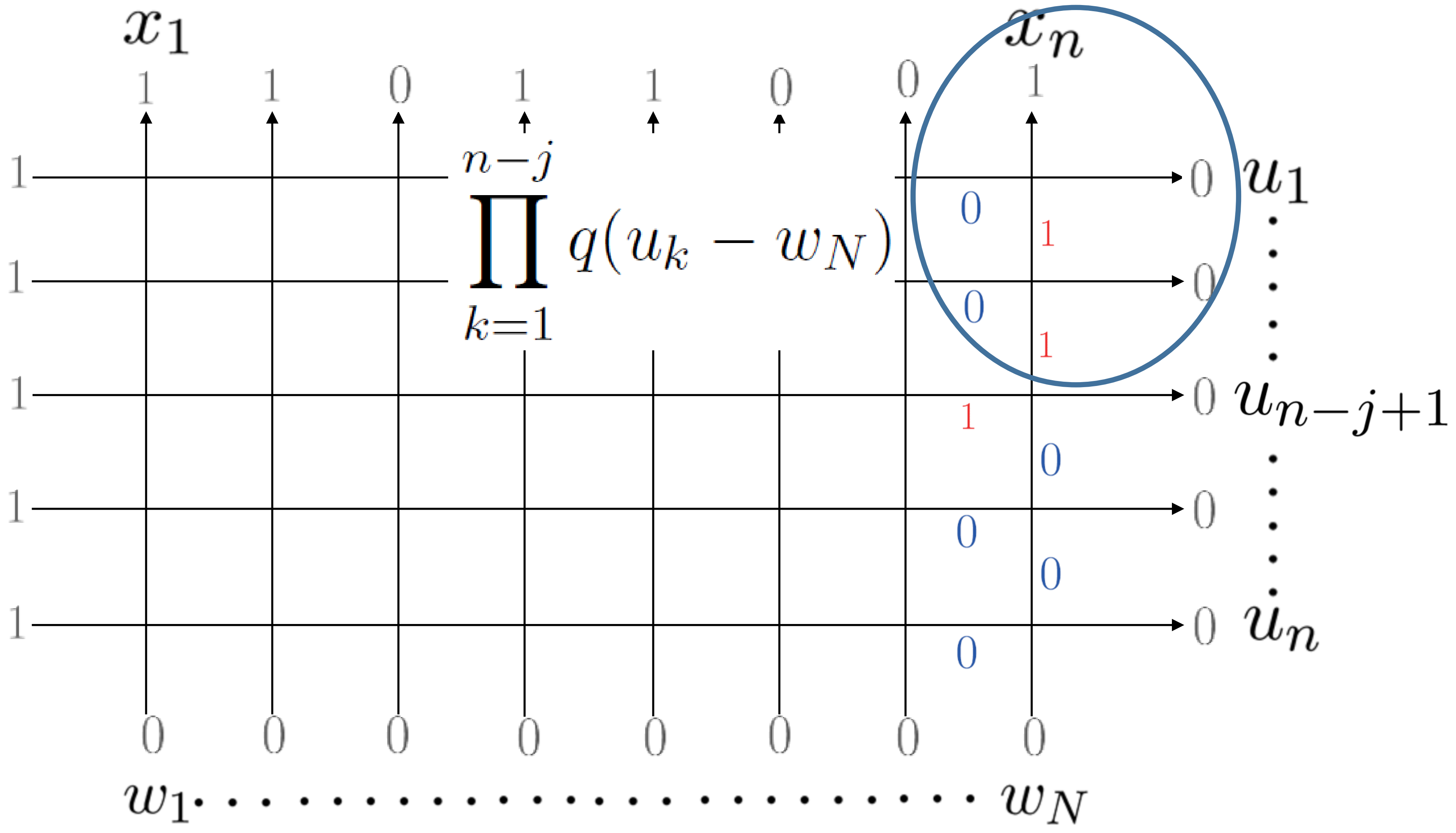
wavefunctions



$$= \sum_{j=1}^{n} \prod_{k=n-j+2}^{n} (u_k - q w_N)$$

wavefunctions

$$= \sum_{j=1}^{n}$$

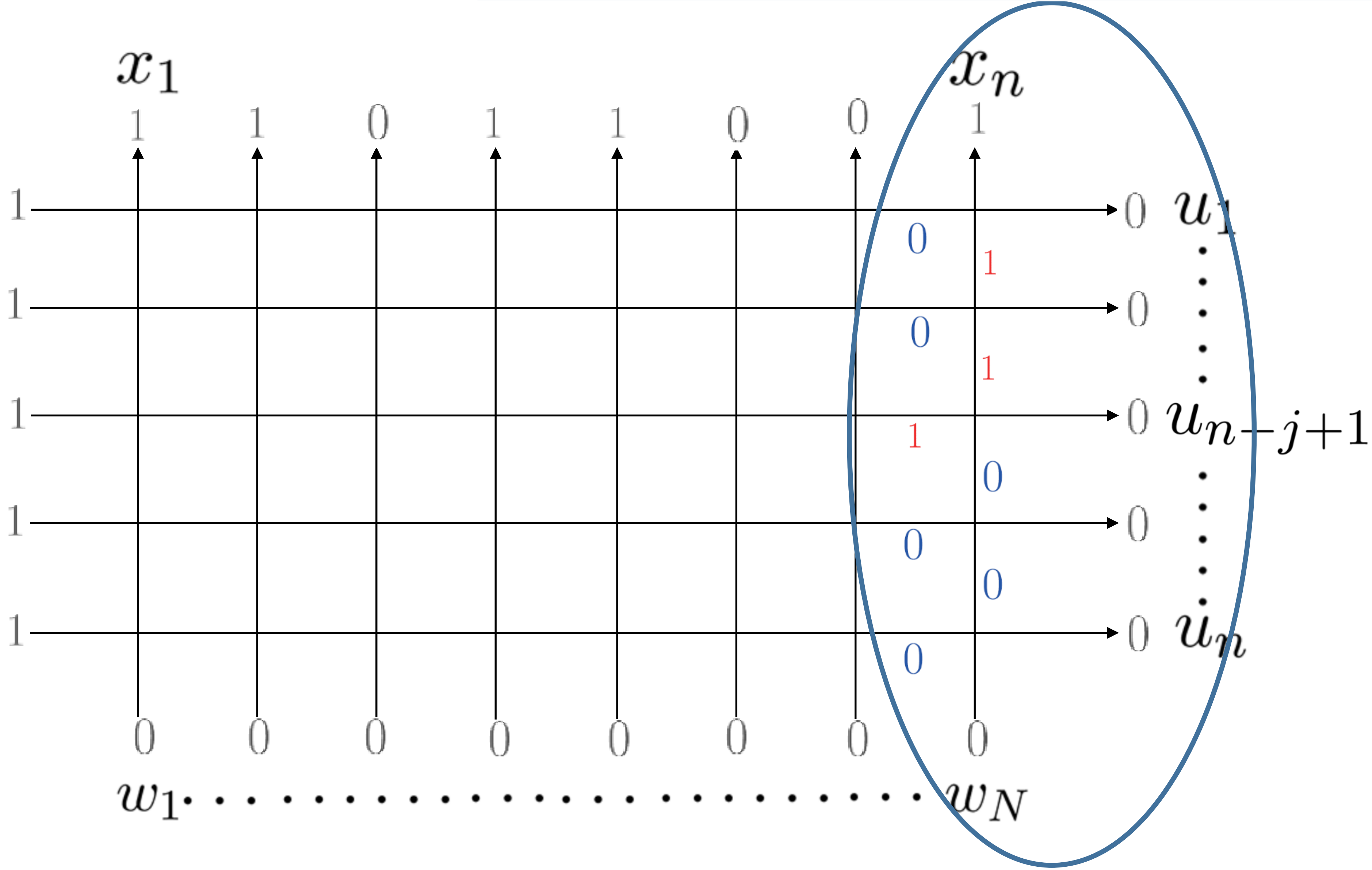


$$\prod_{k=1}^{n-j} q(u_k - w_N)$$

wavefunctions

$$(1-q)u_{n-j+1}\prod_{k=1}^{n-j} q(u_k - w_N) \prod_{k=n-j+2}^{n} (u_k - q w_N)$$

$$= \sum_{j=1}^{n}$$

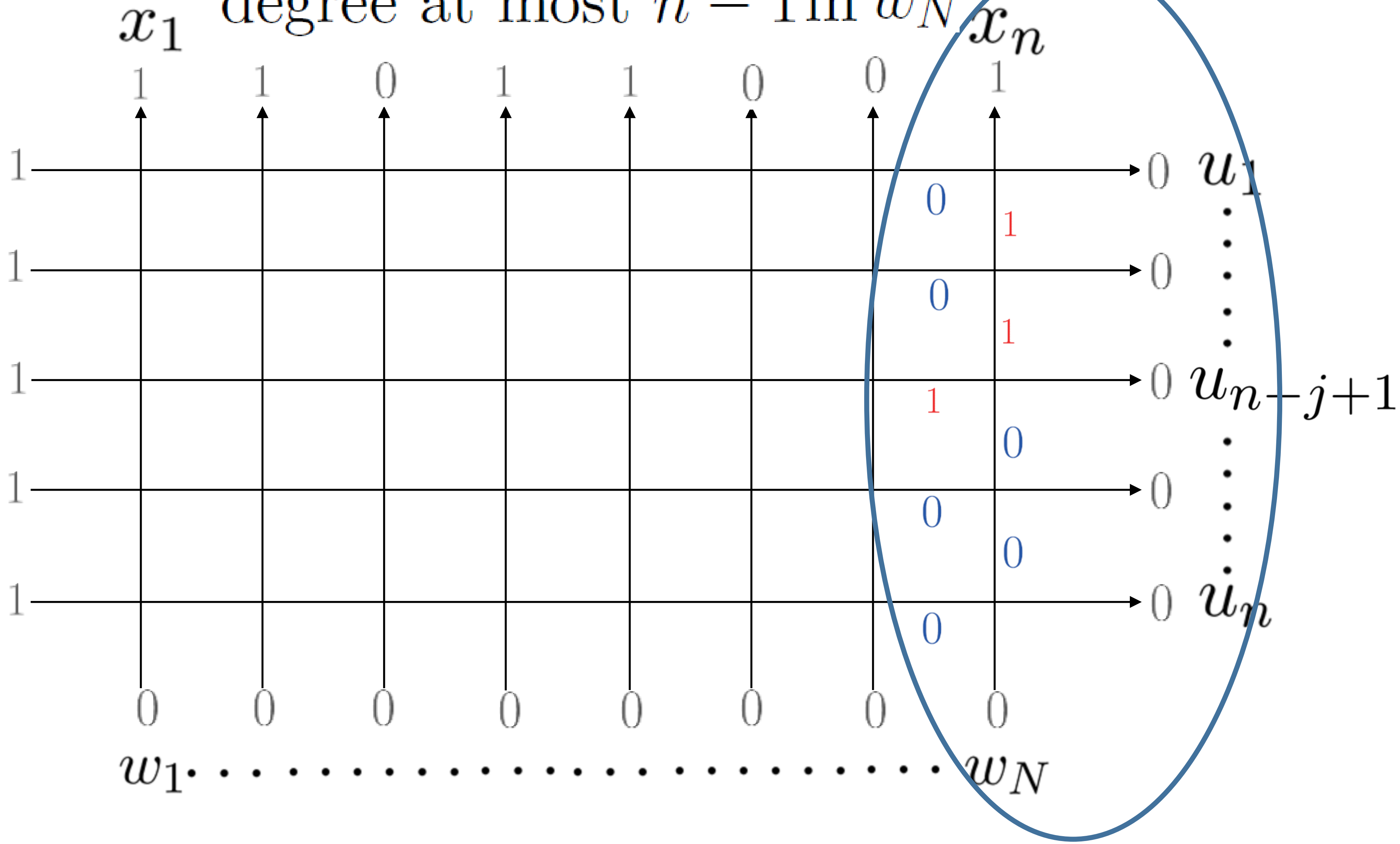


wavefunctions

$$\boxed{(1-q)u_{n-j+1}\prod_{k=1}^{n-j} q(u_k - w_N) \prod_{k=n-j+2}^{n} (u_k - q w_N)}$$

degree at most $n-1$ in w_N

$$= \sum_{j=1}^{n}$$



Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$*.*

(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j*,* $j=1,\dots,n$*.*

(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}
&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\
=&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\
&\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).
\end{aligned}$$

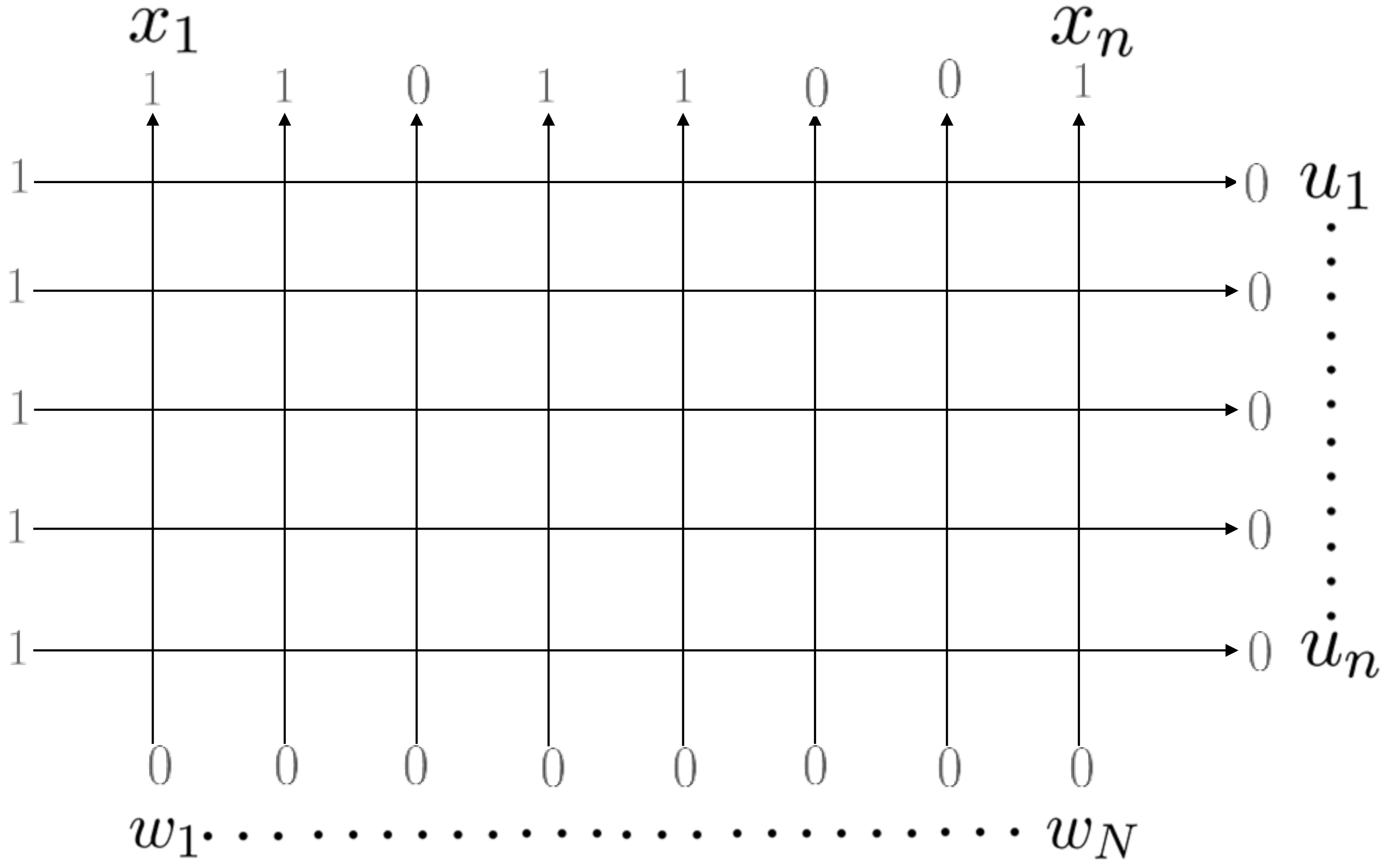
If $x_n \neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}
&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\
=&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).
\end{aligned}$$

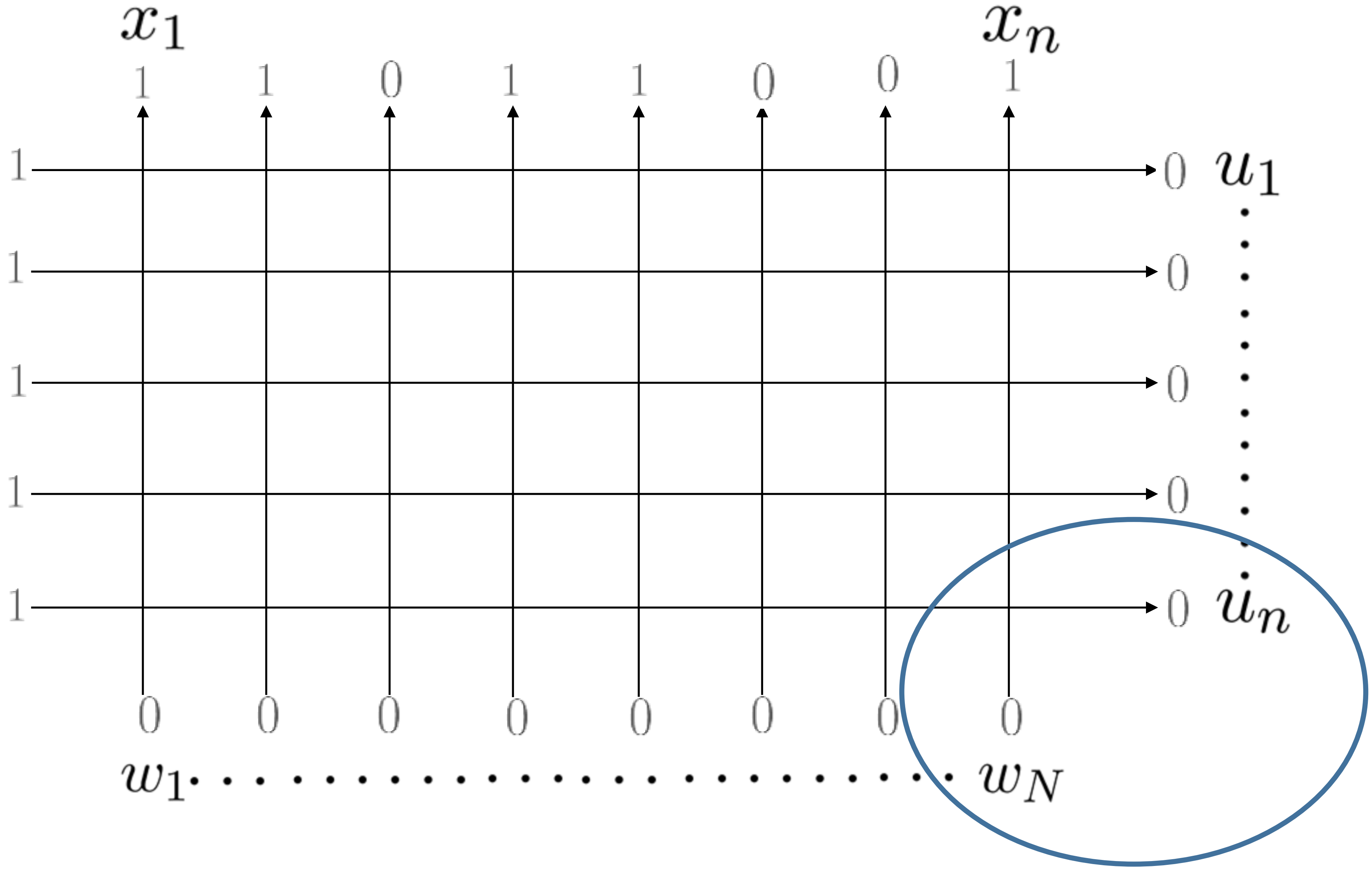
(4) The following holds for the case $n=1$*,* $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

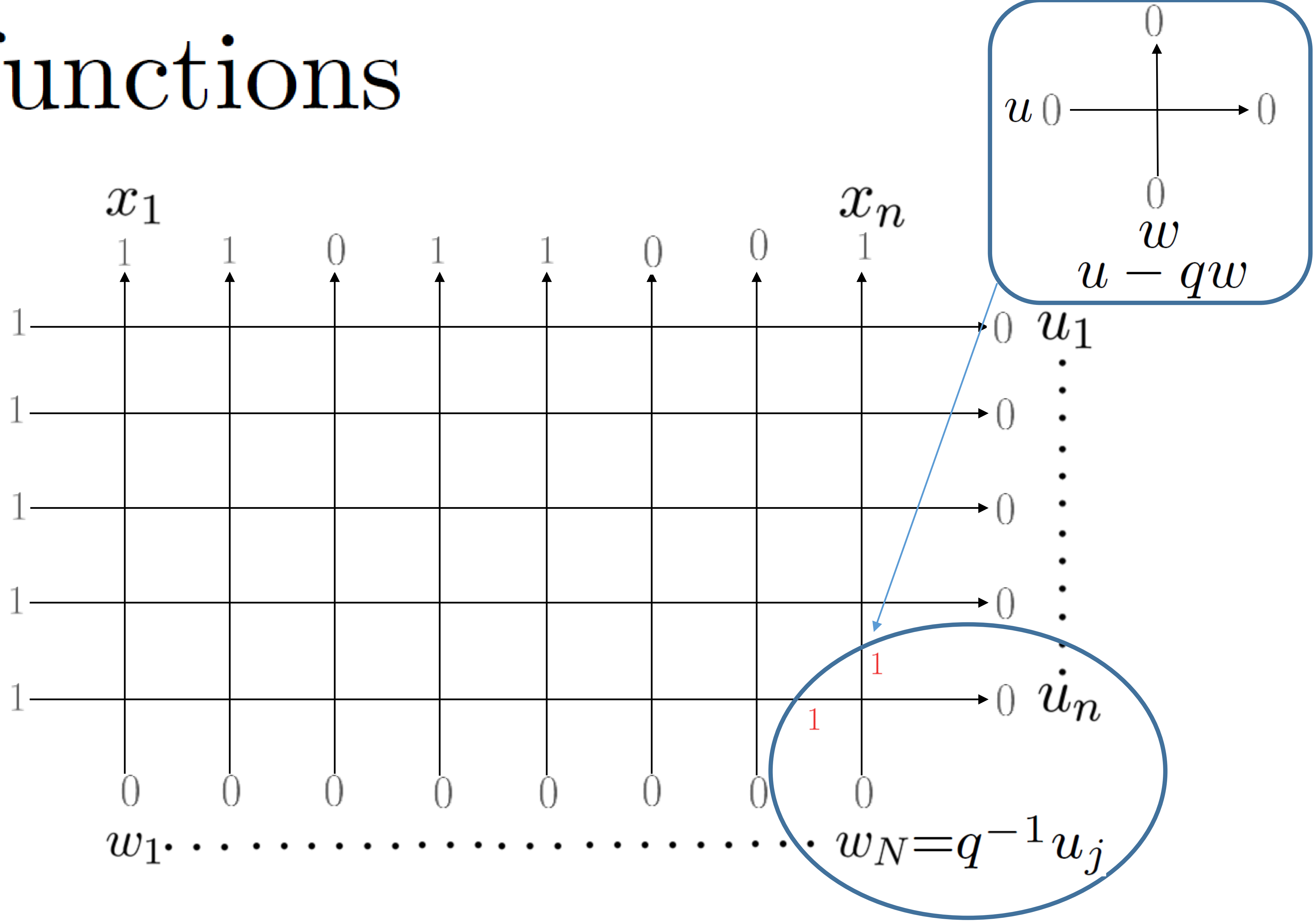
wavefunctions



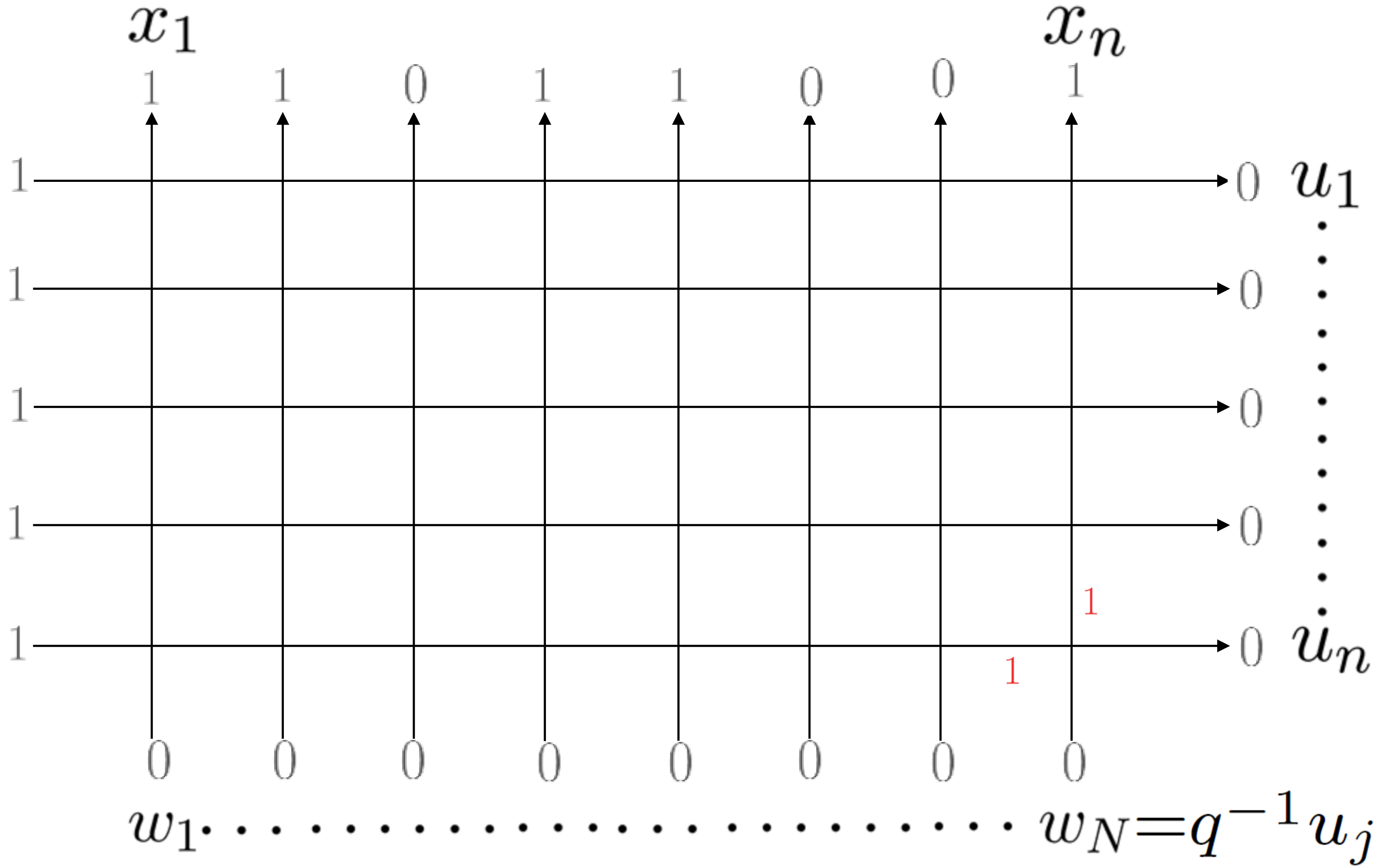
wavefunctions



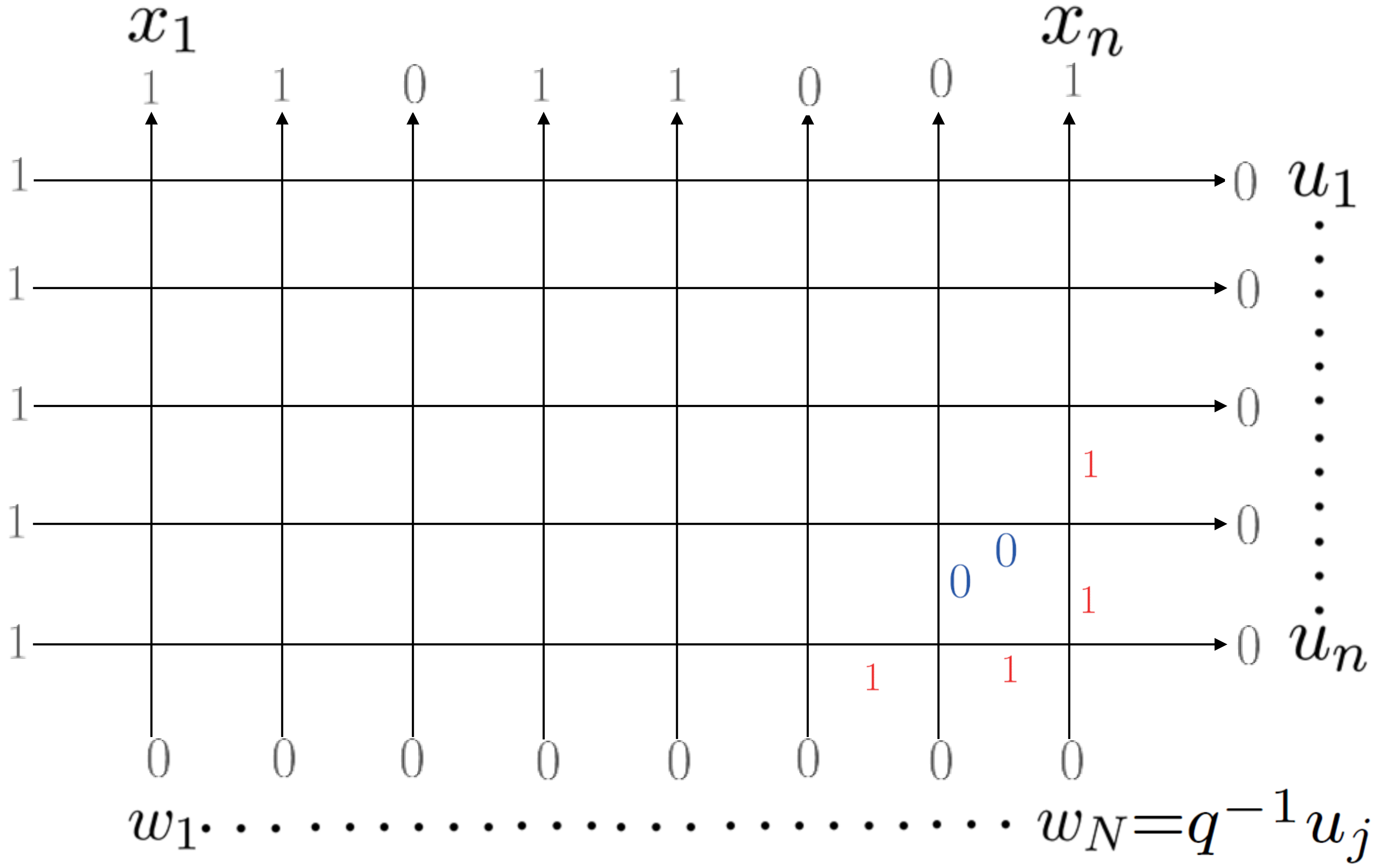
wavefunctions



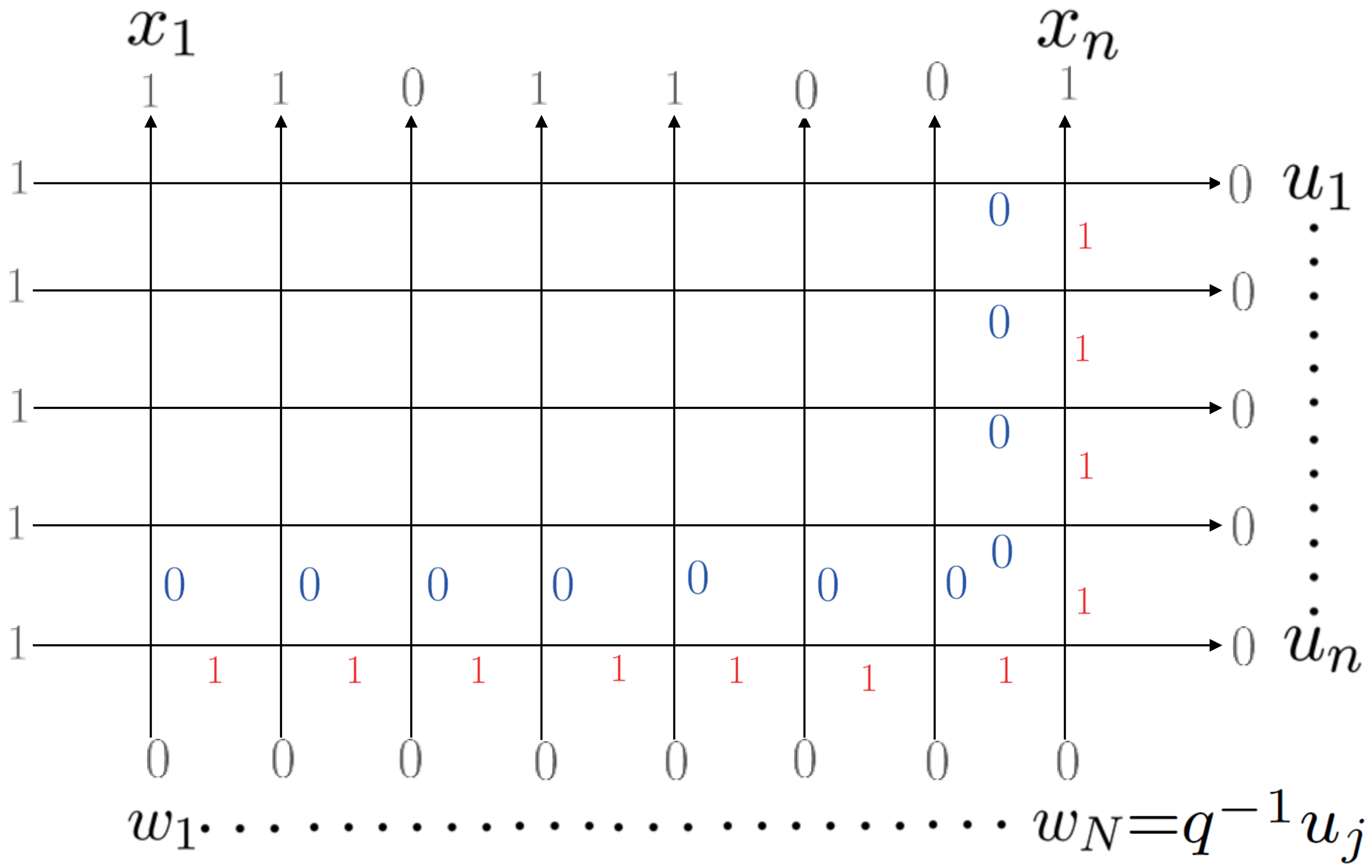
wavefunctions



wavefunctions

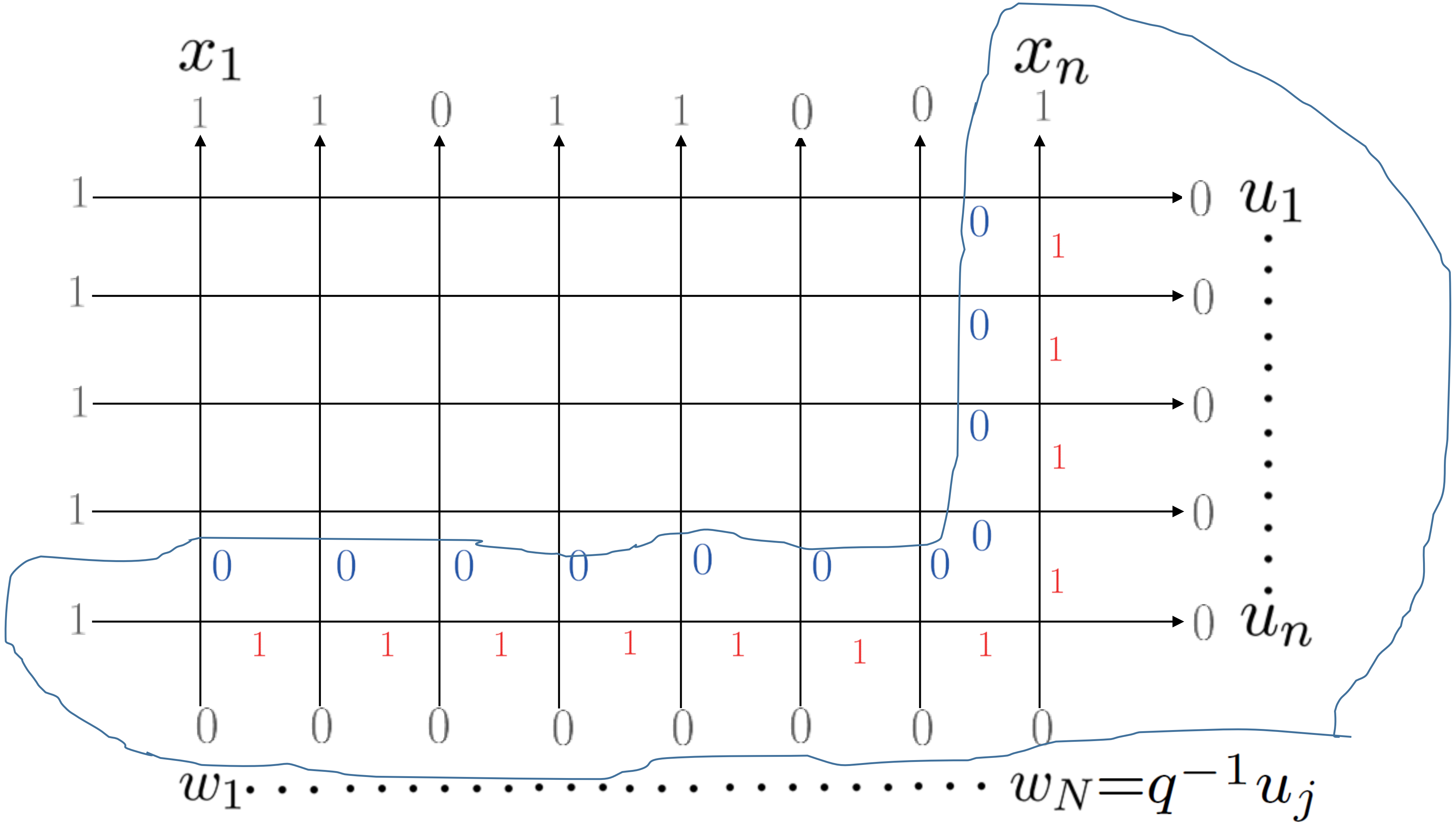


wavefunctions

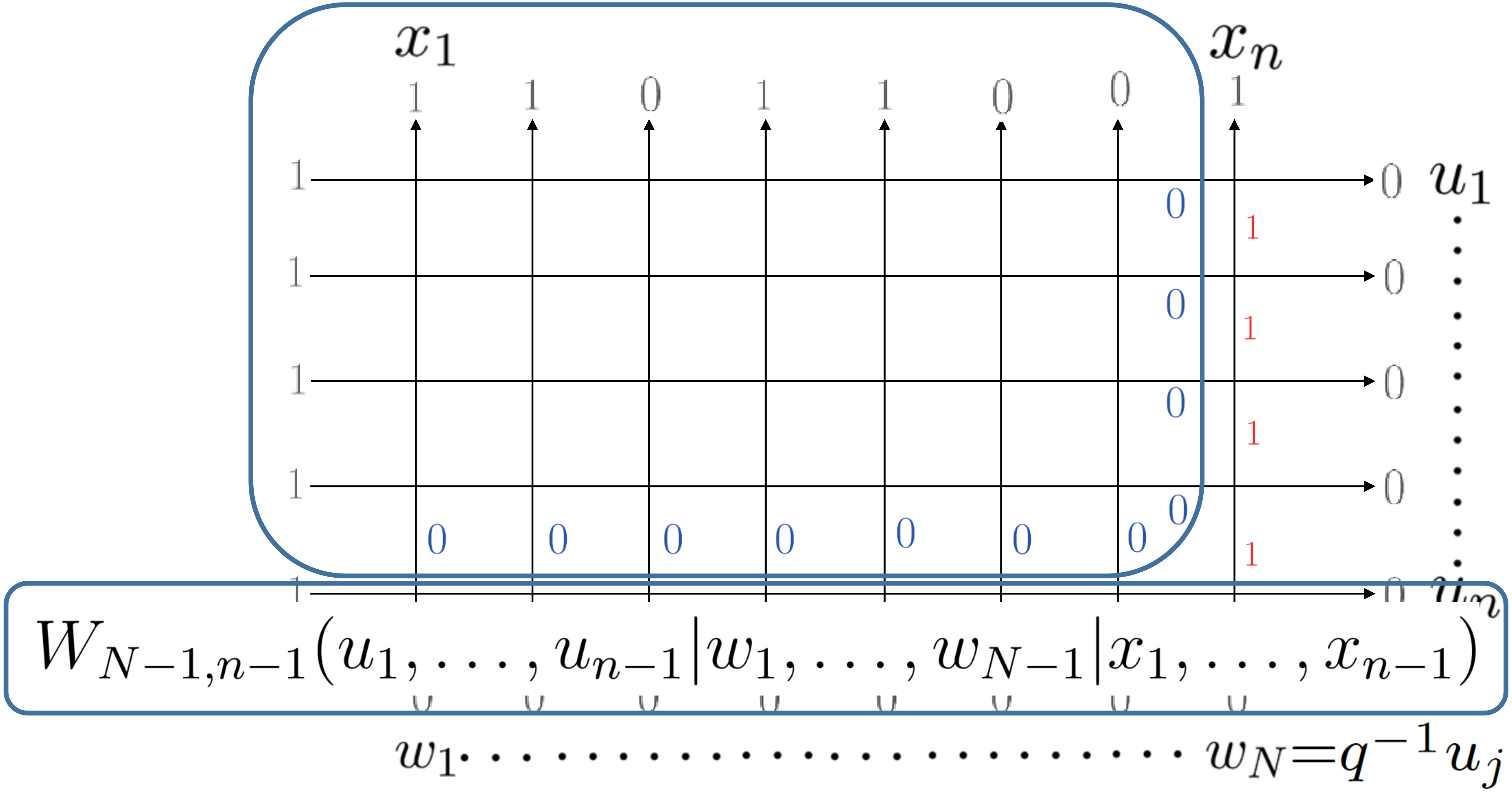


wavefunctions

$$(1-q)u_n \prod_{j=1}^{n-1}(qu_j - u_n) \prod_{j=1}^{N-1}(u_n - w_j)$$



wavefunctions



$$W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1})$$

$w_1 \cdots \cdots w_N = q^{-1}u_j$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\=&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\&\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\=&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$.

(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.

follows from standard train argument

or equivalently, commutativity of B-operators

$$=(1-q)u_n \prod_{j=1} (qu_j-u_n) \prod_{j=1} (u_n-w_j)$$

$$\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).$$

If $x_n \neq N$, *the following factorizations hold for the wavefunctions:*

$$W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$*, the following factorizations hold for the wavefunctions:*

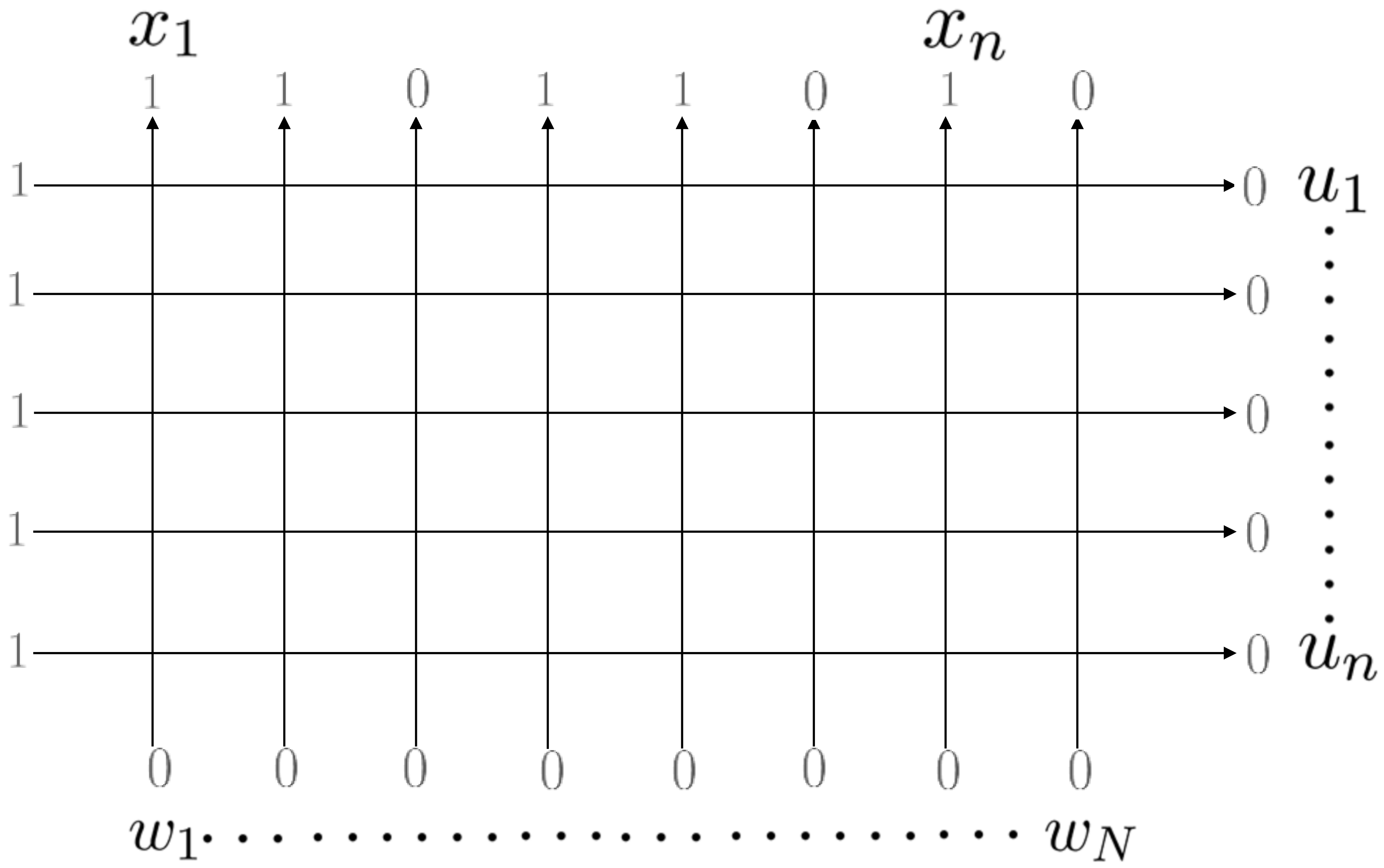
$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

recursive relations between $W_{N,n}$ and $W_{N-1,n}$ $x_n\neq N$

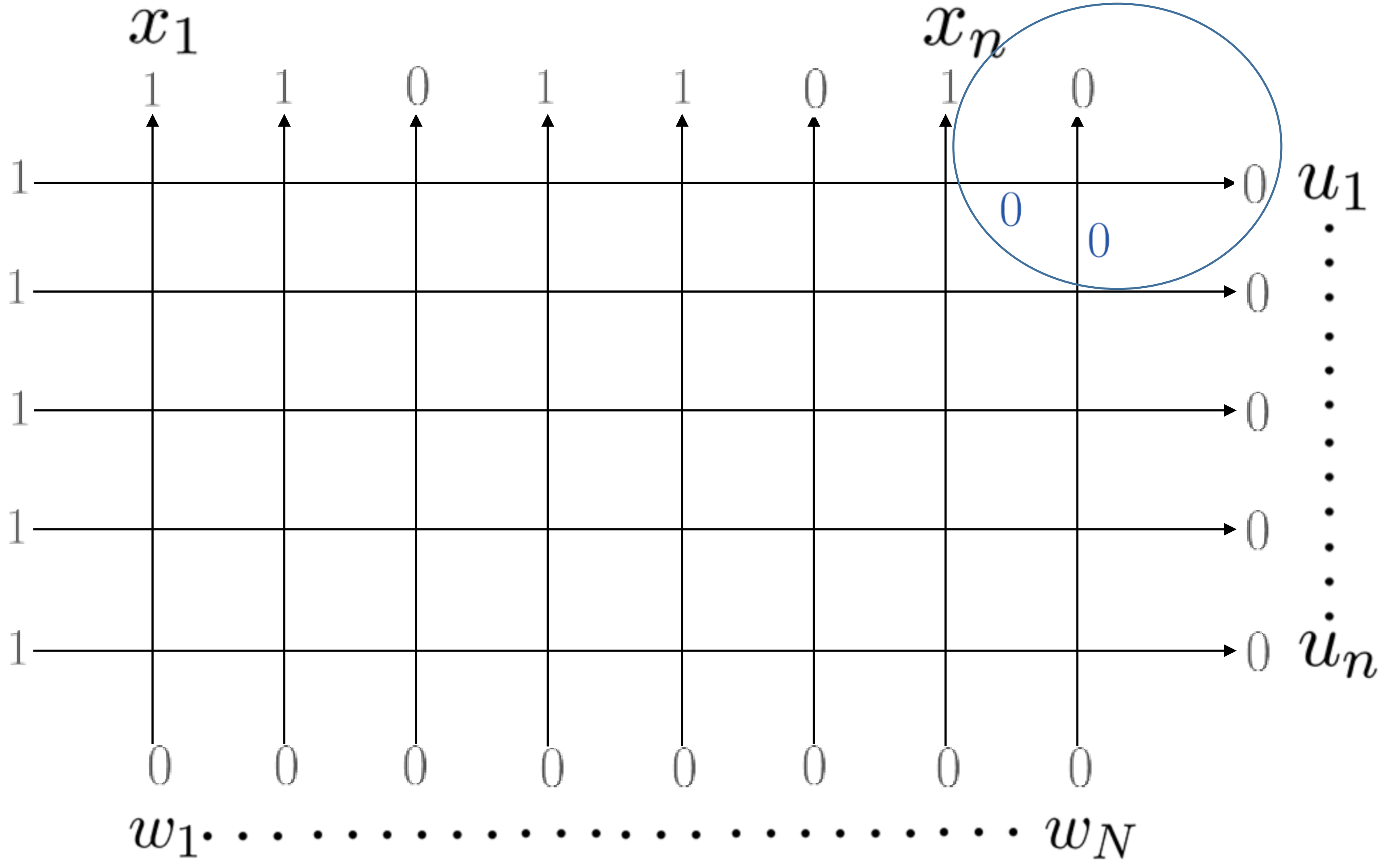
(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

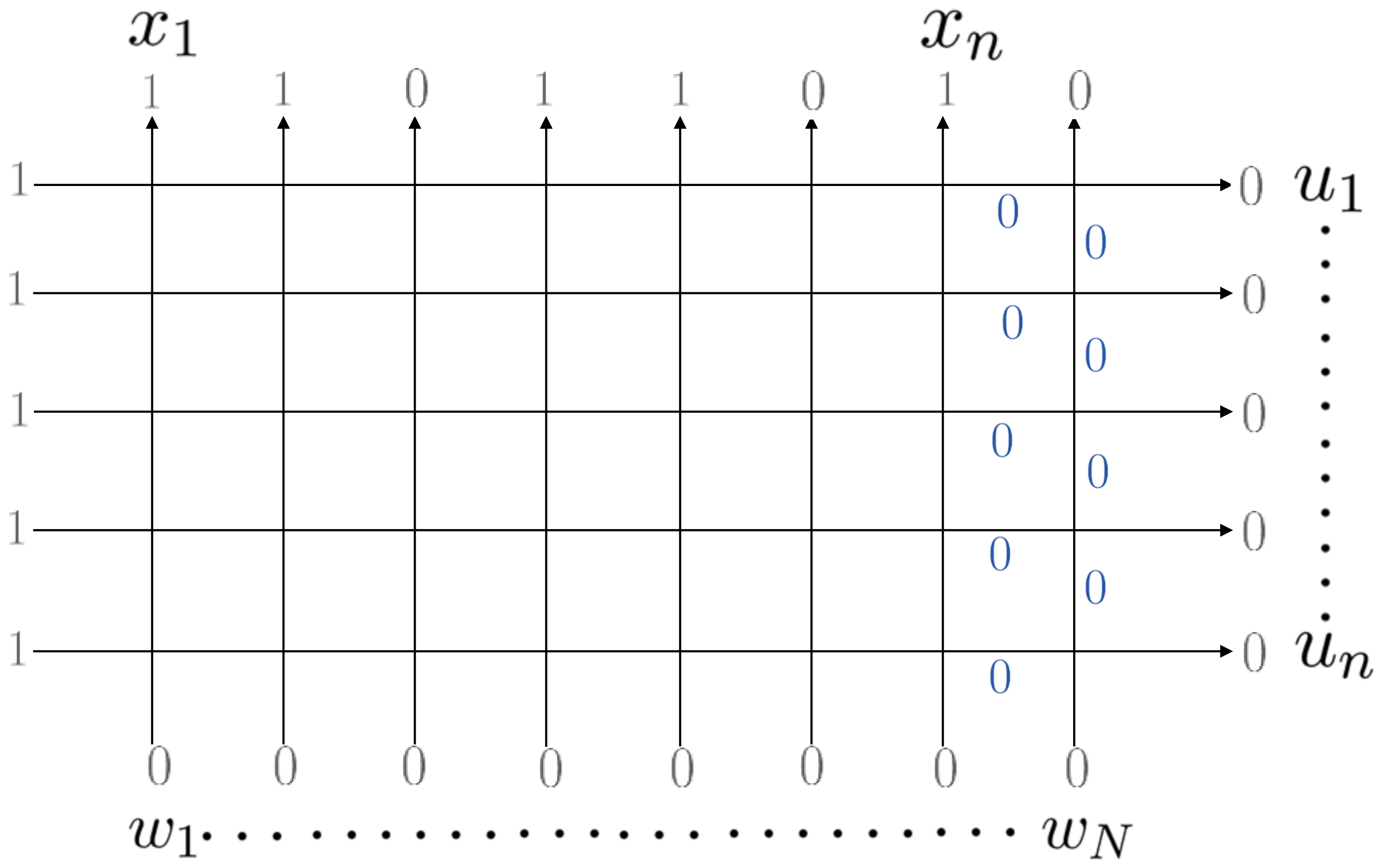
wavefunctions



wavefunctions

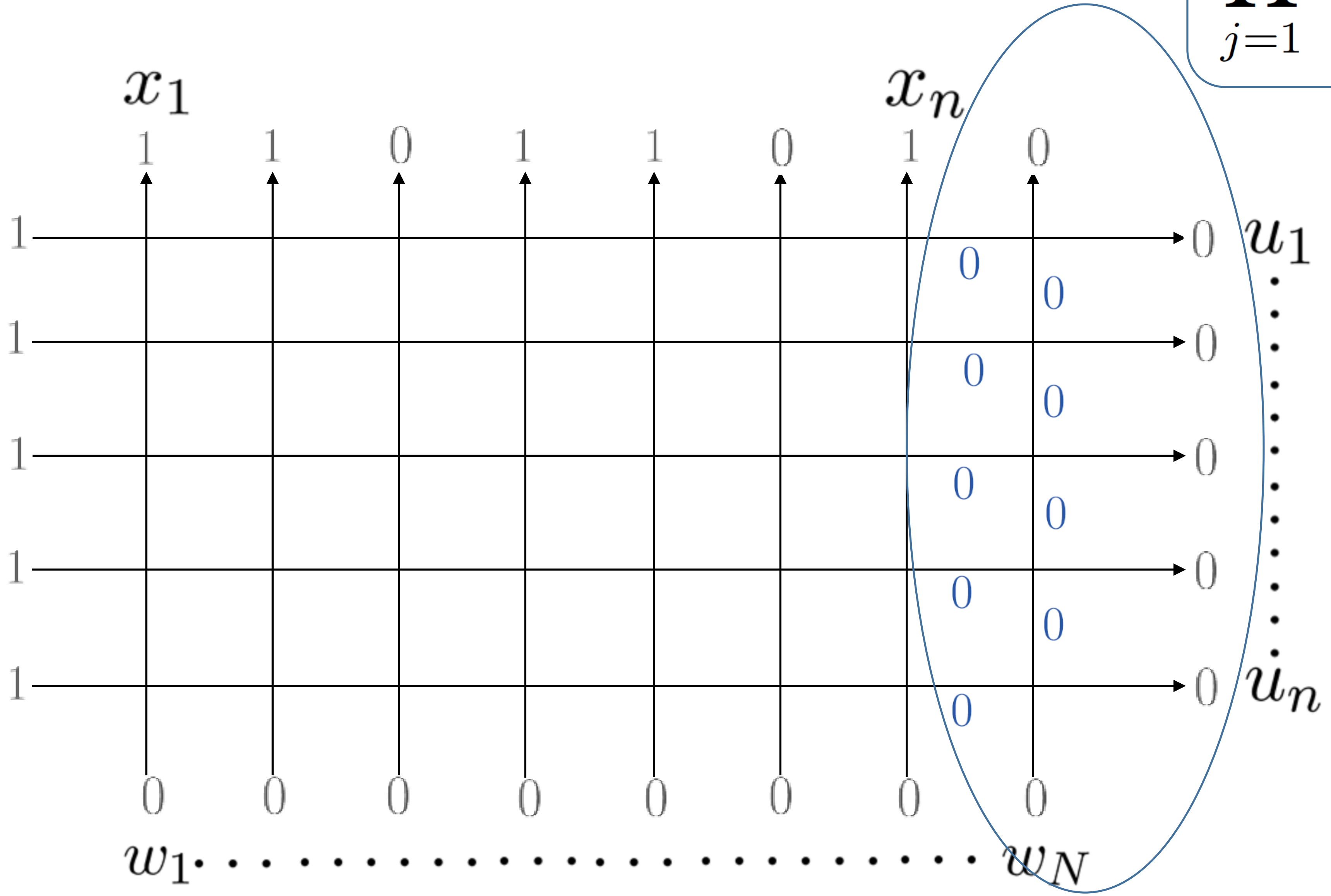


wavefunctions

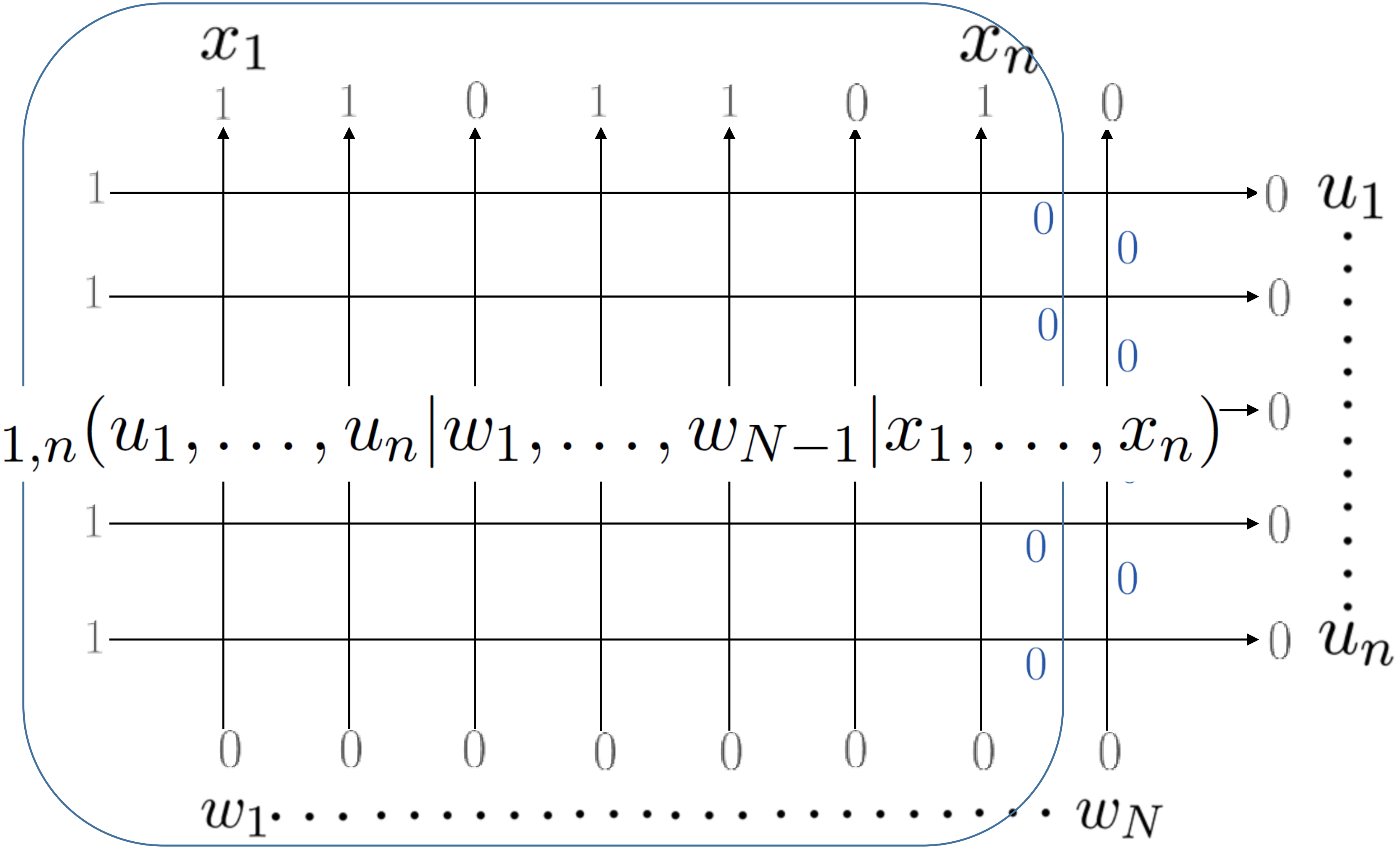


wavefunctions

$$\prod_{j=1}^{n}(u_j - qw_N)$$



wavefunctions



$$W_{N-1,n}(u_1,\ldots,u_n|w_1,\ldots,w_{N-1}|x_1,\ldots,x_n)$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n \neq N$, *the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

recursive relations between $W_{N,n}$ and $W_{N-1,n}$ $x_n \neq N$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\&=(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\&\quad\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$, *the following factorizations hold for the wavefunctions:*

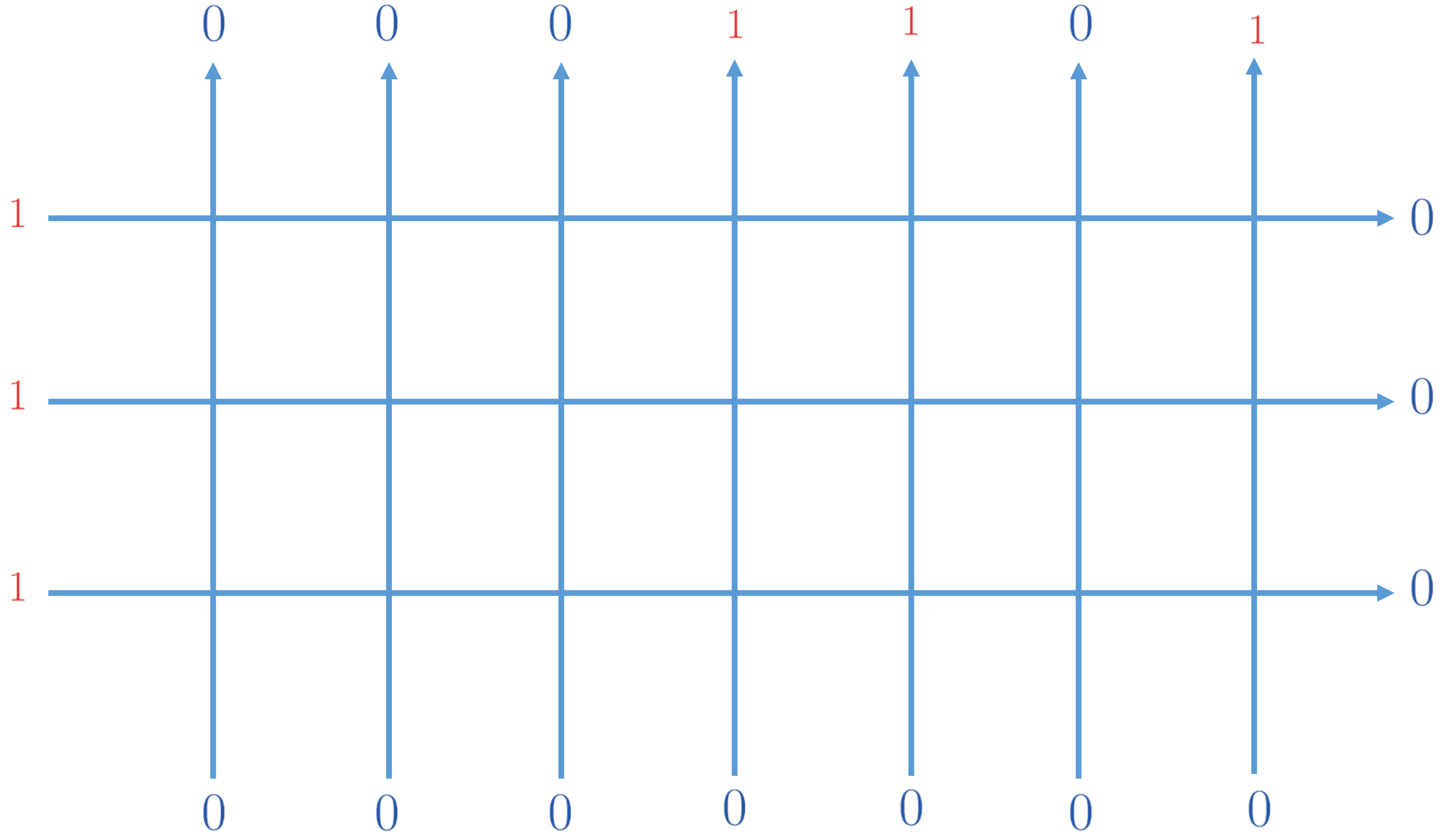
$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\&=\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

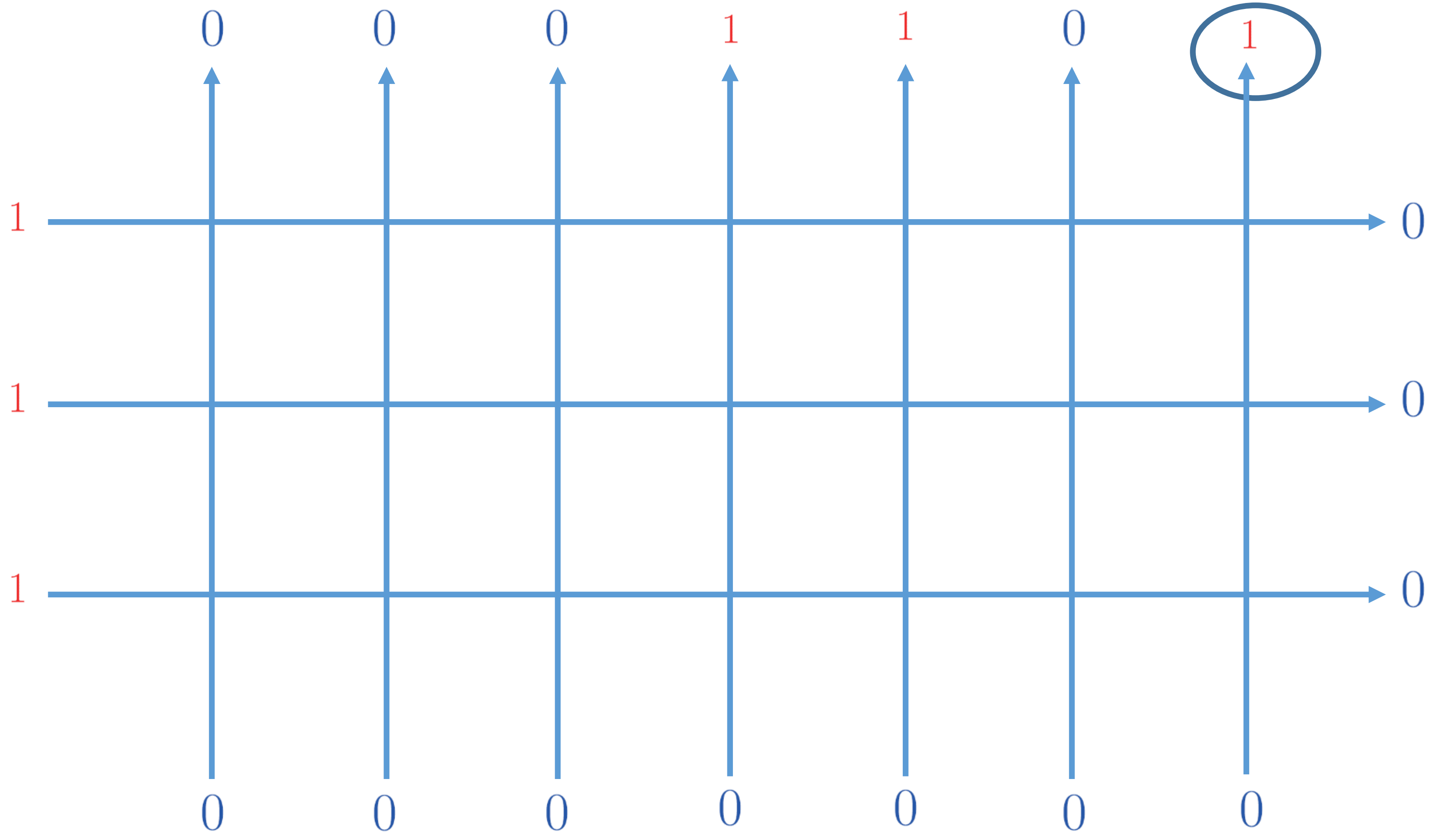
$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

initial condition

$$W_{7,3}(u_1, u_2, u_3 | w_1, w_2, w_3, w_4, w_5, w_6, w_7 | 4, 5, 7)$$

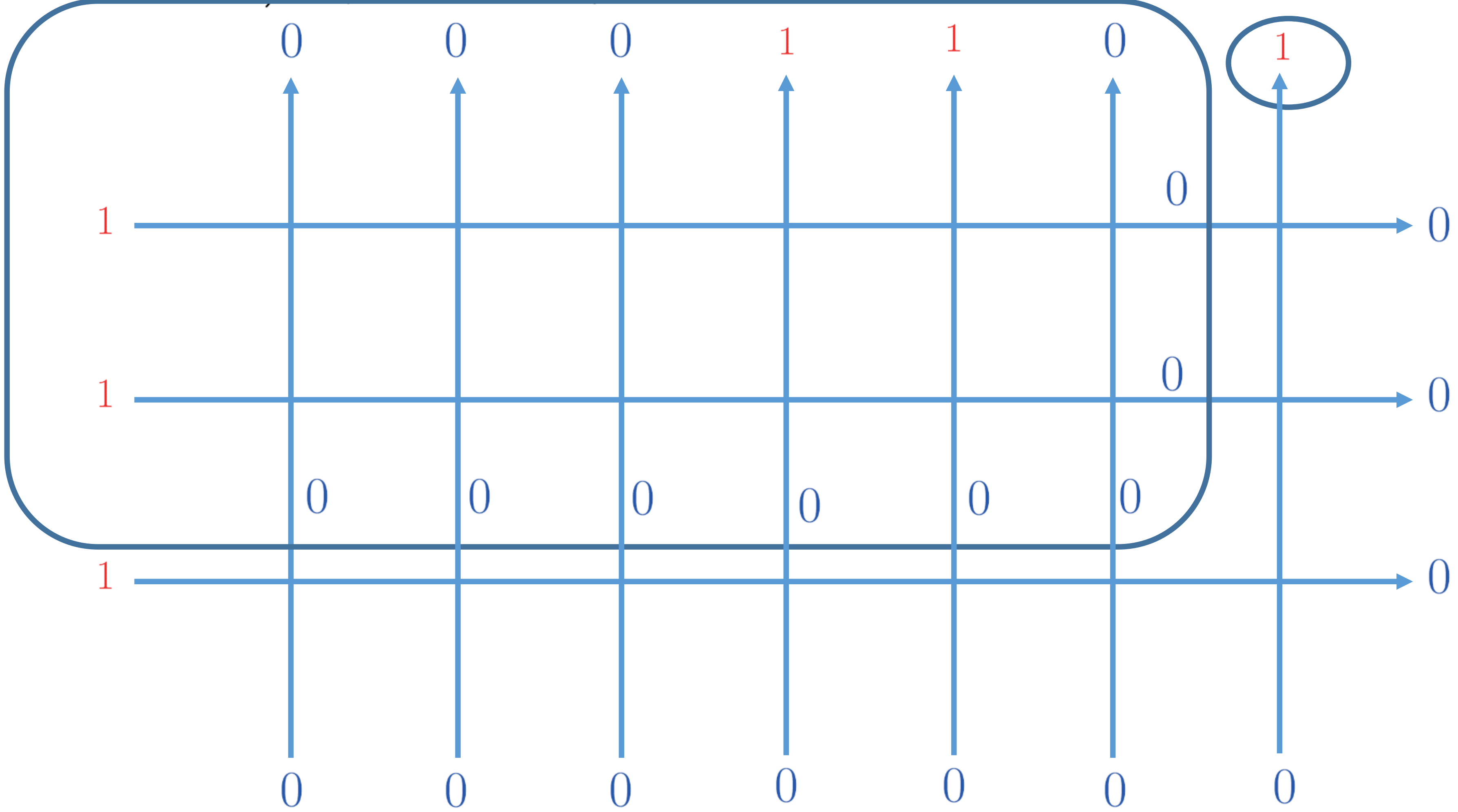


$$W_{7,3}(u_1, u_2, u_3 | w_1, w_2, w_3, w_4, w_5, w_6, w_7 | 4, 5, 7)$$

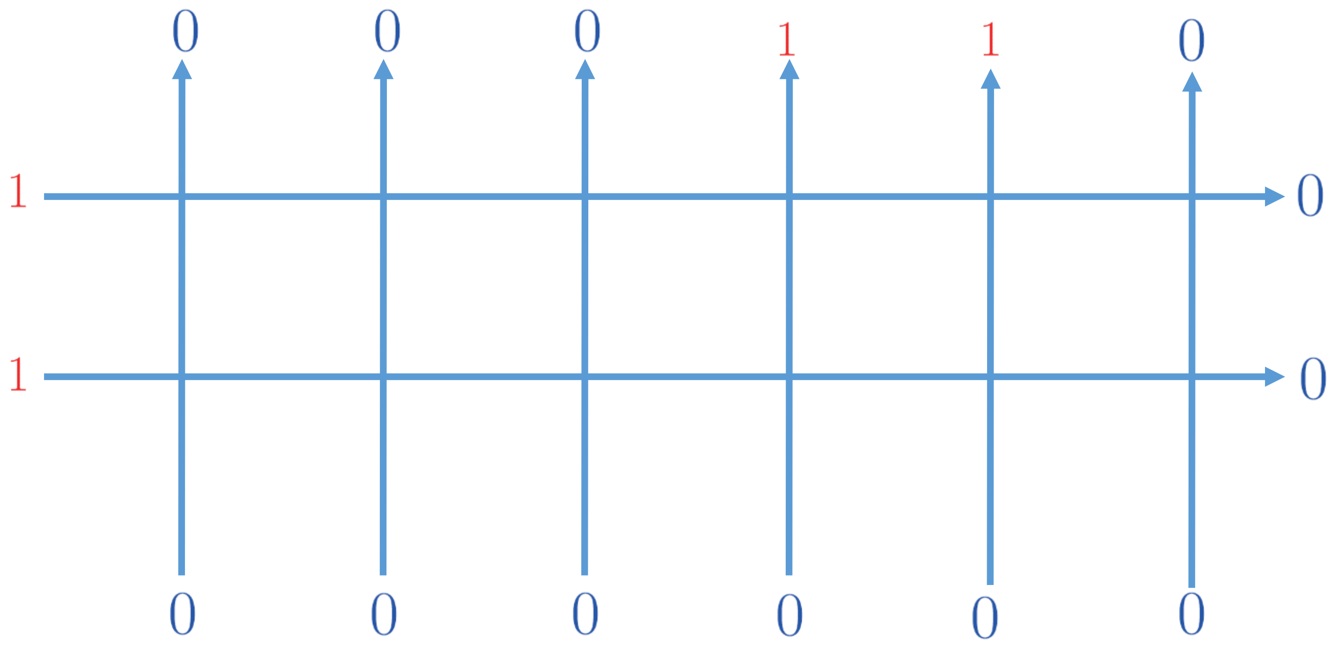


$$W_{7,3}(u_1,u_2,u_3|w_1,w_2,w_3,w_4,w_5,w_6,w_7|4,5,7)$$

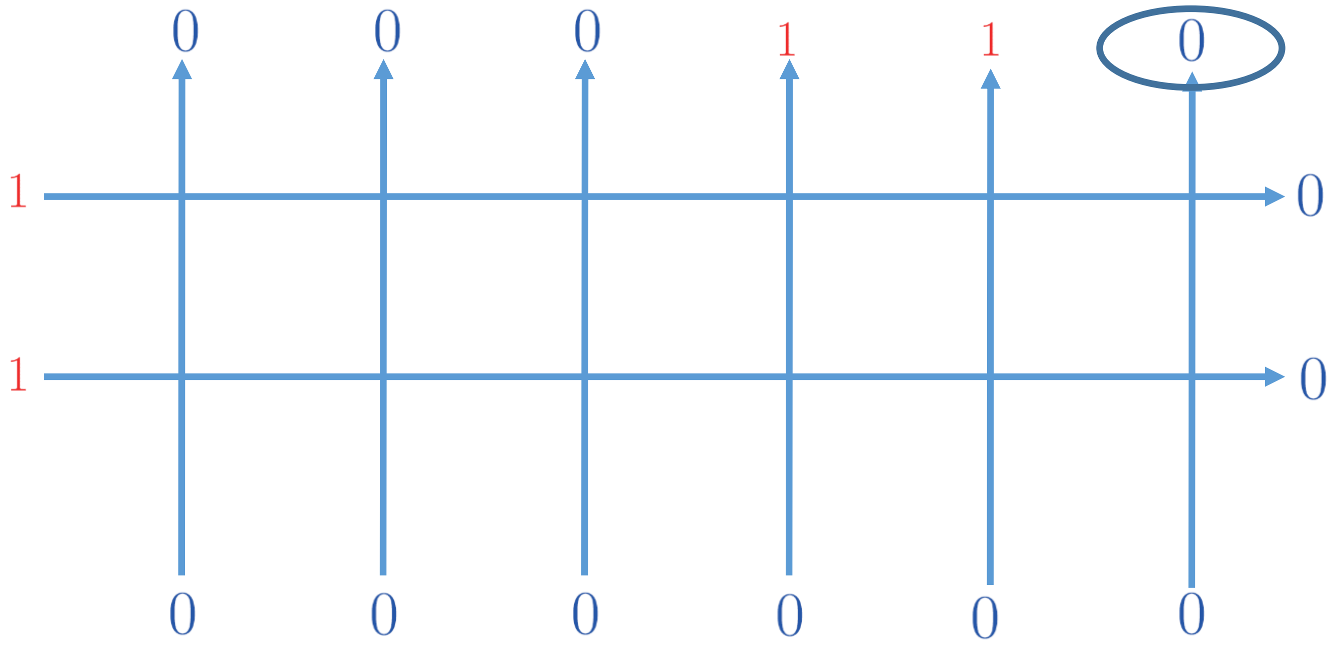
$$\longrightarrow W_{6,2}(u_1,u_2|w_1,w_2,w_3,w_4,w_5,w_6|4,5)$$



$$W_{6,2}(u_1, u_2 | w_1, w_2, w_3, w_4, w_5, w_6 | 4, 5)$$

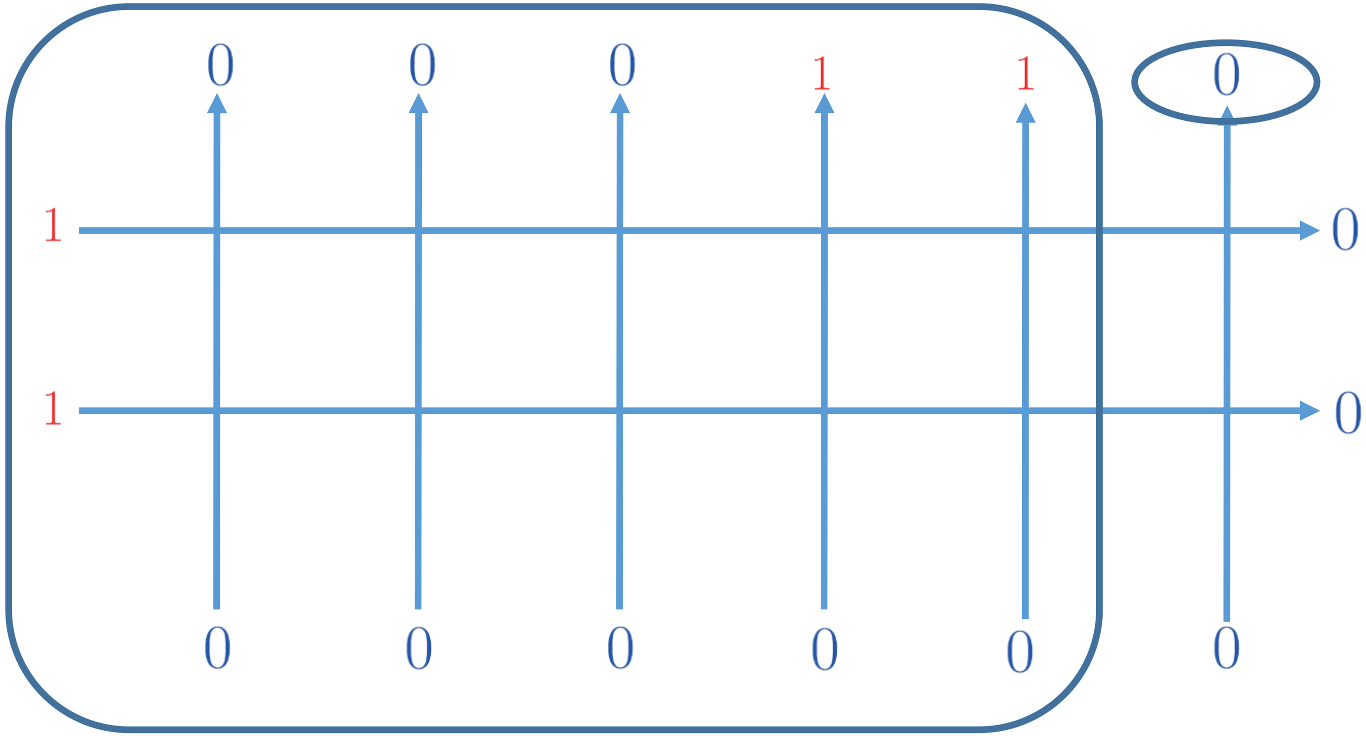


$$W_{6,2}(u_1, u_2 | w_1, w_2, w_3, w_4, w_5, w_6 | 4, 5)$$

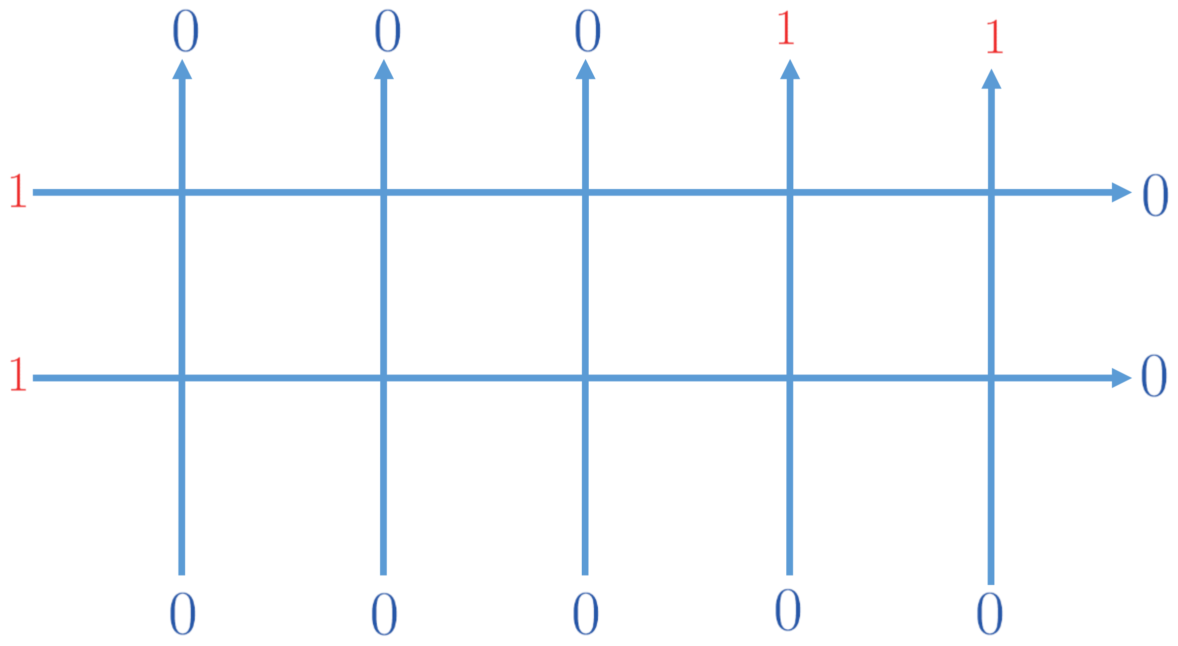


$$W_{6,2}(u_1,u_2|w_1,w_2,w_3,w_4,w_5,w_6|4,5)$$

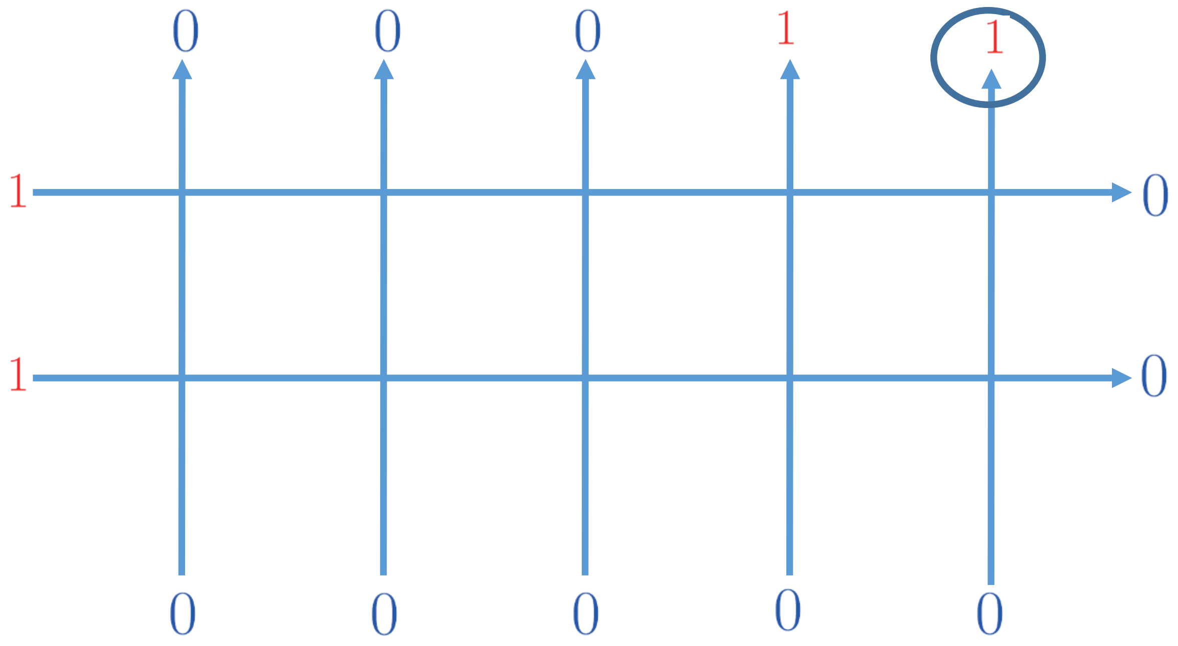
$$\longrightarrow W_{5,2}(u_1,u_2|w_1,w_2,w_3,w_4,w_5|4,5)$$



$$W_{5,2}(u_1, u_2 | w_1, w_2, w_3, w_4, w_5 | 4, 5)$$

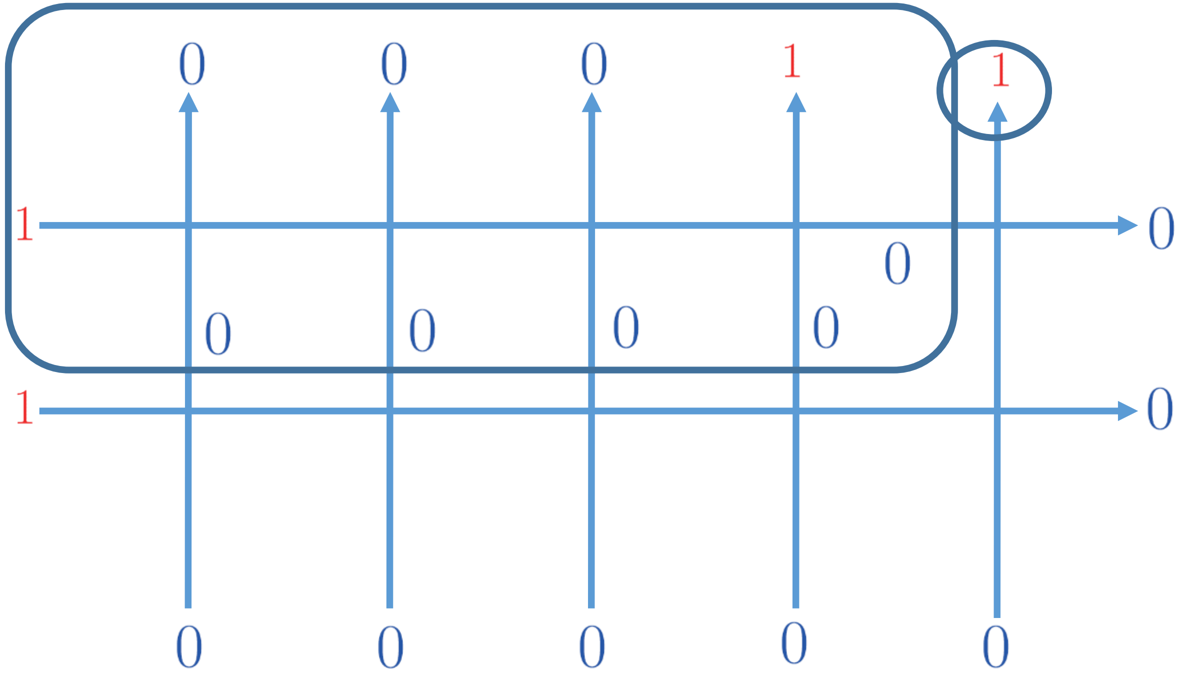


$$W_{5,2}(u_1, u_2 | w_1, w_2, w_3, w_4, w_5 | 4, 5)$$



$$W_{5,2}(u_1, u_2 | w_1, w_2, w_3, w_4, w_5 | 4, 5)$$

$$\longrightarrow W_{4,1}(u_1 | w_1, w_2, w_3, w_4 | 4)$$

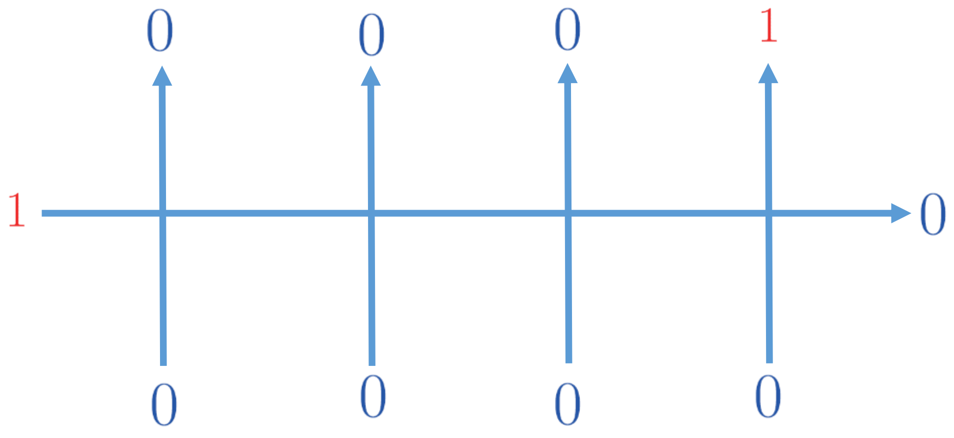


$$W_{7,3}(u_1, u_2, u_3|w_1, w_2, w_3, w_4, w_5, w_6, w_7|4,5,7)$$

$$\longrightarrow W_{6,2}(u_1, u_2|w_1, w_2, w_3, w_4, w_5, w_6|4,5)$$

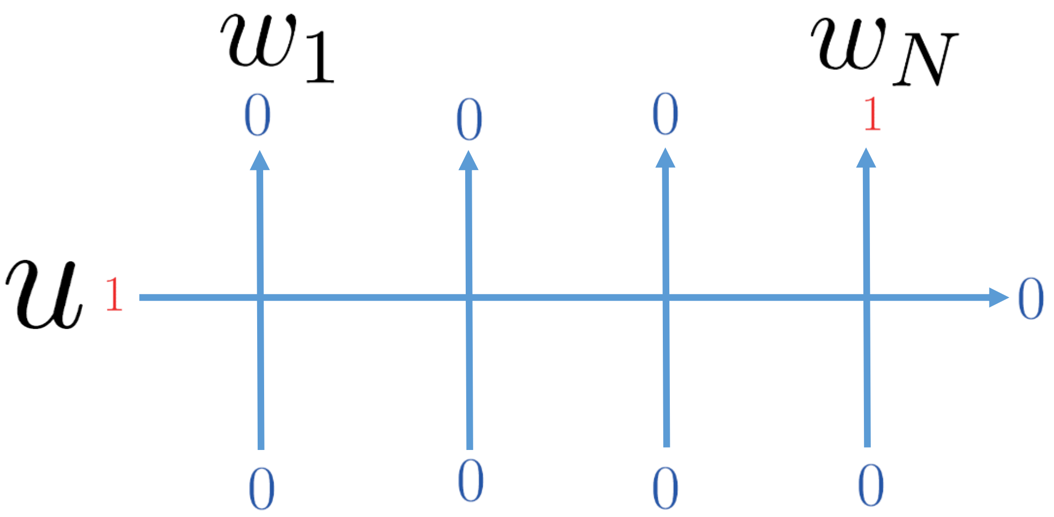
$$\longrightarrow W_{5,2}(u_1, u_2|w_1, w_2, w_3, w_4, w_5|4,5)$$

$$\longrightarrow W_{4,1}(u_1|w_1, w_2, w_3, w_4|4)$$

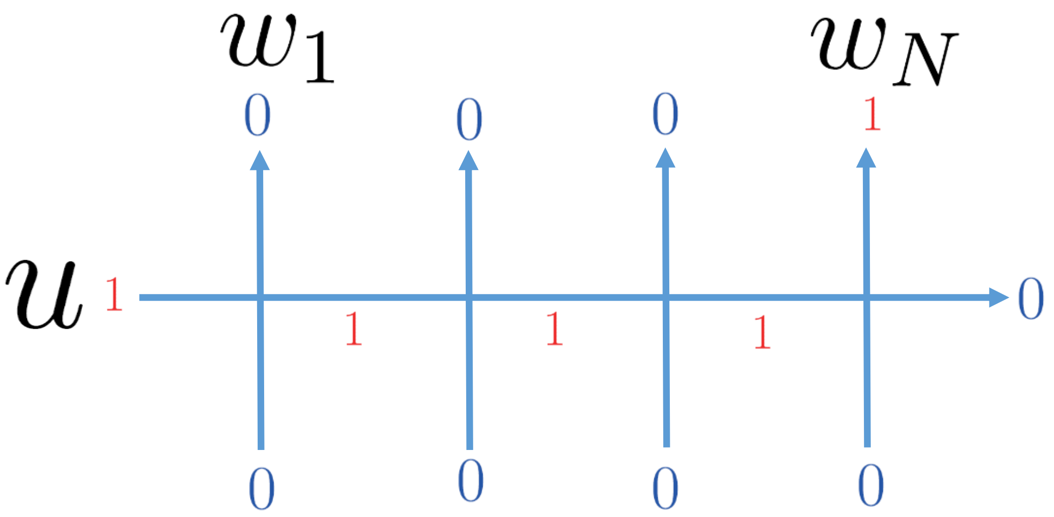


initial condition

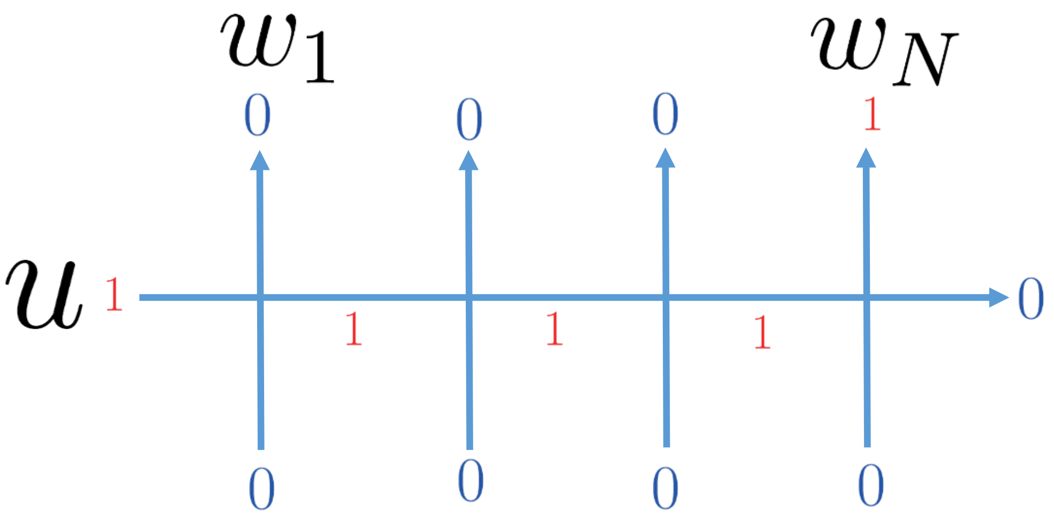
$$W_{N,1}(u|w_1,\ldots,w_N|N)$$



$$W_{N,1}(u|w_1,\dots,w_N|N)$$



$$W_{N,1}(u|w_1,\ldots,w_N|N)$$



$$(1-q)u\prod_{k=1}^{N-1}(u-w_k)$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$, *the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

initial condition

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n \neq N$.
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.
(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n \neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

show

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

satisfies Korepin’s lemma

show

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

satisfies Korepin's lemma

$$\longrightarrow W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$.

(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$.

(3) The following recursive relations between the wavefunctions hold if $x_n=N$:

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$, *the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\boxed{\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)}\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$\Big\downarrow\; x_n=N$$

$$\boxed{\prod_{j=1}^{n-1}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\boxed{\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)}\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$x_n = N$$

$$\boxed{\prod_{j=1}^{n-1}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)}$$

only the summands satisfying $\sigma(n) = n$

survive after $w_N = q^{-1}u_n$

$$
\begin{aligned}
&F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n} \\
&= \sum_{\sigma'\in S_{n-1}} \prod_{j=1}^{n-1} \prod_{k=x_j+1}^{N-1} (u_{\sigma'(j)} - qw_k) \prod_{j=1}^{n-1} (u_{\sigma'(j)} - u_n) \\
&\quad \times \prod_{1\le j<k\le n-1} \frac{qu_{\sigma'(j)} - u_{\sigma'(k)}}{u_{\sigma'(j)} - u_{\sigma'(k)}} \prod_{j=1}^{n-1} \frac{qu_{\sigma'(j)} - u_n}{u_{\sigma'(j)} - u_n} \prod_{j=1}^{n-1} \prod_{k=1}^{x_j-1} (u_{\sigma'(j)} - w_k) \prod_{k=1}^{N-1} (u_n - w_k) \\
&\quad \times (1-q)u_n \prod_{j=1}^{n-1} (1-q)u_{\sigma'(j)}.
\end{aligned}
$$

$$
\begin{aligned}
&F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\big|_{w_N=q^{-1}u_n}\\
&=\sum_{\sigma'\in S_{n-1}}\prod_{j=1}^{n-1}\prod_{k=x_j+1}^{N-1}(u_{\sigma'(j)}-qw_k)\boxed{\prod_{j=1}^{n-1}(u_{\sigma'(j)}-u_n)}\\
&\times\prod_{1\le j<k\le n-1}\frac{qu_{\sigma'(j)}-u_{\sigma'(k)}}{u_{\sigma'(j)}-u_{\sigma'(k)}}\boxed{\prod_{j=1}^{n-1}\frac{qu_{\sigma'(j)}-u_n}{u_{\sigma'(j)}-u_n}}\prod_{j=1}^{n-1}\prod_{k=1}^{x_j-1}(u_{\sigma'(j)}-w_k)\prod_{k=1}^{N-1}(u_n-w_k)\\
&\times(1-q)u_n\prod_{j=1}^{n-1}(1-q)u_{\sigma'(j)}.
\end{aligned}
$$

$$
\boxed{\prod_{j=1}^{n-1}(qu_j-u_n)}
$$

$$
\begin{aligned}
&F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\
=&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\sum_{\sigma'\in S_{n-1}}\prod_{j=1}^{n-1}\prod_{k=x_j+1}^{N-1}(u_{\sigma'(j)}-qw_k)\\
&\times\prod_{1\le j<k\le n-1}\frac{qu_{\sigma'(j)}-u_{\sigma'(k)}}{u_{\sigma'(j)}-u_{\sigma'(k)}}\prod_{j=1}^{n-1}\prod_{k=1}^{x_j-1}(u_{\sigma'(j)}-w_k)\prod_{j=1}^{n-1}(1-q)u_{\sigma'(j)}.
\end{aligned}
$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}$$

$$=(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\boxed{\sum_{\sigma'\in S_{n-1}}\prod_{j=1}^{n-1}\prod_{k=x_j+1}^{N-1}(u_{\sigma'(j)}-qw_k)}$$

$$\boxed{\times\prod_{1\le j<k\le n-1}\frac{qu_{\sigma'(j)}-u_{\sigma'(k)}}{u_{\sigma'(j)}-u_{\sigma'(k)}}\prod_{j=1}^{n-1}\prod_{k=1}^{x_j-1}(u_{\sigma'(j)}-w_k)\prod_{j=1}^{n-1}(1-q)u_{\sigma'(j)}.}$$

$$\boxed{F_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1})}$$

$$
\begin{aligned}
&F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\
=&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\
&\times F_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1})
\end{aligned}
$$

Korepin's lemma

The wavefunctions $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *satisfies the following properties.*

(1) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is a polynomial of degree* $n-1$ *in* w_N *if* $x_n=N$ *and degree* n *if* $x_n\neq N$*.*
(2) $W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$ *is symmetric with respect to* u_j, $j=1,\dots,n$*.*
(3) The following recursive relations between the wavefunctions hold if $x_n=N$*:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)|_{w_N=q^{-1}u_n}\\ =&(1-q)u_n\prod_{j=1}^{n-1}(qu_j-u_n)\prod_{j=1}^{N-1}(u_n-w_j)\\ &\times W_{N-1,n-1}(u_1,\dots,u_{n-1}|w_1,\dots,w_{N-1}|x_1,\dots,x_{n-1}).\end{aligned}$$

If $x_n\neq N$*, the following factorizations hold for the wavefunctions:*

$$\begin{aligned}&W_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\ =&\prod_{j=1}^{n}(u_j-qw_N)W_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n).\end{aligned}$$

(4) The following holds for the case $n=1$, $x_1=N$

$$W_{N,1}(u|w_1,\dots,w_N|N)=(1-q)u\prod_{k=1}^{N-1}(u-w_k).$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\boxed{\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)}\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$\Big\downarrow \quad x_n \neq N$$

$$\boxed{\prod_{j=1}^{n}\prod_{k=x_j+1}^{N-1}(u_{\sigma(j)}-qw_k)\prod_{j=1}^{n}(u_{\sigma(j)}-qw_N)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$= \sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N-1}(u_{\sigma(j)}-qw_k)\prod_{j=1}^{n}(u_{\sigma(j)}-qw_N)$$

$$\times \prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-qw_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

$$=\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N-1}(u_{\sigma(j)}-qw_k)\boxed{\prod_{j=1}^{n}(u_{\sigma(j)}-qw_N)}$$

$$\times\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(\quad\parallel\quad-qw_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

$$\boxed{\prod_{j=1}^{n}(u_j-qw_N)}$$

$$F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)$$

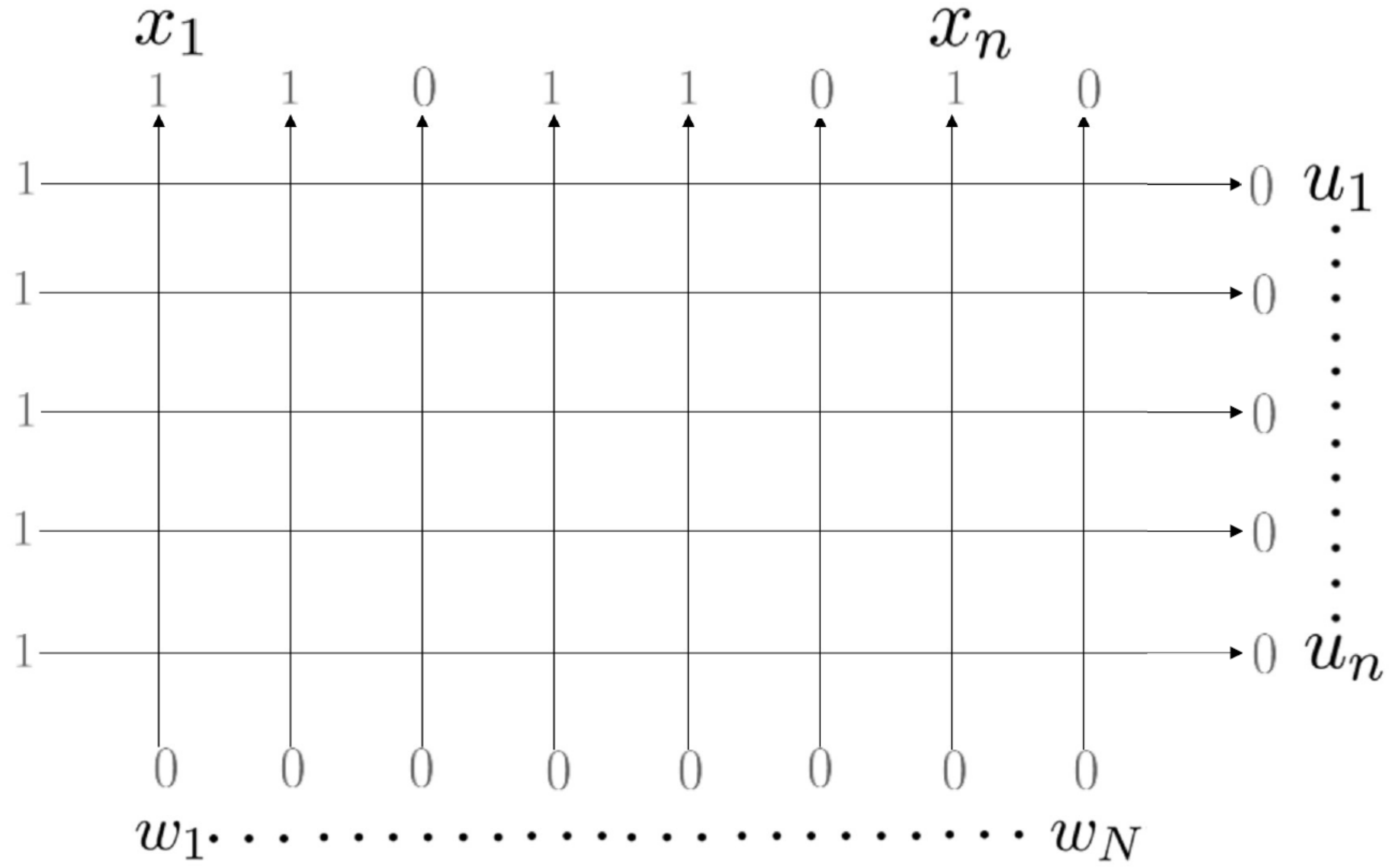
$$=\prod_{j=1}^{n}(u_j-qw_N)\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N-1}(u_{\sigma(j)}-qw_k)$$

$$\times\prod_{1\leq j<k\leq n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-qw_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

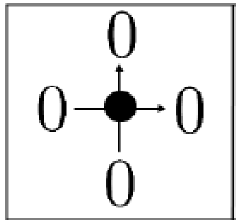
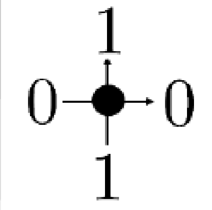
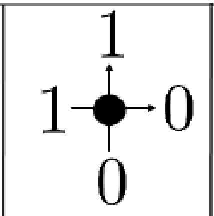
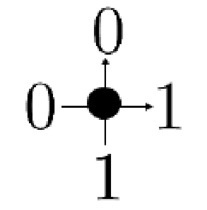
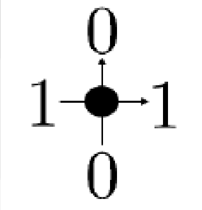
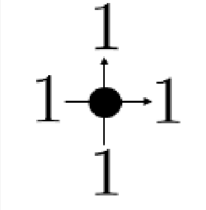
$$
\begin{aligned}
&F_{N,n}(u_1,\dots,u_n|w_1,\dots,w_N|x_1,\dots,x_n)\\
&=\prod_{j=1}^{n}(u_j-qw_N)\sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N-1}(u_{\sigma(j)}-qw_k)\\
&\quad\times\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-qw_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}\\
&=\prod_{j=1}^{n}(u_j-qw_N)F_{N-1,n}(u_1,\dots,u_n|w_1,\dots,w_{N-1}|x_1,\dots,x_n)
\end{aligned}
$$

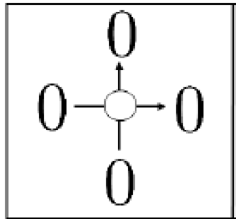
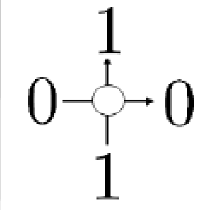
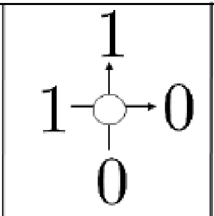
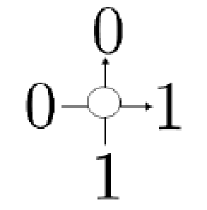
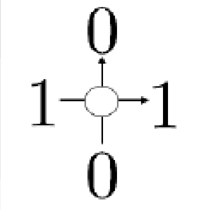
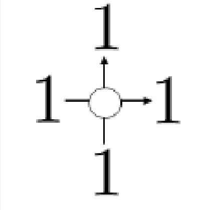
$$W_{N,n}(u_1,\ldots,u_n|w_1,\ldots,w_N|x_1,\ldots,x_n)$$

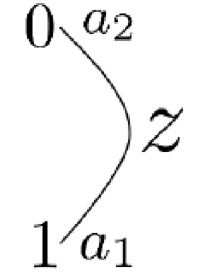
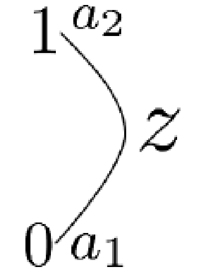
$$= \sum_{\sigma\in S_n}\prod_{j=1}^{n}\prod_{k=x_j+1}^{N}(u_{\sigma(j)}-qw_k)\prod_{1\le j<k\le n}\frac{qu_{\sigma(j)}-u_{\sigma(k)}}{u_{\sigma(j)}-u_{\sigma(k)}}\prod_{j=1}^{n}\prod_{k=1}^{x_j-1}(u_{\sigma(j)}-w_k)\prod_{j=1}^{n}(1-q)u_{\sigma(j)}$$

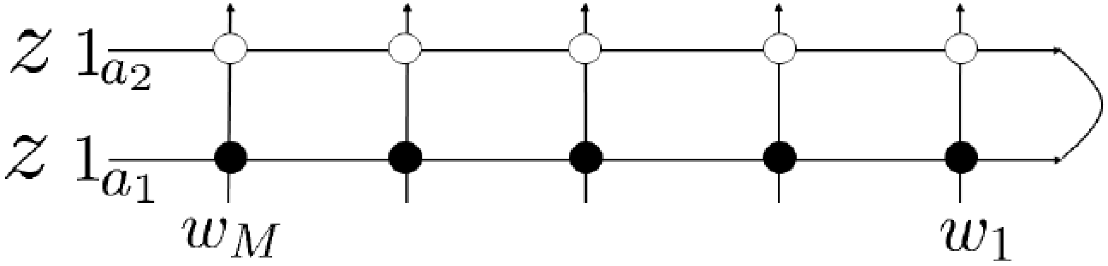


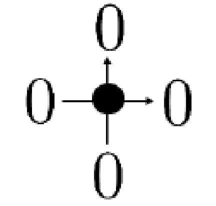
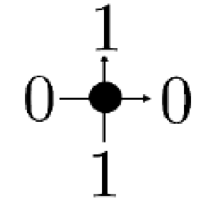
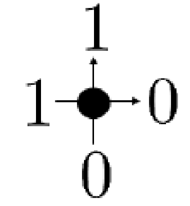
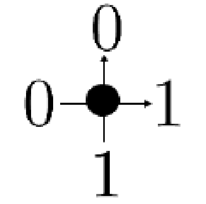
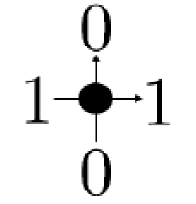
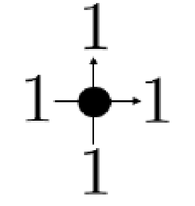
reflecting boundary

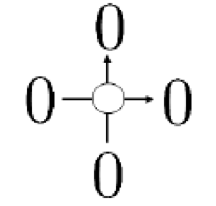
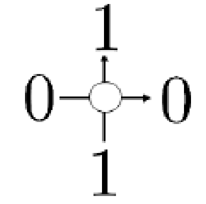
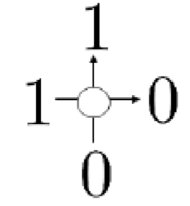
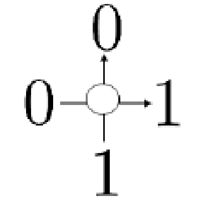
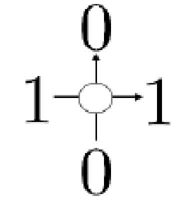
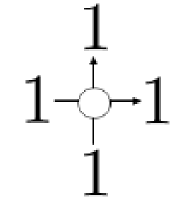
					
$az^{-1}w_j - a^{-1}z$	$az - a^{-1}z^{-1}w_j$	$a^2 - a^{-2}$	$(a^2 - a^{-2})w_j$	$az - a^{-1}z^{-1}w_j$	$az^{-1}w_j - a^{-1}z$

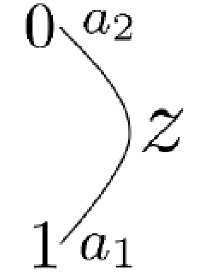
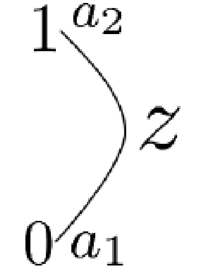
					
$azw_j - a^{-1}z^{-1}$	$az^{-1} - a^{-1}zw_j$	$a^2 - a^{-2}$	$(a^2 - a^{-2})w_j$	$az^{-1} - a^{-1}zw_j$	$azw_j - a^{-1}z^{-1}$

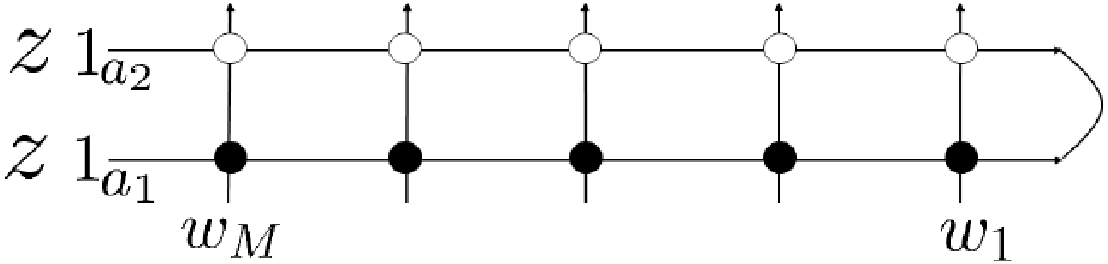
	
$baz - b^{-1}a^{-1}z^{-1}$	$ba^{-1}z^{-1} - b^{-1}az$



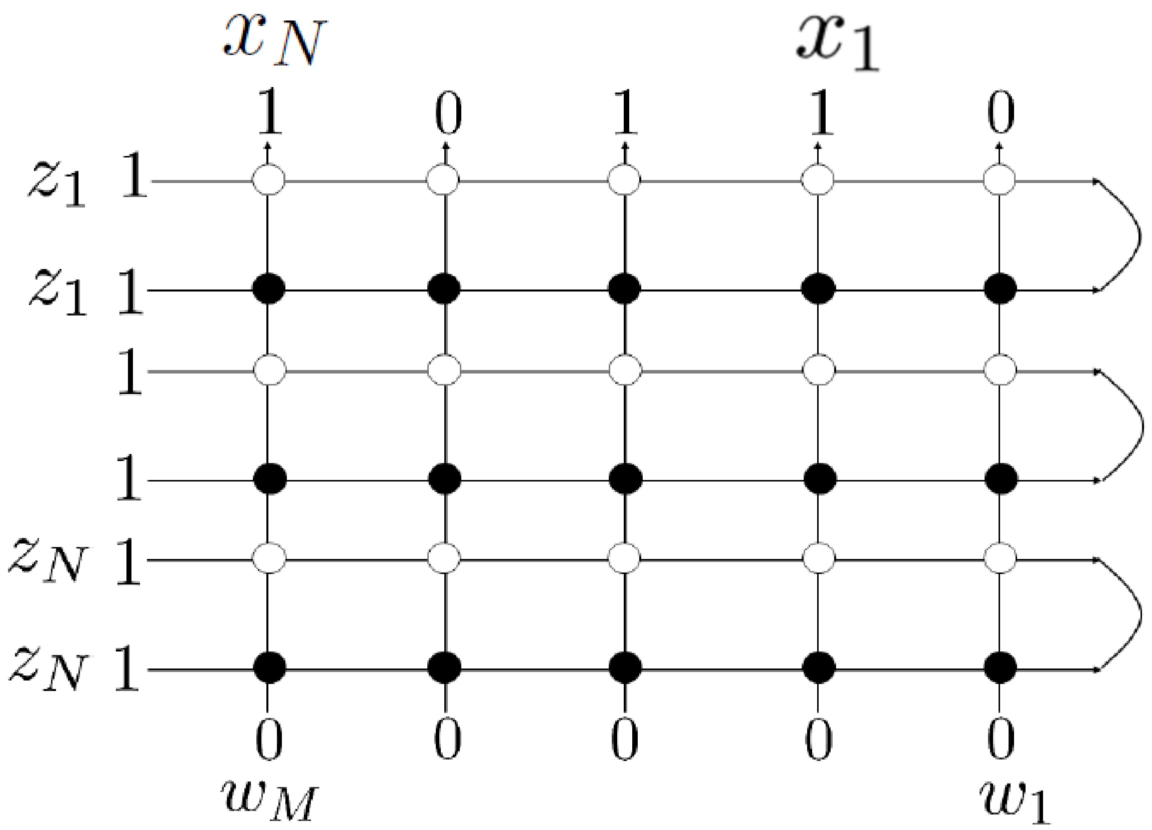
					
$az^{-1}w_j - a^{-1}z$	$az - a^{-1}z^{-1}w_j$	$a^2 - a^{-2}$	$(a^2 - a^{-2})w_j$	$az - a^{-1}z^{-1}w_j$	$az^{-1}w_j - a^{-1}z$

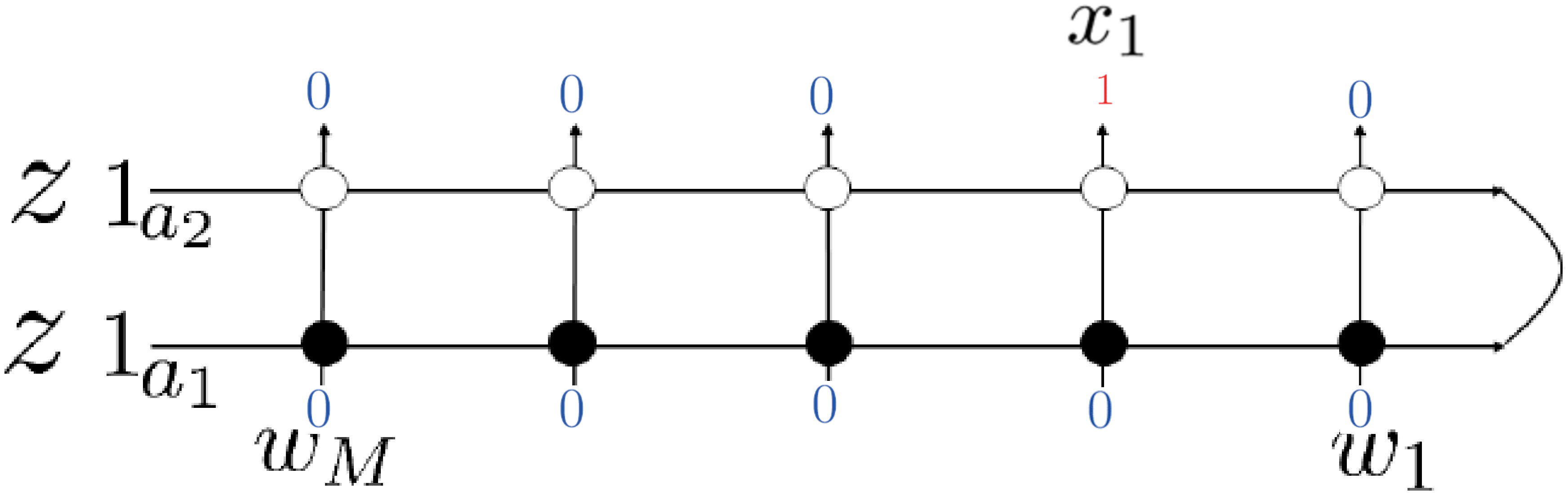
					
$azw_j - a^{-1}z^{-1}$	$az^{-1} - a^{-1}zw_j$	$a^2 - a^{-2}$	$(a^2 - a^{-2})w_j$	$az^{-1} - a^{-1}zw_j$	$azw_j - a^{-1}z^{-1}$

	
$baz - b^{-1}a^{-1}z^{-1}$	$ba^{-1}z^{-1} - b^{-1}az$



$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$





$$= (a^2 - a^{-2})(-a^2 z^2 + a^{-2} z^{-2}) \sum_{\tau=\pm 1} \frac{1}{z^{2\tau} - z^{-2\tau}} (b a^{-1} z^{-\tau} - b^{-1} a z^{\tau})$$

$$\times \prod_{j=1}^{M} (a z^{-\tau} - a^{-1} z^{\tau} w_j) \prod_{j=1}^{x_1-1} (a z^{-\tau} w_j - a^{-1} z^{\tau}) \prod_{j=x_1+1}^{M} (a z^{\tau} - a^{-1} z^{-\tau} w_j)$$

one-particle case

one needs to show

$$(a^2 - a^{-2})(z^2 - z^{-2}) \sum_{j=1}^{x_1-1} w_j \prod_{k=1}^{j-1} (az^{-1} - a^{-1} z w_k)(az^{-1} w_k - a^{-1} z)$$

$$\times \prod_{k=j+1}^{x_1-1} (azw_k - a^{-1} z^{-1})(az - a^{-1} z^{-1} w_k)$$

$$= \prod_{k=1}^{x_1-1} (azw_k - a^{-1} z^{-1})(az - a^{-1} z^{-1} w_k) - \prod_{k=1}^{x_1-1} (az^{-1} w_k - a^{-1} z)(az^{-1} - a^{-1} z w_k)$$

Korepin's lemma

The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *satisfies the following properties.*

(1) When $x_N = M$*, the wavefunctions* $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is a polynomial of degree* $2N-1$ *in* w_M*.*
(2) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is symmetric with respect to* $z_1,\dots,z_N$*, i.e.,*

$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)=W_{M,N}(z_{\sigma(1)},\dots,z_{\sigma(N)}|w_1,\dots,w_M|x_1,\dots,x_N),$$

for $\sigma\in S_N$*.*
(3) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *with* z_i *replaced by* z_i^{-1} *is connected with the original one by*

$$\frac{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{z_i\longleftrightarrow z_i^{-1}}}{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)}=\frac{a^2z_i^{-2}-a^{-2}z_i^2}{a^2z_i^2-a^{-2}z_i^{-2}}.$$

Korepin's lemma

(4) The following recursive relations between the wavefunctions hold if $x_N = M$

$$
\begin{aligned}
&W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{w_M=a^2z_N^2}\\
=&(a^2-a^{-2})(ba^{-1}z_N^{-1}-b^{-1}az_N)\prod_{j=1}^{N}(a^3z_N^2z_j-a^{-1}z_j^{-1})\prod_{j=1}^{N-1}(a^3z_N^2z_j^{-1}-a^{-1}z_j)\\
&\times\prod_{j=1}^{M-1}(az_N^{-1}w_j-a^{-1}z_N)(az_N^{-1}-a^{-1}z_Nw_j)\\
&\times W_{M-1,N-1}(z_1,\dots,z_{N-1}|w_1,\dots,w_{M-1}|x_1,\dots,x_{N-1}).
\end{aligned}
$$

If $x_N \neq M$*, the following factorizations hold for the wavefunctions*

$$
\begin{aligned}
&W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)\\
=&\prod_{j=1}^{N}(az_j^{-1}-a^{-1}z_jw_M)(az_j-a^{-1}z_j^{-1}w_M)W_{M-1,N}(z_1,\dots,z_N|w_1,\dots,w_{M-1}|x_1,\dots,x_N).
\end{aligned}
$$

(5) The following holds for the case $N=1$, $x_1=M$

$$
\begin{aligned}
W_{M,1}(z|w_1,\dots,w_M|M)=&(a^2-a^{-2})(-a^2z^2+a^{-2}z^{-2})\sum_{\tau=\pm1}\frac{1}{z^{2\tau}-z^{-2\tau}}(ba^{-1}z^{-\tau}-b^{-1}az^{\tau})\\
&\times\prod_{j=1}^{M}(az^{-\tau}-a^{-1}z^{\tau}w_j)\prod_{j=1}^{M-1}(az^{-\tau}w_j-a^{-1}z^{\tau}).
\end{aligned}
$$

Korepin's lemma

The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *satisfies the following properties.*

(1) When $x_N = M$*, the wavefunctions* $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is a polynomial of degree* $2N-1$ *in* w_M*.*

(2) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is symmetric with respect to* $z_1,\dots,z_N$*, i.e.,*

$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N) = W_{M,N}(z_{\sigma(1)},\dots,z_{\sigma(N)}|w_1,\dots,w_M|x_1,\dots,x_N),$$

for $\sigma \in S_N$*.*

(3) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *with* z_i *replaced by* z_i^{-1} *is connected with the original one by*

$$\frac{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{z_i \longleftrightarrow z_i^{-1}}}{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)} = \frac{a^2 z_i^{-2} - a^{-2} z_i^2}{a^2 z_i^2 - a^{-2} z_i^{-2}}.$$

Sklyanin

Algebraic Bethe ansatz for reflecting boundary conditions

follows from commutativity of the B-operators for double-row monodromy matrices

compare with the approach presented in Chenyang Zhong's talk

Korepin's lemma

The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *satisfies the following properties.*

(1) When $x_N=M$*, the wavefunctions* $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is a polynomial of degree* $2N-1$ *in* w_M*.*
(2) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *is symmetric with respect to* $z_1,\dots,z_N$*, i.e.,*

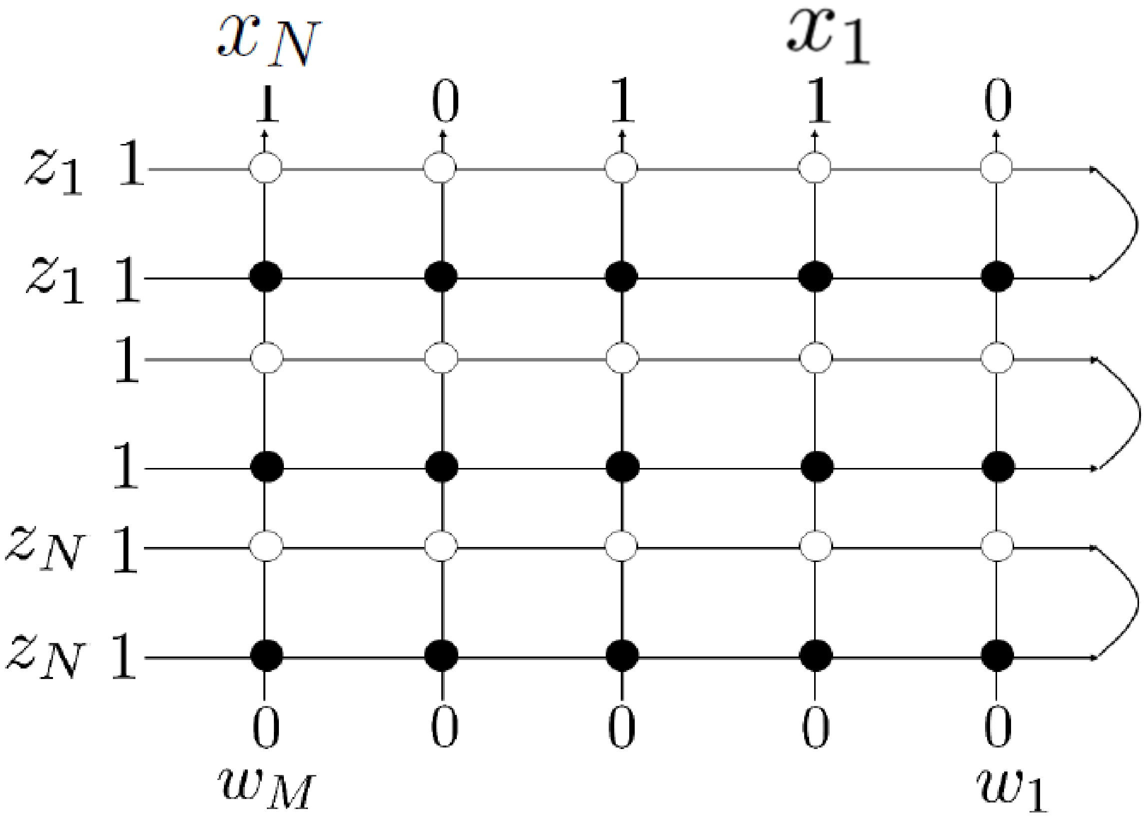
$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)=W_{M,N}(z_{\sigma(1)},\dots,z_{\sigma(N)}|w_1,\dots,w_M|x_1,\dots,x_N),$$

for $\sigma\in S_N$*.*

(3) The wavefunctions $W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$ *with* z_i *replaced by* z_i^{-1} *is connected with the original one by*

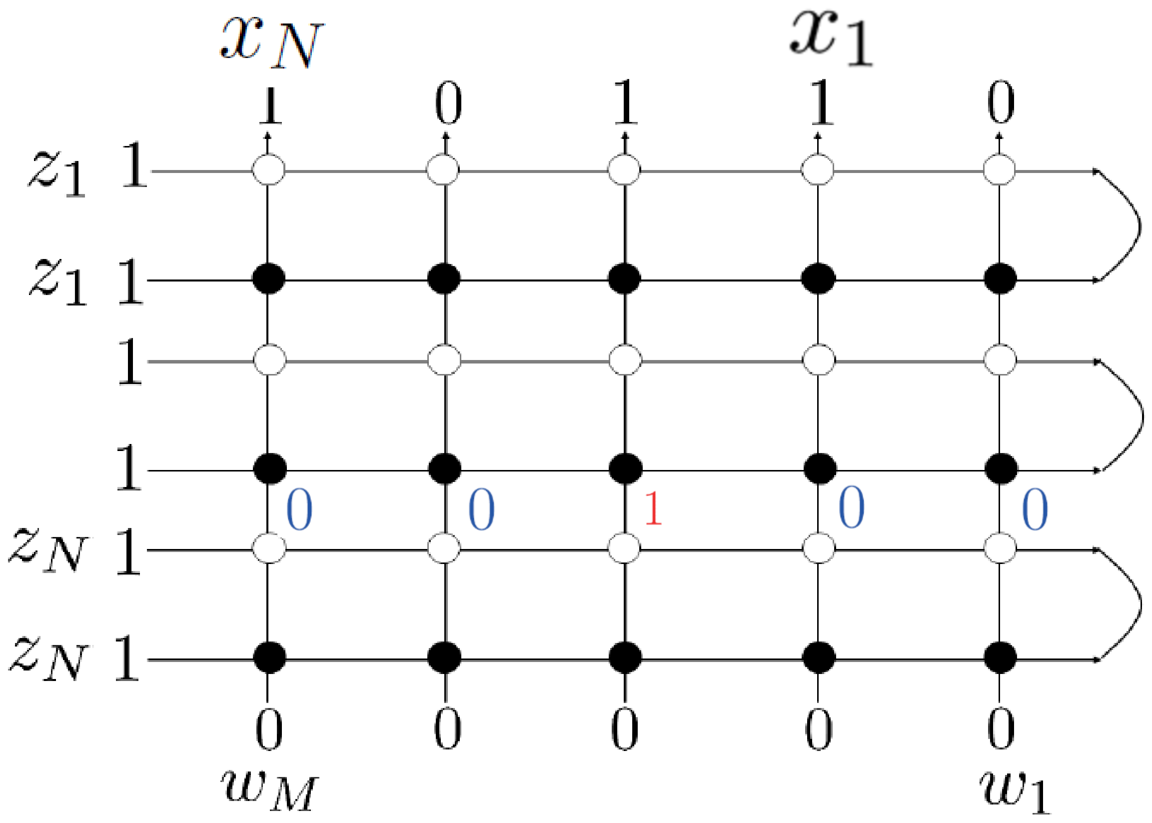
$$\frac{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{z_i\longleftrightarrow z_i^{-1}}}{W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)}=\frac{a^2z_i^{-2}-a^{-2}z_i^2}{a^2z_i^2-a^{-2}z_i^{-2}}.$$

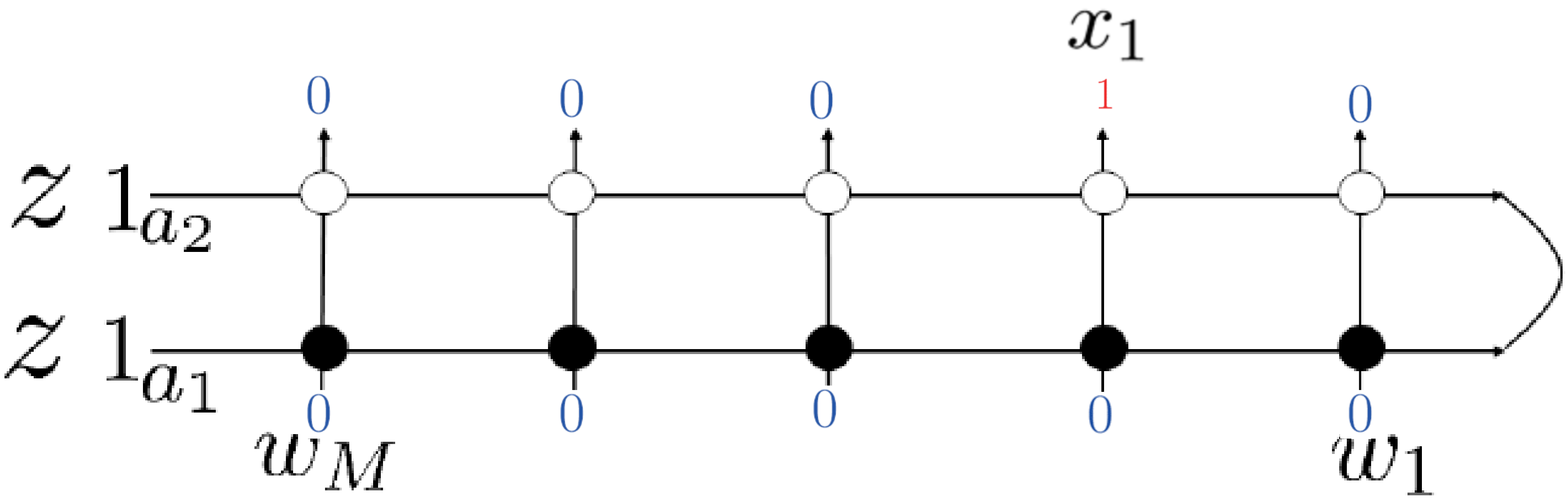
$$W_{M,N}(z_1,\ldots,z_N|w_1,\ldots,w_M|x_1,\ldots,x_N)$$



$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$

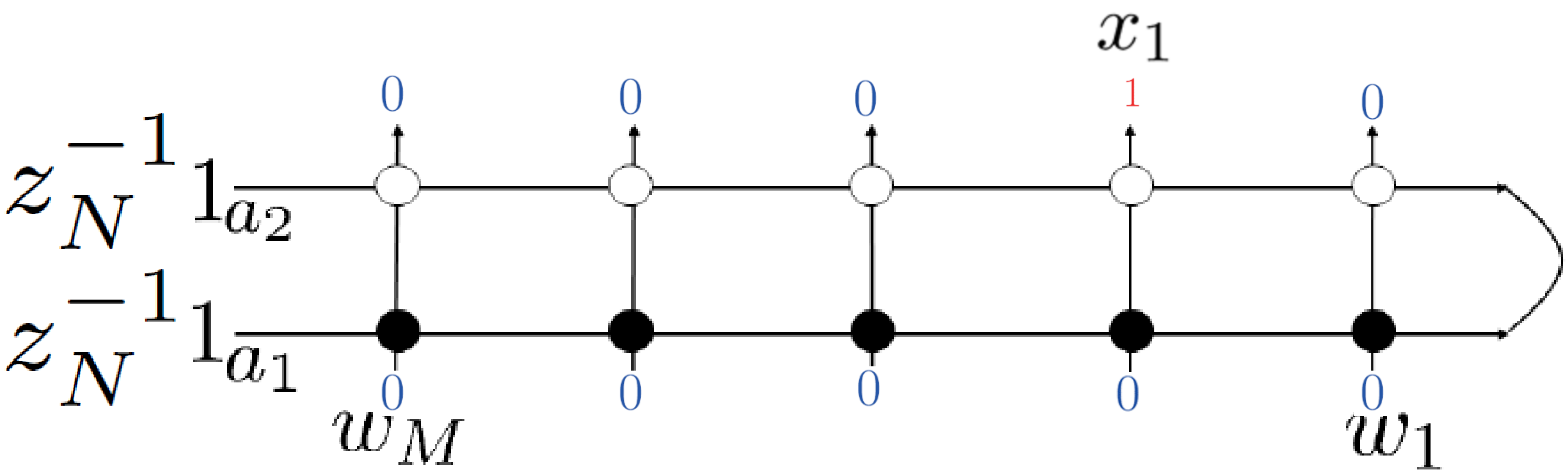
$$=\sum_x$$

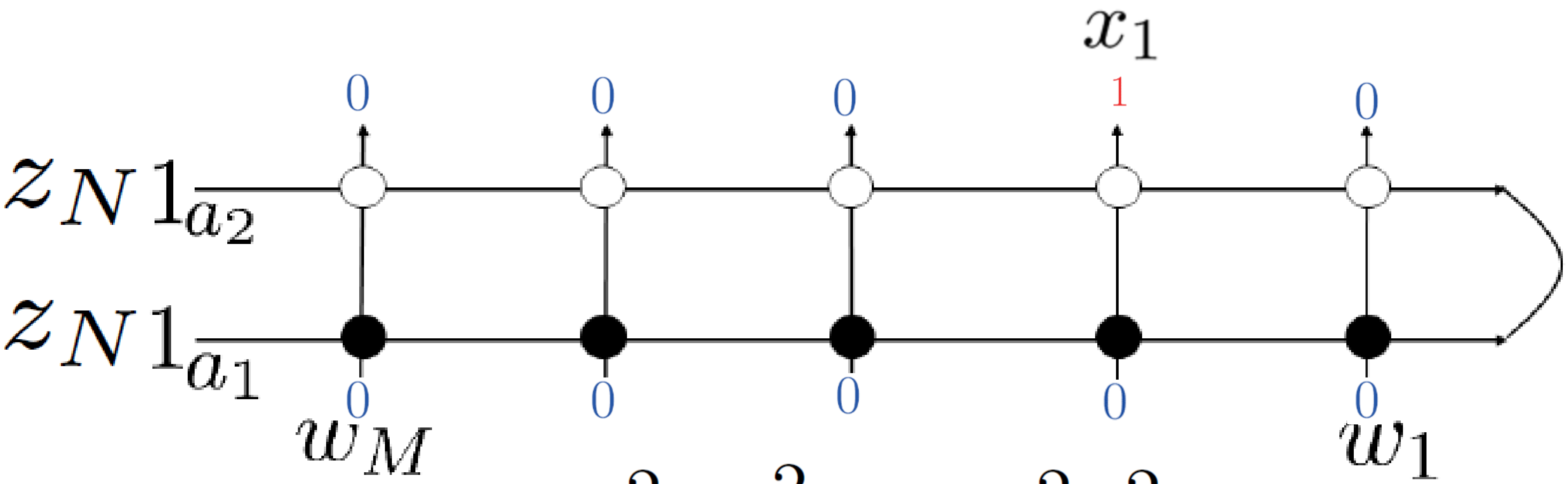




$$= (a^2 - a^{-2})(-a^2 z^2 + a^{-2} z^{-2}) \sum_{\tau=\pm1} \frac{1}{z^{2\tau} - z^{-2\tau}} (b a^{-1} z^{-\tau} - b^{-1} a z^{\tau})$$

$$\times \prod_{j=1}^{M} (a z^{-\tau} - a^{-1} z^{\tau} w_j) \prod_{j=1}^{x_1-1} (a z^{-\tau} w_j - a^{-1} z^{\tau}) \prod_{j=x_1+1}^{M} (a z^{\tau} - a^{-1} z^{-\tau} w_j)$$

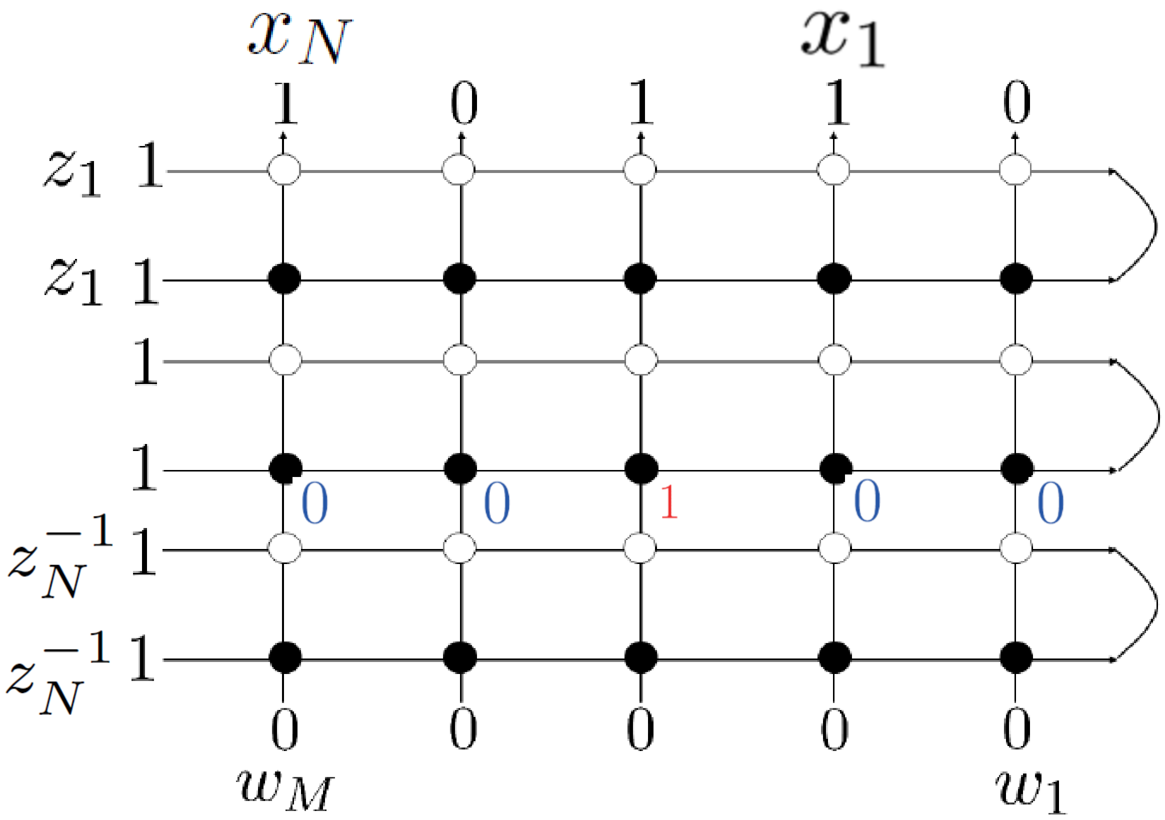




$$= \frac{a^2 z_N^{-2} - a^{-2} z_N^{2}}{a^2 z_N^{2} - a^{-2} z_N^{-2}}$$

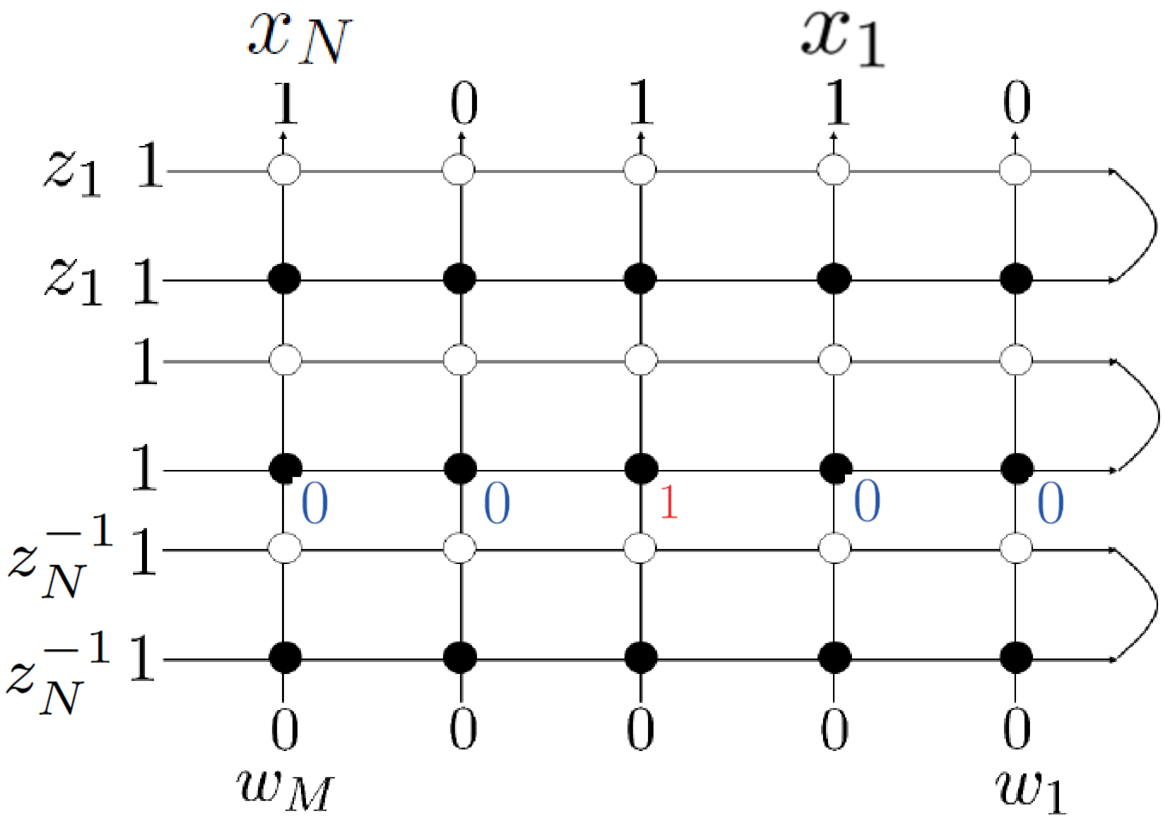
$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$

$$=\sum_x \frac{a^2 z_N^2 - a^{-2} z_N^{-2}}{a^2 z_N^{-2} - a^{-2} z_N^2}$$



$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$

$$=\sum_x \frac{a^2 z_N^2 - a^{-2} z_N^{-2}}{a^2 z_N^{-2} - a^{-2} z_N^2}$$



$$=\frac{a^2 z_N^2 - a^{-2} z_N^{-2}}{a^2 z_N^{-2} - a^{-2} z_N^2} W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{z_N \longleftrightarrow z_N^{-1}}$$

see also Tsuchiya, Kuperberg, Ivanov

Korepin's lemma

(4) The following recursive relations between the wavefunctions hold if $x_N = M$

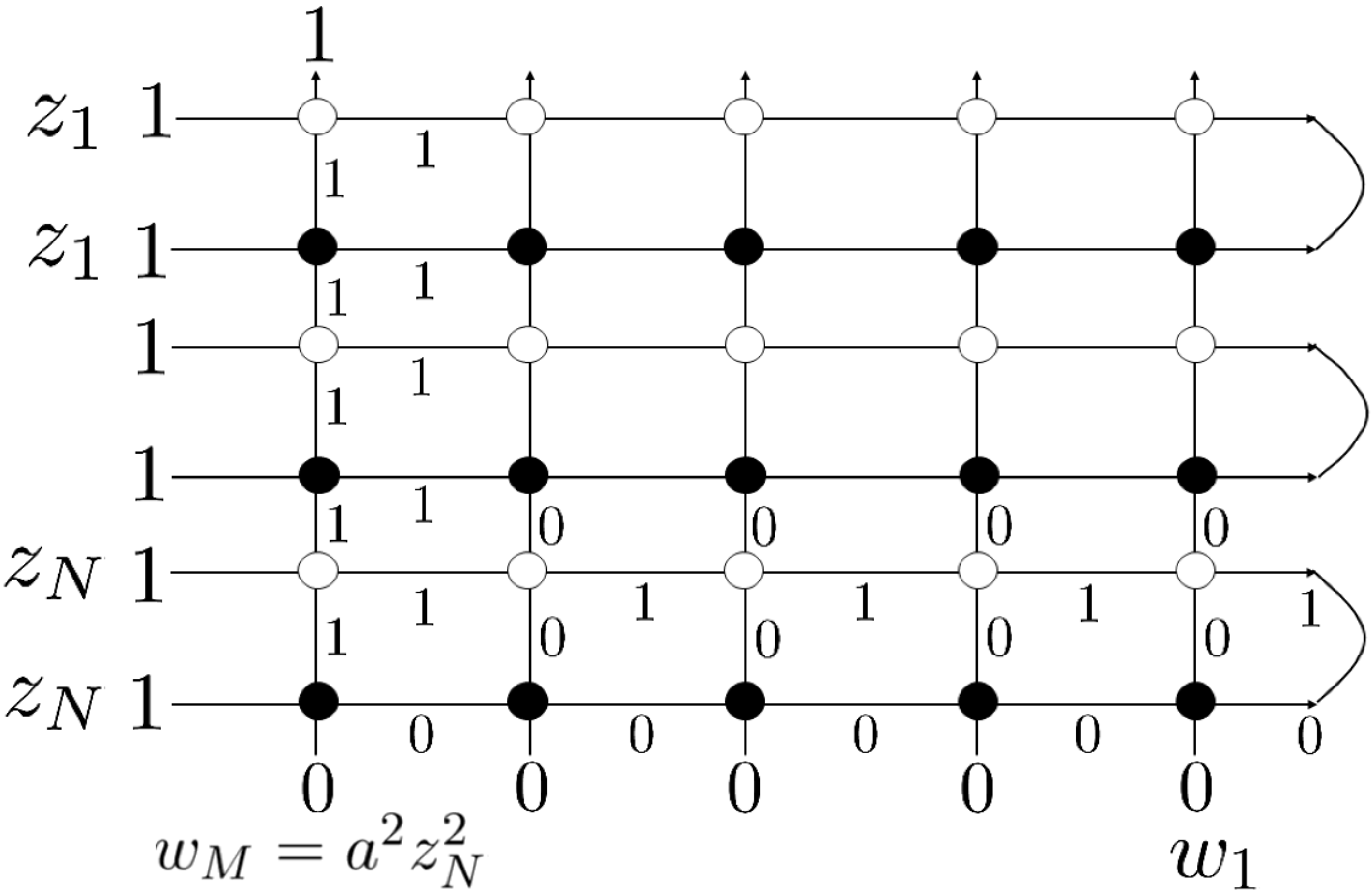
$$
\begin{aligned}
&W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{w_M=a^2 z_N^2} \\
=&(a^2-a^{-2})(ba^{-1}z_N^{-1}-b^{-1}az_N)\prod_{j=1}^{N}(a^3 z_N^2 z_j - a^{-1} z_j^{-1}) \prod_{j=1}^{N-1}(a^3 z_N^2 z_j^{-1} - a^{-1} z_j) \\
&\times \prod_{j=1}^{M-1}(a z_N^{-1} w_j - a^{-1} z_N)(a z_N^{-1} - a^{-1} z_N w_j) \\
&\times W_{M-1,N-1}(z_1,\dots,z_{N-1}|w_1,\dots,w_{M-1}|x_1,\dots,x_{N-1}).
\end{aligned}
$$

If $x_N \neq M$*, the following factorizations hold for the wavefunctions*

$$
\begin{aligned}
&W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N) \\
=&\prod_{j=1}^{N}(a z_j^{-1} - a^{-1} z_j w_M)(a z_j - a^{-1} z_j^{-1} w_M) W_{M-1,N}(z_1,\dots,z_N|w_1,\dots,w_{M-1}|x_1,\dots,x_N).
\end{aligned}
$$

(5) The following holds for the case $N=1$, $x_1 = M$

$$
\begin{aligned}
W_{M,1}(z|w_1,\dots,w_M|M) =& (a^2-a^{-2})(-a^2 z^2 + a^{-2} z^{-2}) \sum_{\tau=\pm 1} \frac{1}{z^{2\tau}-z^{-2\tau}} (ba^{-1} z^{-\tau} - b^{-1} a z^{\tau}) \\
&\times \prod_{j=1}^{M}(a z^{-\tau} - a^{-1} z^{\tau} w_j) \prod_{j=1}^{M-1}(a z^{-\tau} w_j - a^{-1} z^{\tau}).
\end{aligned}
$$


1
z_1 1
z_1 1
1
1
z_N 1
z_N 1
1
1
1
1
1
1
1
1
1 1 1 1 1
0 0 0 0
0 0 0 0
0 0 0 0 0
0 0 0 0 0
$w_M = a^2 z_N^2$
w_1

Korepin's lemma

(4) The following recursive relations between the wavefunctions hold if $x_N = M$

$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)|_{w_M=a^2z_N^2}$$

$$=(a^2-a^{-2})(ba^{-1}z_N^{-1}-b^{-1}az_N)\prod_{j=1}^{N}(a^3z_N^2z_j-a^{-1}z_j^{-1})\prod_{j=1}^{N-1}(a^3z_N^2z_j^{-1}-a^{-1}z_j)$$

$$\times\prod_{j=1}^{M-1}(az_N^{-1}w_j-a^{-1}z_N)(az_N^{-1}-a^{-1}z_Nw_j)$$

$$\times W_{M-1,N-1}(z_1,\dots,z_{N-1}|w_1,\dots,w_{M-1}|x_1,\dots,x_{N-1}).$$

If $x_N \neq M$*, the following factorizations hold for the wavefunctions*

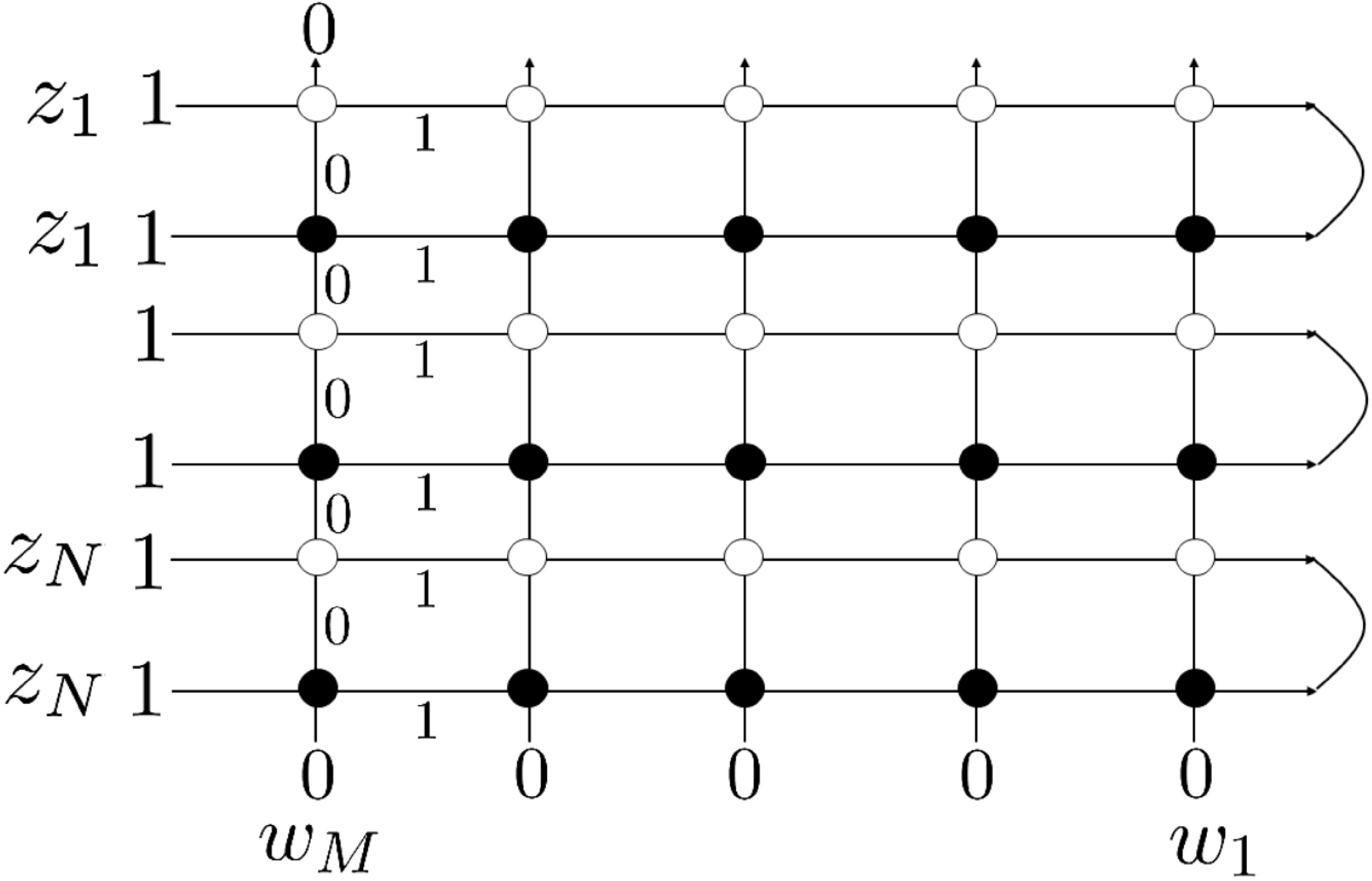
$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$

$$=\prod_{j=1}^{N}(az_j^{-1}-a^{-1}z_jw_M)(az_j-a^{-1}z_j^{-1}w_M)W_{M-1,N}(z_1,\dots,z_N|w_1,\dots,w_{M-1}|x_1,\dots,x_N).$$

(5) The following holds for the case $N=1$, $x_1=M$

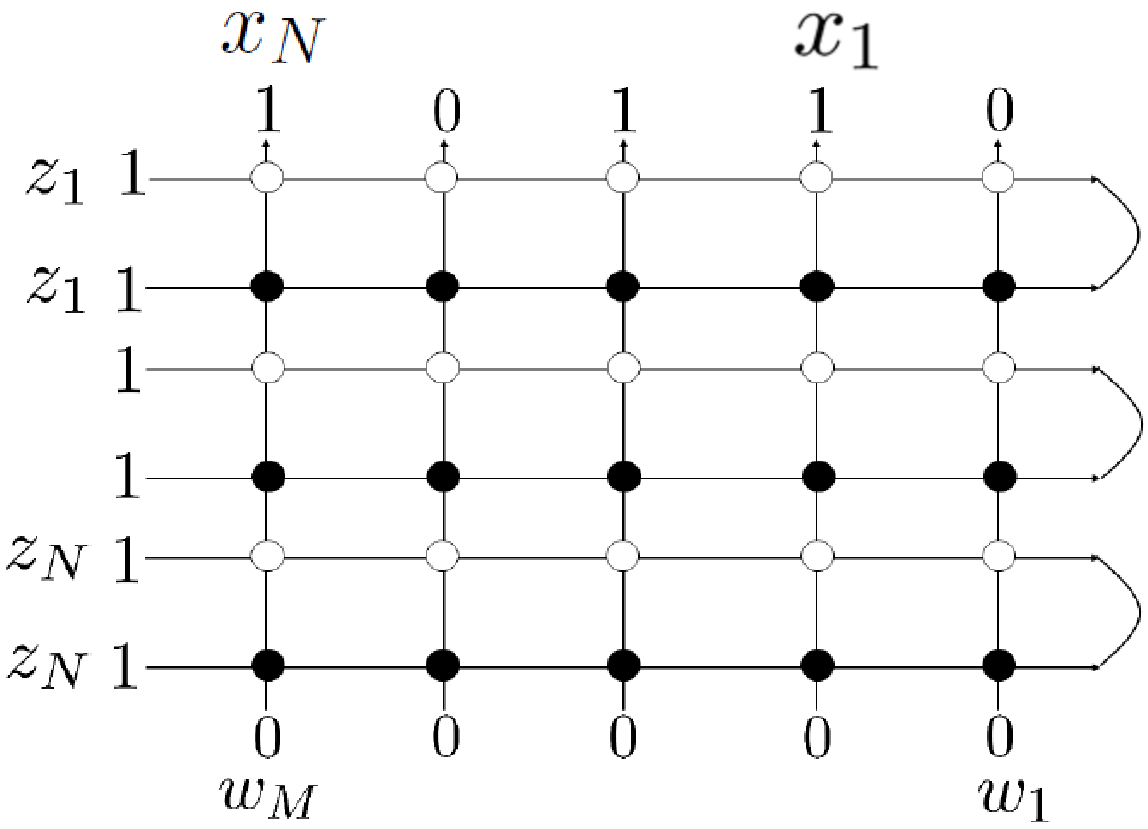
$$W_{M,1}(z|w_1,\dots,w_M|M)=(a^2-a^{-2})(-a^2z^2+a^{-2}z^{-2})\sum_{\tau=\pm1}\frac{1}{z^{2\tau}-z^{-2\tau}}(ba^{-1}z^{-\tau}-b^{-1}az^{\tau})$$

$$\times\prod_{j=1}^{M}(az^{-\tau}-a^{-1}z^{\tau}w_j)\prod_{j=1}^{M-1}(az^{-\tau}w_j-a^{-1}z^{\tau}).$$

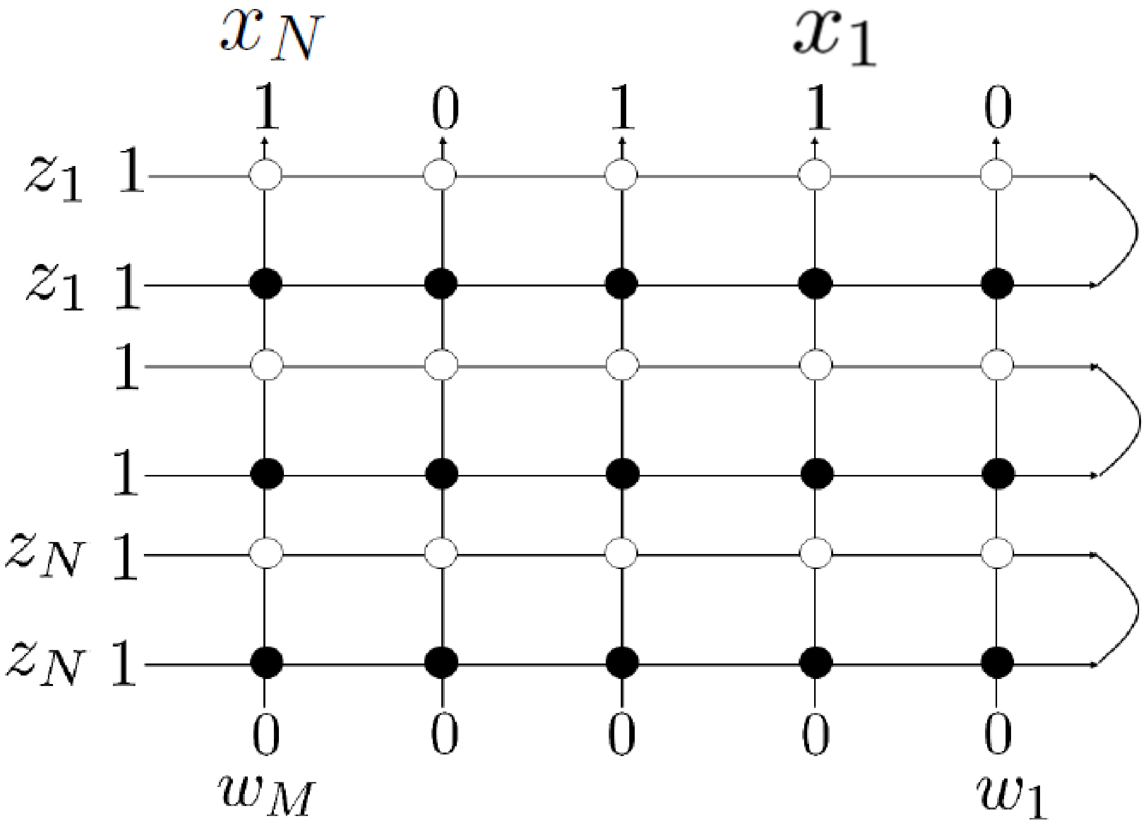

0
z_1 1
1
0
z_1 1
1
0
1
1
0
1
1
0
z_N 1
1
0
z_N 1
1
0
0
0
0
0
w_M
w_1

$$
\begin{aligned}
&W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)\\
=&(a^2-a^{-2})^N\prod_{j=1}^{N}(-a^2z_j^2+a^{-2}z_j^{-2})\sum_{\sigma\in S_N}\sum_{\tau_1=\pm1,\dots,\tau_N=\pm1}\prod_{j=1}^{N}\frac{1}{z_j^{2\tau_j}-z_j^{-2\tau_j}}\\
&\times\prod_{1\le j<k\le N}\frac{(a^2z_{\sigma(j)}^{\tau_{\sigma(j)}}z_{\sigma(k)}^{\tau_{\sigma(k)}}-a^{-2}z_{\sigma(j)}^{-\tau_{\sigma(j)}}z_{\sigma(k)}^{-\tau_{\sigma(k)}})(a^2z_{\sigma(j)}^{-\tau_{\sigma(j)}}z_{\sigma(k)}^{\tau_{\sigma(k)}}-a^{-2}z_{\sigma(j)}^{\tau_{\sigma(j)}}z_{\sigma(k)}^{-\tau_{\sigma(k)}})}{(z_{\sigma(j)}^{\tau_{\sigma(j)}}z_{\sigma(k)}^{\tau_{\sigma(k)}}-z_{\sigma(j)}^{-\tau_{\sigma(j)}}z_{\sigma(k)}^{-\tau_{\sigma(k)}})(z_{\sigma(j)}^{-\tau_{\sigma(j)}}z_{\sigma(k)}^{\tau_{\sigma(k)}}-z_{\sigma(j)}^{\tau_{\sigma(j)}}z_{\sigma(k)}^{-\tau_{\sigma(k)}})}\\
&\times\prod_{k=1}^{N}(ba^{-1}z_{\sigma(k)}^{-\tau_{\sigma(k)}}-b^{-1}az_{\sigma(k)}^{\tau_{\sigma(k)}})\prod_{k=1}^{N}\prod_{j=1}^{M}(az_{\sigma(k)}^{-\tau_{\sigma(k)}}-a^{-1}z_{\sigma(k)}^{\tau_{\sigma(k)}}w_j)\\
&\times\prod_{k=1}^{N}\prod_{j=1}^{x_k-1}(az_{\sigma(k)}^{-\tau_{\sigma(k)}}w_j-a^{-1}z_{\sigma(k)}^{\tau_{\sigma(k)}})\prod_{k=1}^{N}\prod_{j=x_k+1}^{M}(az_{\sigma(k)}^{\tau_{\sigma(k)}}-a^{-1}z_{\sigma(k)}^{-\tau_{\sigma(k)}}w_j).
\end{aligned}
$$

$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$



$$W_{M,N}(z_1,\dots,z_N|w_1,\dots,w_M|x_1,\dots,x_N)$$



homogeneous limit $w_1 = \cdots = w_M = 1$

off-shell Bethe wavefunctions for open XXZ chain

coordinate Bethe ansatz wavefunctions

for open XXZ chain

Alcaraz-Barber-Batchelor-Baxter-Quispel

$$\sum_P \epsilon_P A(K'_1, \ldots, K'_N) \mathrm{e}^{\sum_{k=1}^N iK'_k x'_k}$$

$$\begin{aligned} A(K'_1, \ldots, K'_N) =& \prod_{j=1}^N \beta(-K'_j) \prod_{1\leq j<k\leq N} B(-K'_j, K'_k)\mathrm{e}^{-iK'_k}, \\ \beta(K') =& (1+(p'-\Delta)\mathrm{e}^{-iK'})\mathrm{e}^{i(M+1)K'}, \\ B(-K'_j, K'_k) =& s(-K'_j, K'_k)s(K'_k, K'_j), \\ s(K'_1, K'_2) =& 1-2\Delta\mathrm{e}^{iK'_2}+\mathrm{e}^{i(K'_1+K'_2)}. \end{aligned}$$

Sklyanin

Algebraic Bethe ansatz for reflecting boundary conditions

$$W_{M,N}(z_1,\dots,z_N|1,\dots,1|x_1,\dots,x_N)$$

$$=(b-b^{-1})^N \prod_{1\le j<k\le N} \frac{(a^2+a^{-2}-z_j^2-z_j^{-2})(a^2+a^{-2}-z_k^{-2}-z_k^{-2})}{(a^2-a^{-2})^2(z_k^2+z_k^{-2}-z_j^2-z_j^{-2})}$$

$$\times \prod_{j=1}^{N} \frac{(-a^2 z_j^2+a^{-2}z_j^{-2})(a^2+a^{-2}-z_j^2-z_j^{-2})^M}{z_j^2-z_j^{-2}} f_{M,N}(K_1,\dots,K_N|x_1,\dots,x_N)$$

$$f_{M,N}(K_1,\dots,K_N|x_1,\dots,x_N) = \sum_{\sigma\in S_N} \sum_{\tau_1=\pm 1,\dots,\tau_N=\pm 1} \mathrm{sgn}(\sigma)(-1)^{|\tau|}$$

$$\times \prod_{1\le j<k\le N} (1-2\Delta \mathrm{e}^{i\tau_{\sigma(j)}K_{\sigma(j)}} + \mathrm{e}^{i\tau_{\sigma(j)}K_{\sigma(j)}-i\tau_{\sigma(k)}K_{\sigma(k)}})$$

$$\times (1-2\Delta \mathrm{e}^{i\tau_{\sigma(k)}K_{\sigma(k)}} + \mathrm{e}^{i\tau_{\sigma(j)}K_{\sigma(j)}+i\tau_{\sigma(k)}K_{\sigma(k)}})\mathrm{e}^{-i\tau_{\sigma(j)}K_{\sigma(j)}}$$

$$\times \prod_{k=1}^{N} \mathrm{e}^{-i\tau_{\sigma(k)}K_{\sigma(k)}x_k}(1+(p'-\Delta)\mathrm{e}^{i\tau_{\sigma(k)}K_{\sigma(k)}}).$$

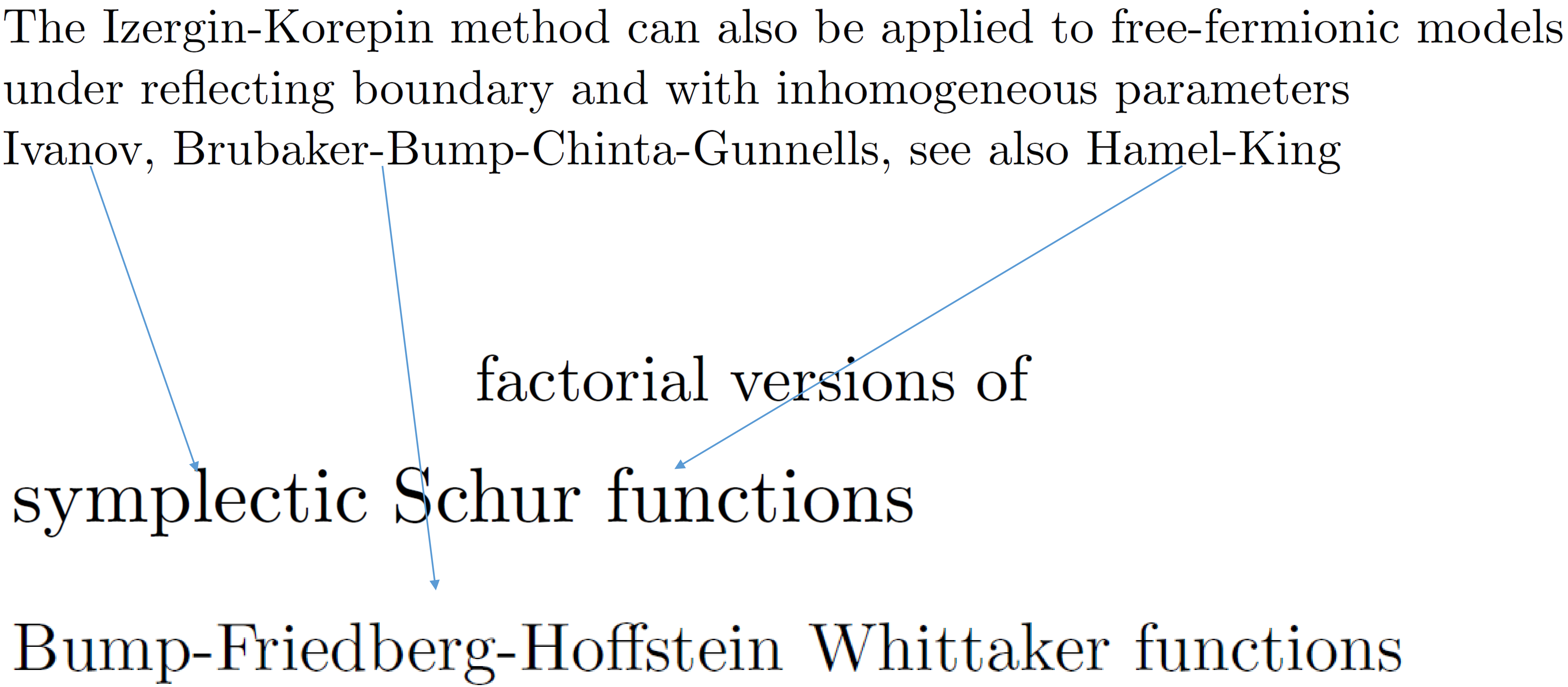
$$\mathrm{e}^{iK_\ell} = \frac{az_\ell - a^{-1}z_\ell^{-1}}{az_\ell^{-1}-a^{-1}z_\ell}, \ell = 1,\dots,N, \Delta = -\frac{a^2+a^{-2}}{2},\ p' = -\frac{b+b^{-1}}{2(b-b^{-1})}(a^2-a^{-2})$$

$$
\begin{aligned}
&f_{M,N}(K_1,\dots,K_N|x_1,\dots,x_N)\\
&=\sum_P \epsilon_P \prod_{1\le j<k\le N}(1-2\Delta \mathrm{e}^{iK_j}+\mathrm{e}^{iK_j-iK_k})(1-2\Delta \mathrm{e}^{iK_k}+\mathrm{e}^{iK_j+iK_k})\mathrm{e}^{-iK_j}\\
&\quad\times\prod_{k=1}^{N}\mathrm{e}^{-iK_kx_k}(1+(p'-\Delta)\mathrm{e}^{iK_k}),
\end{aligned}
$$

where the sum means that we take sum over all permutations and negations of $K_1,\dots,K_N$, and ϵ_P changes sign at each such "mutation". Relabelling the momentums and positions of down spins as $K'_j=K_{N+1-j}$, $x'_j=M+1-x_{N+1-j}$ $(j=1,\dots,N)$

$$
\begin{aligned}
&=\sum_P \epsilon_P \prod_{k=1}^{N}(1+(p'-\Delta)\mathrm{e}^{iK'_k})\mathrm{e}^{-i(M+1)K'_k}\\
&\quad\times\prod_{1\le j<k\le N}(1-2\Delta \mathrm{e}^{iK'_k}+\mathrm{e}^{iK'_k-iK'_j})(1-2\Delta \mathrm{e}^{iK'_j}+\mathrm{e}^{iK'_j+iK'_k})\mathrm{e}^{-iK'_k}\prod_{k=1}^{N}\mathrm{e}^{iK'_kx'_k}\\
&=\sum_P \epsilon_P A(K'_1,\dots,K'_N)\mathrm{e}^{\sum_{k=1}^{N}iK'_kx'_k},
\end{aligned}
$$

see Annales de l'Institut Henri Poincaré D, 7 (2020), 165 for details



The Izergin-Korepin method can also be applied to free-fermionic models under reflecting boundary and with inhomogeneous parameters

Ivanov, Brubaker-Bump-Chinta-Gunnells, see also Hamel-King

factorial versions of

symplectic Schur functions

Bump-Friedberg-Hoffstein Whittaker functions

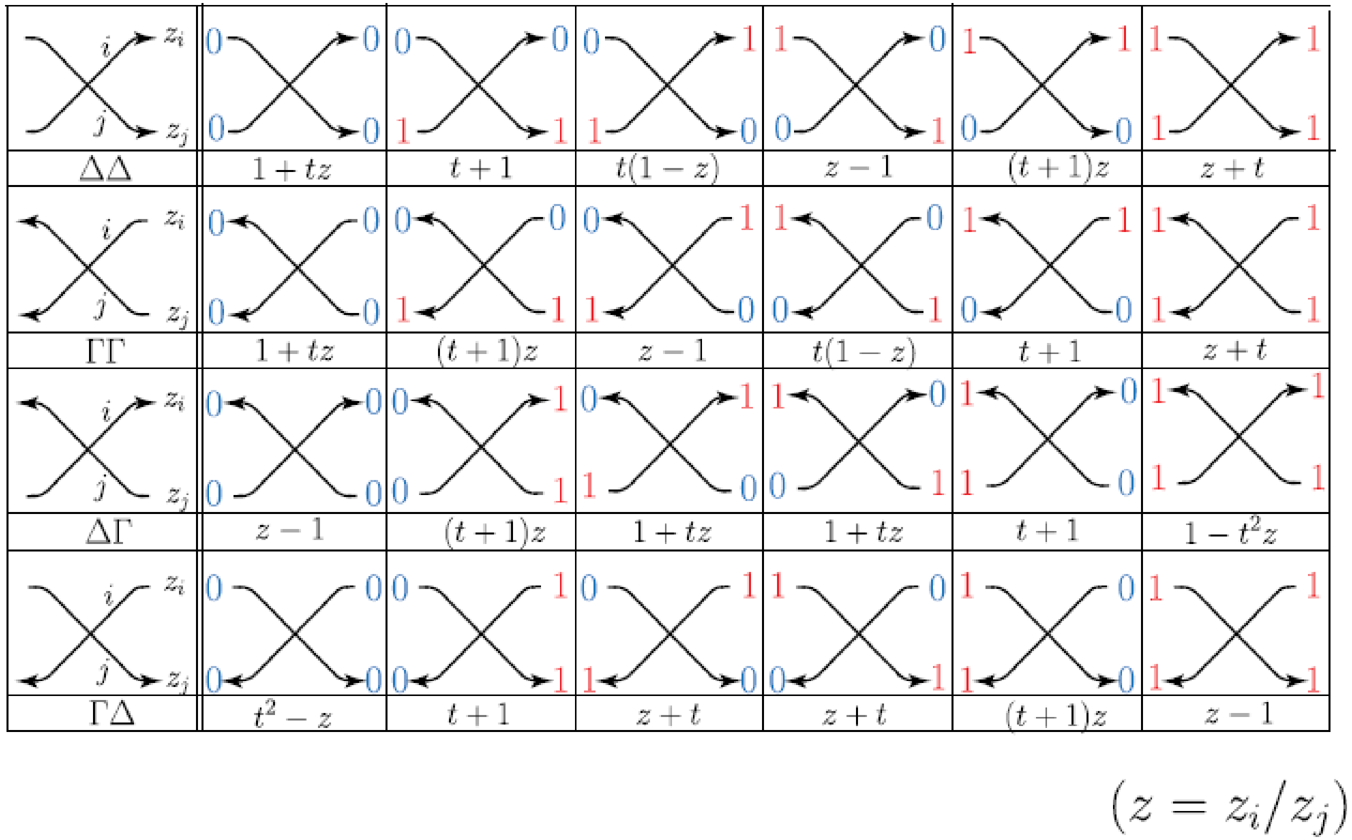


Fig. 3. *R*-matrices.

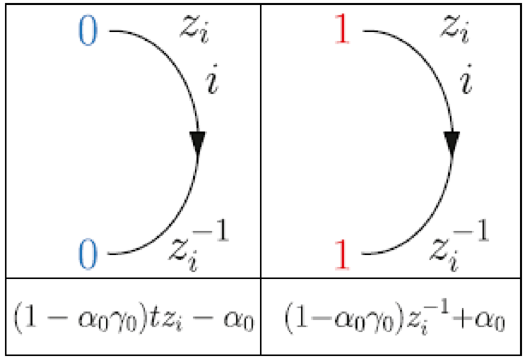


Fig. 4. The type I *K*-matrix (3.3).

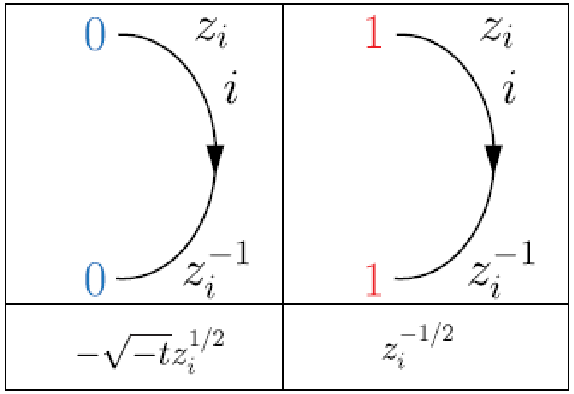
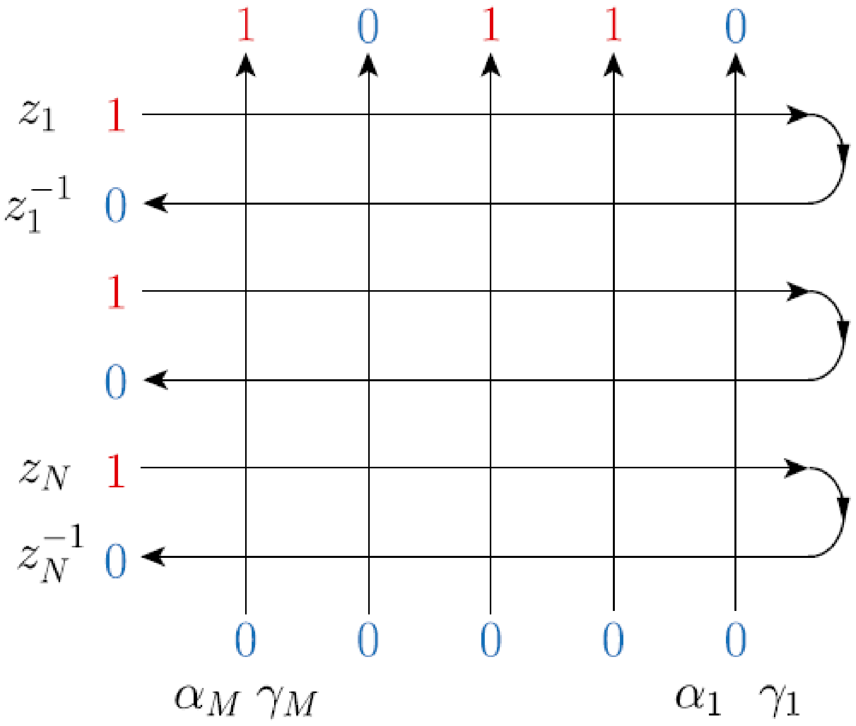


Fig. 5. The type II *K*-matrix (3.5).

factorial versions of symplectic Schur functions



$$= \prod_{j=1}^{N} z_j^{j-1-N}(1+tz_j^2) \prod_{1\le j<k\le N} (1+tz_jz_k)(1+tz_jz_k^{-1}) sp_\lambda(\{z\}_N|\{\overline{\alpha}\}|\{\overline{\gamma}\})$$

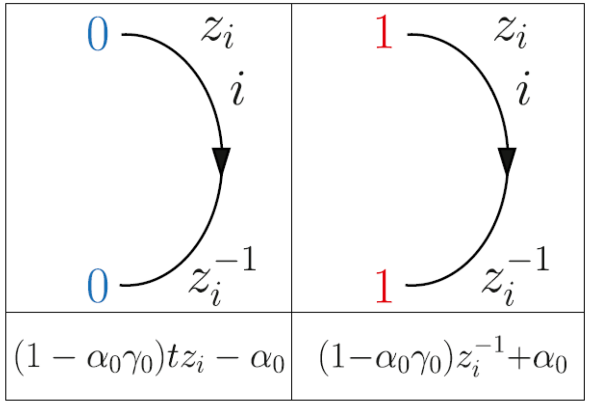


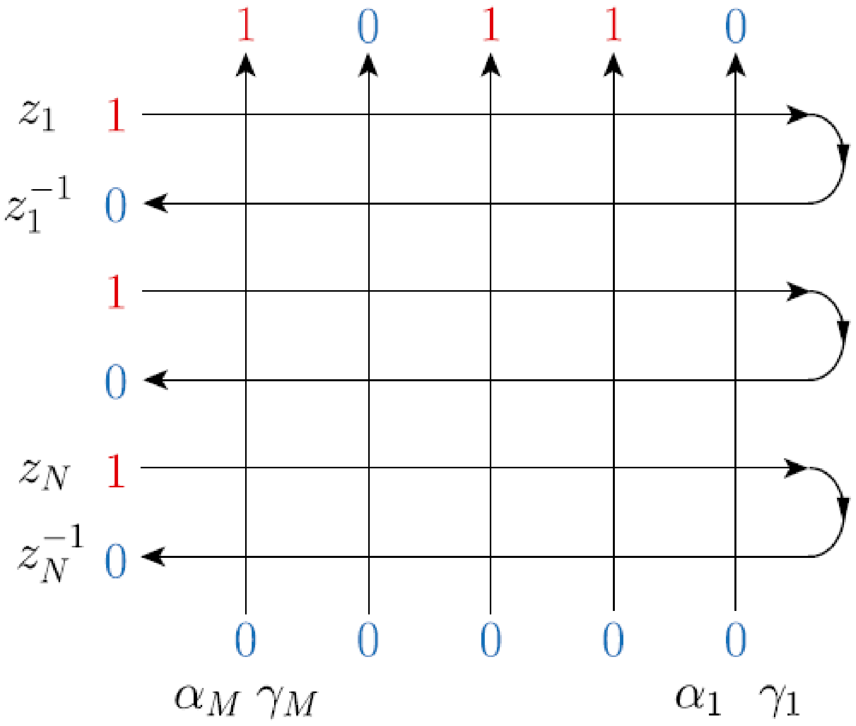
Fig. 4. The type I K-matrix (3.3).

$$sp_\lambda(\{z\}_N|\{\overline{\alpha}\}|\{\overline{\gamma}\}) = \frac{G_{\lambda+\delta}(\{z\}_N|\{\overline{\alpha}\}|\{\overline{\gamma}\})}{\det_N(z_k^{N-j+1} - z_k^{-N+j-1})}$$

$$G_\mu(\{z\}_N|\{\overline{\alpha}\}|\{\overline{\gamma}\}) = \det_N(g_{\mu_j}(z_k|\{\overline{\alpha}\}|\{\overline{\gamma}\}) - g_{\mu_j}(z_k^{-1}|\{\overline{\alpha}\}|\{\overline{\gamma}\}))$$

$$g_\mu(z|\{\overline{\alpha}\}|\{\overline{\gamma}\}) = \prod_{j=0}^{\mu}\{\alpha_j + (1-\alpha_j\gamma_j)z\} \prod_{j=\mu+2}^{M}(1-\gamma_j z)\prod_{j=1}^{M}(1-\gamma_j z^{-1})$$

factorial versions of Bump-Friedberg-Hoffstein Whittaker functions



$$= \prod_{j=1}^{N} z_j^{j-1/2-N}(1-\sqrt{-t}z_j) \prod_{1\le j<k\le N}(1+tz_jz_k)(1+tz_jz_k^{-1})o_\lambda^+(\{z\}_N|\{\alpha\}|\{\gamma\}|t)$$

$$o_\lambda^\pm(\{z\}_N|\{\alpha\}|\{\gamma\}|t) = \frac{H^\pm_{\lambda+\delta}(\{z\}_N|\{\alpha\}|\{\gamma\}|t)}{\det_N(z_k^{N-j+1}-z_k^{-N+j-1})}$$

$$H_\mu^\pm(\{z\}_N|\{\alpha\}|\{\gamma\}|t) = \det_N(h^\pm_{\mu_j}(z_k|\{\alpha\}|\{\gamma\}|t) - h^\pm_{\mu_j}(z_k^{-1}|\{\alpha\}|\{\gamma\}|t))$$

$$h_\mu^\pm(z|\{\alpha\}|\{\gamma\}|t) = (z\pm\sqrt{-t})\prod_{j=1}^{\mu}(\alpha_j+(1-\alpha_j\gamma_j)z)\prod_{j=\mu+2}^{M}(1-\gamma_jz)\prod_{j=1}^{M}(1-\gamma_jz^{-1})$$

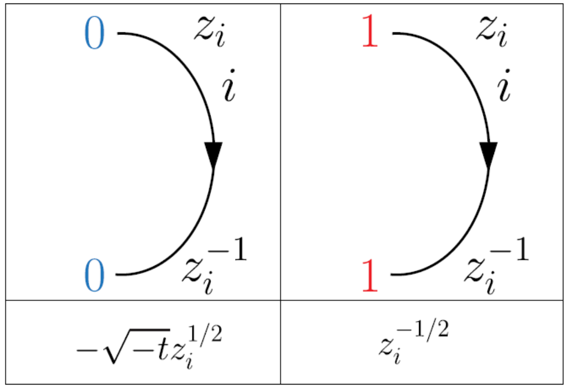


Fig. 5. The type II *K*-matrix (3.5).

evaluate the domain wall boundary partition functions in two ways

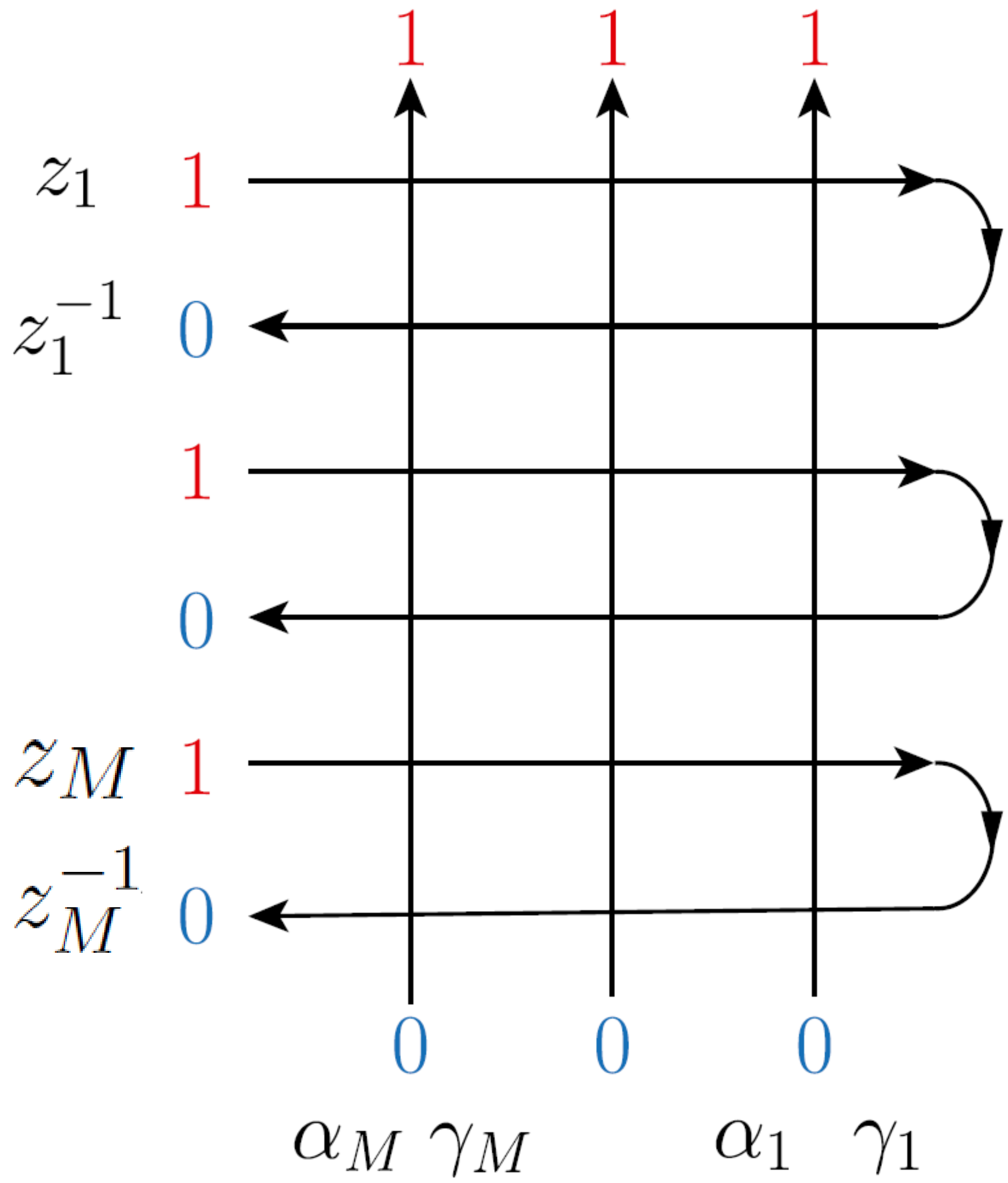
- factorized expression
- cut the domain wall boundary partition functions in the middle

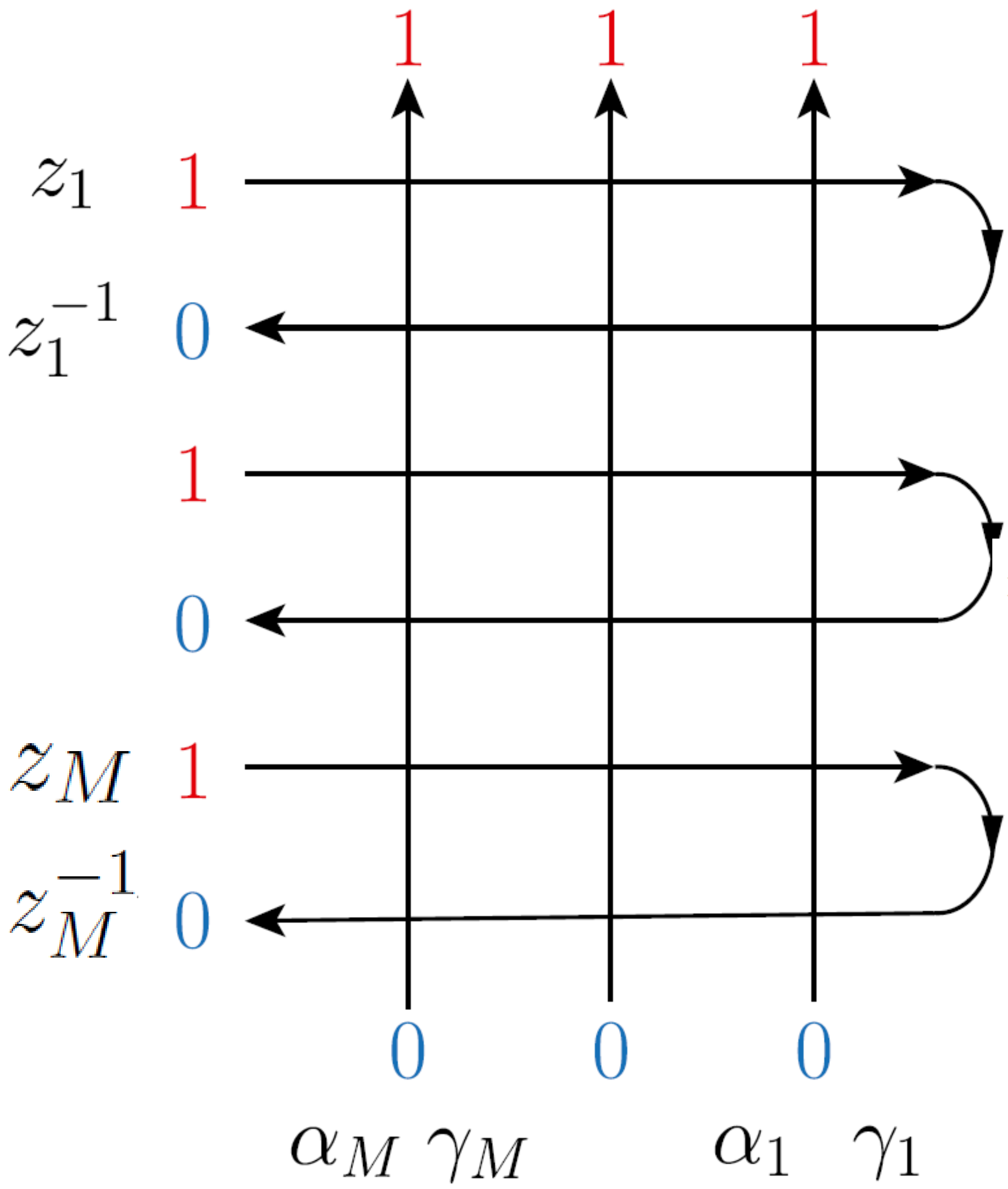
→ dual Cauchy formulas

idea presented in Brubaker's talk

Brubaker-Bump-Friedberg, Bump-McNamara-Nakasuji

view as functions in t


1
1
1
z_1
1
z_1^{-1}
0
1
0
z_M
1
z_M^{-1}
0
0
0
0
α_M γ_M
α_1 γ_1



$$= \prod_{j=1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j)(1+\sqrt{-t}\gamma_j)$$

$$\times \prod_{1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1}) \prod_{1\le j<k\le M}\{1+\alpha_j(\gamma_k-\gamma_j)\} \prod_{1\le j<k\le M}(1-\gamma_j\gamma_k)$$

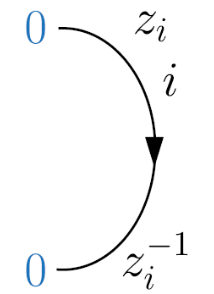
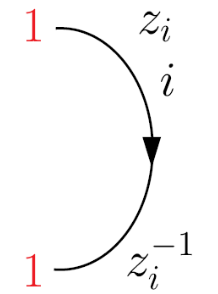
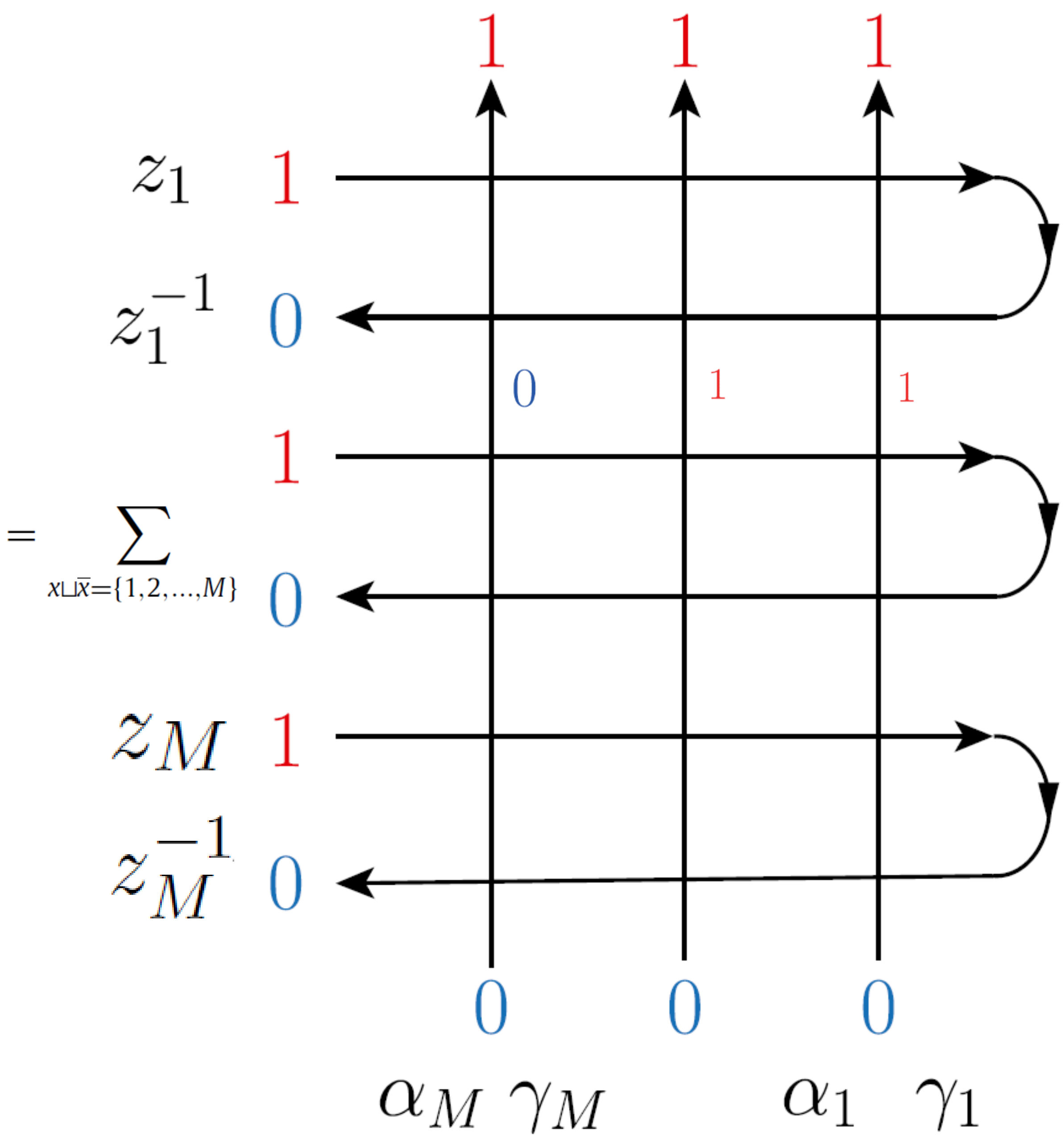
	
$-\sqrt{-t}z_i^{1/2}$	$z_i^{-1/2}$

Fig. 5. The type II K-matrix (3.5).



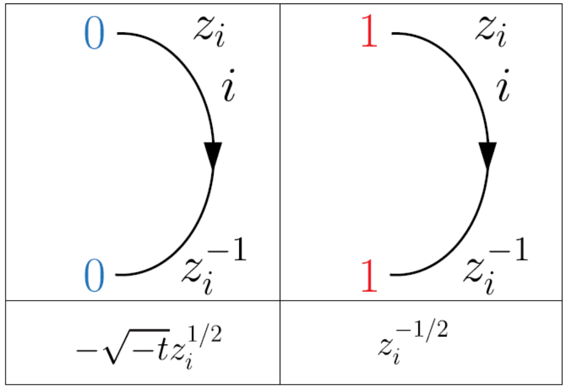
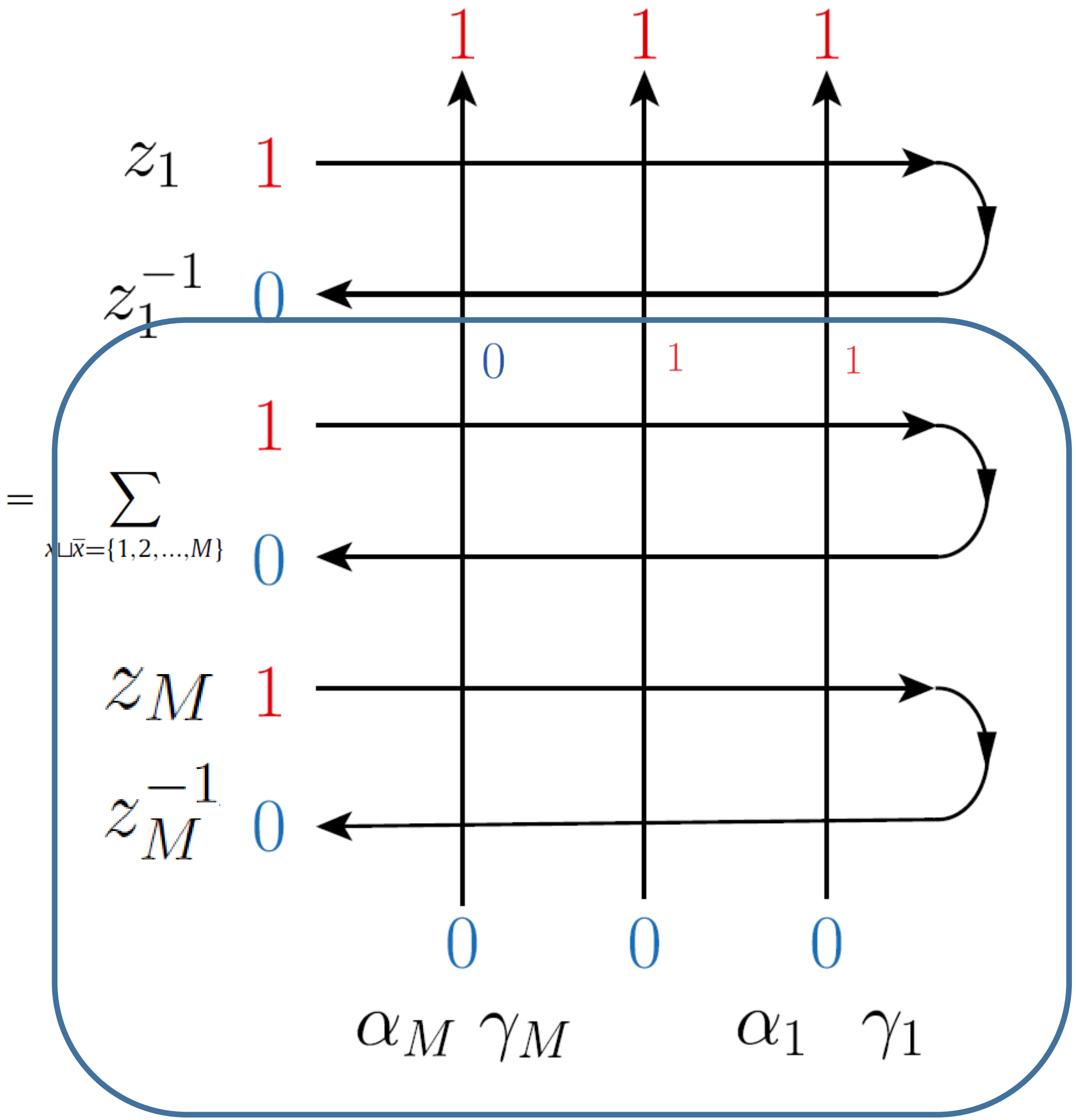


Fig. 5. The type II K-matrix (3.5).



$$\prod_{j=M-N+1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j)\prod_{M-N+1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1})$$

$$\times\, o_\lambda^{+}(z_{M-N+1},\dots,z_M|\{\alpha\}|\{\gamma\}|t).$$

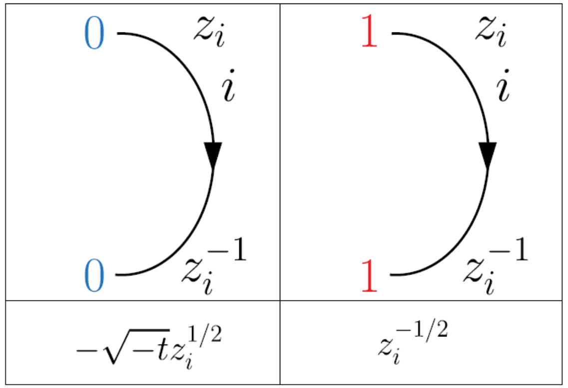
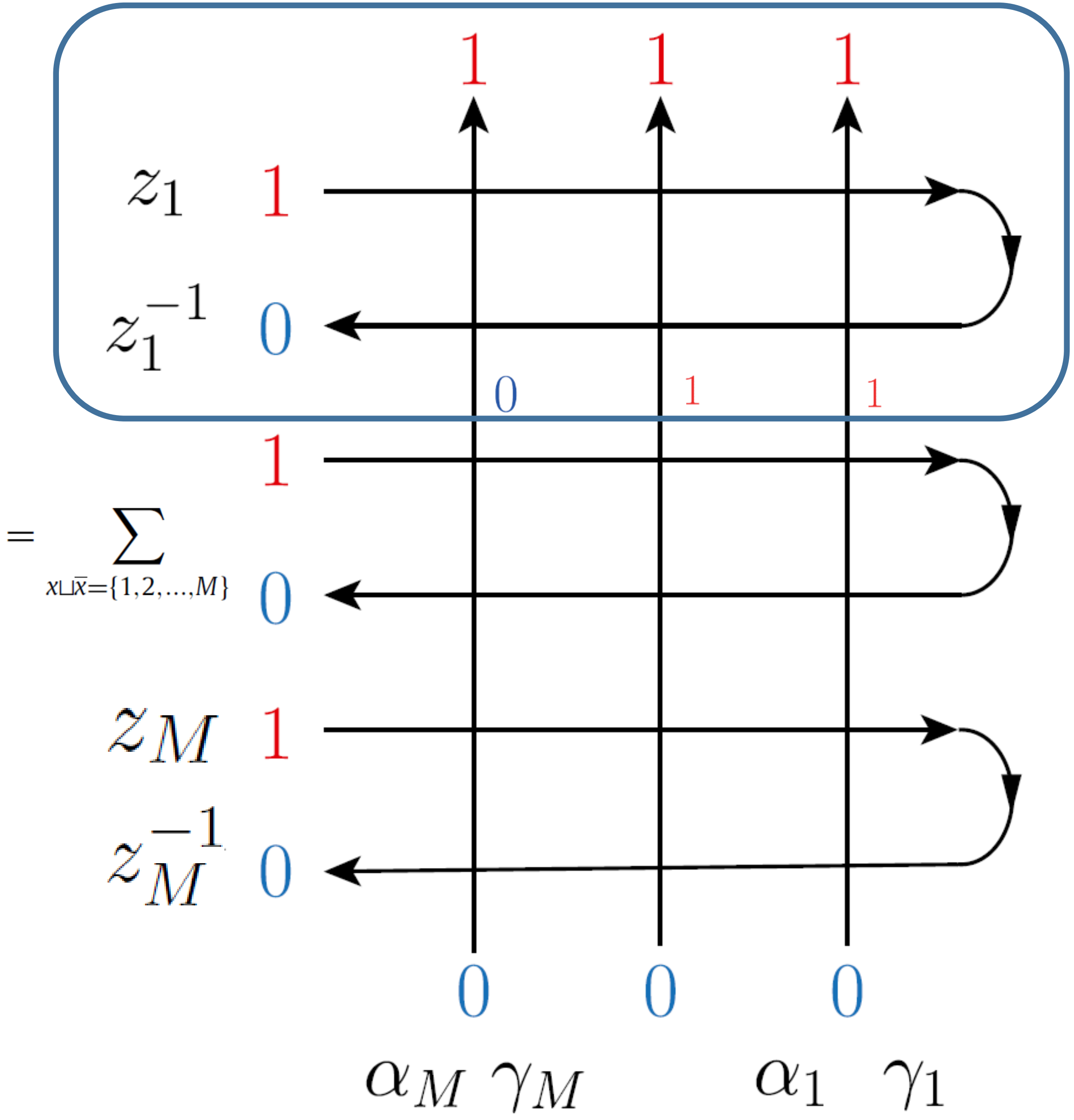


Fig. 5. The type II K-matrix (3.5).



$$t^{N(M-N)}\prod_{j=1}^{M-N} z_j^{j-1/2-M+N}(1-\sqrt{-t}z_j)\prod_{1\le j<k\le M-N}(1+tz_jz_k)(1+tz_jz_k^{-1})$$

$$\times\prod_{j=M-N+1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j)\prod_{M-N+1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1})$$

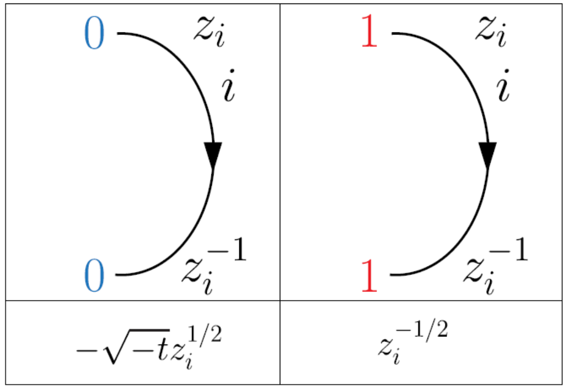
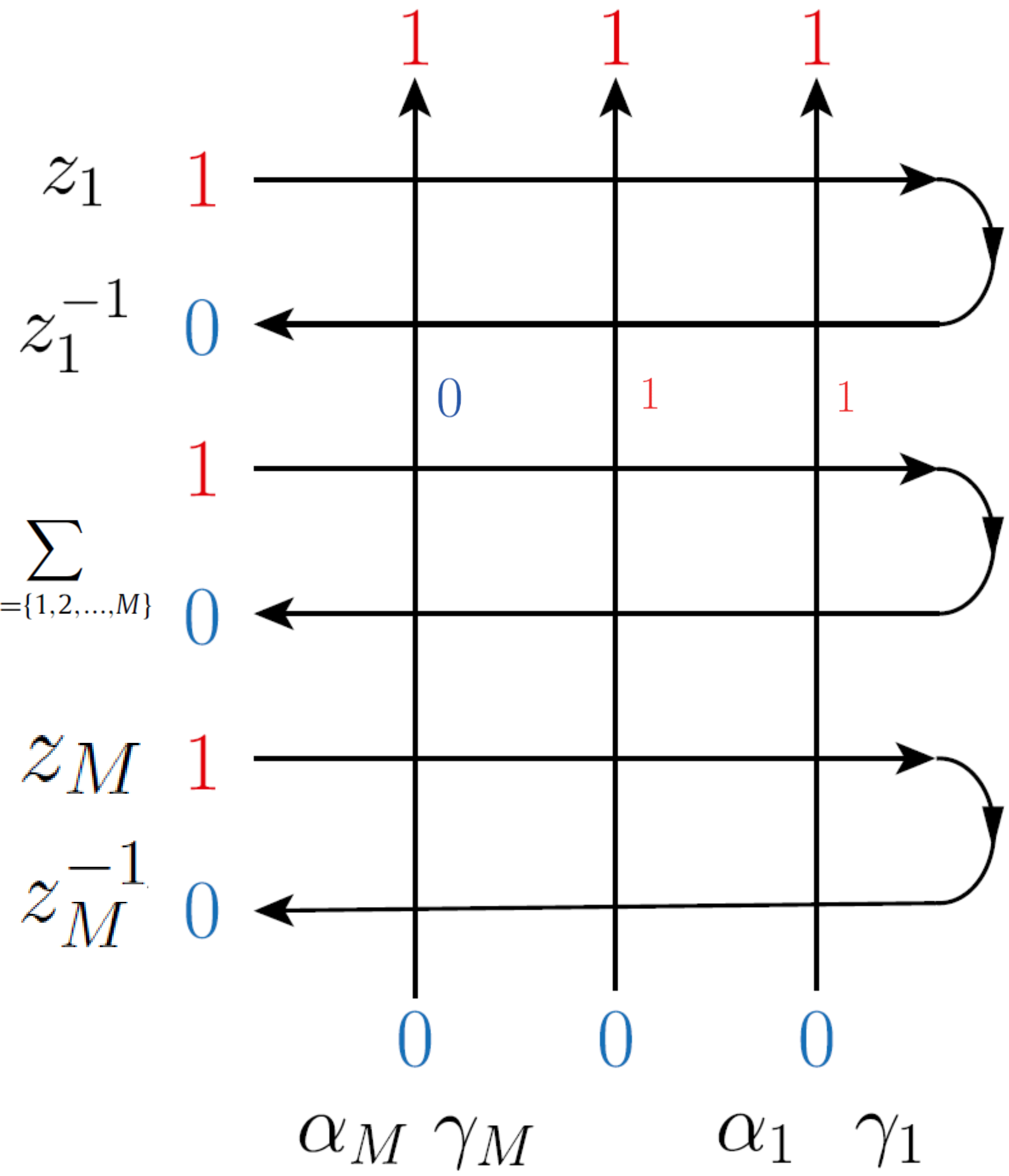


Fig. 5. The type II K-matrix (3.5).

$$= \sum_{x\sqcup\bar{x}=\{1,2,\dots,M\}}$$



$$= \sum_{\lambda\subseteq(M-N)^N} t^{N(M-N)} \prod_{j=1}^{M-N} z_j^{j-1/2-M+N}(1-\sqrt{-t}z_j) \prod_{1\le j<k\le M-N}(1+tz_jz_k)(1+tz_jz_k^{-1})$$

$$\times\, o_{\bar{\lambda}}^{-}(tz_1,\dots,tz_{M-N}|\{-\alpha\}|\{-\gamma\}|t)$$

$$\times \prod_{j=M-N+1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j) \prod_{M-N+1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1})$$

$$\times\, o_{\lambda}^{+}(z_{M-N+1},\dots,z_M|\{\alpha\}|\{\gamma\}|t).$$

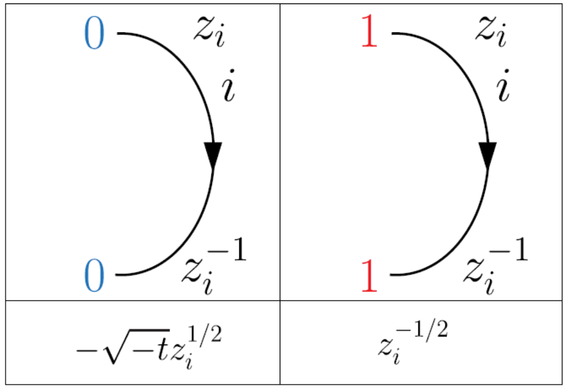


Fig. 5. The type II K-matrix (3.5).

$$\begin{aligned}&\prod_{j=1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j)(1+\sqrt{-t}\gamma_j) \prod_{1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1}) \prod_{1\le j<k\le M}\{1+\alpha_j(\gamma_k-\gamma_j)\} \prod_{1\le j<k\le M}(1-\gamma_j\gamma_k)\\
&\qquad = \sum_{\lambda\subseteq(M-N)^N} t^{N(M-N)} \prod_{j=1}^{M-N} z_j^{j-1/2-M+N}(1-\sqrt{-t}z_j) \prod_{1\le j<k\le M-N}(1+tz_jz_k)(1+tz_jz_k^{-1})\\
&\qquad\quad \times o_{\bar{\lambda}}^{-}(tz_1,\dots,tz_{M-N}|\{-\alpha\}|\{-\gamma\}|t)\\
&\qquad\quad \times \prod_{j=M-N+1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j) \prod_{M-N+1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1})\\
&\qquad\quad \times o_{\lambda}^{+}(z_{M-N+1},\dots,z_M|\{\alpha\}|\{\gamma\}|t).\end{aligned}$$

$$\begin{aligned}
&\prod_{j=1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j)(1+\sqrt{-t}\gamma_j) \prod_{1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1}) \prod_{1\le j<k\le M}\{1+\alpha_j(\gamma_k-\gamma_j)\} \prod_{1\le j<k\le M}(1-\gamma_j\gamma_k) \\
&= \sum_{\lambda\subseteq(M-N)^N} t^{N(M-N)} \prod_{j=1}^{M-N} z_j^{j-1/2-M+N}(1-\sqrt{-t}z_j) \prod_{1\le j<k\le M-N}(1+tz_jz_k)(1+tz_jz_k^{-1}) \\
&\quad \times o_\lambda^-(tz_1,\dots,tz_{M-N}|\{-\alpha\}|\{-\gamma\}|t) \\
&\quad \times \prod_{j=M-N+1}^{M} z_j^{j-1/2-M}(1-\sqrt{-t}z_j) \prod_{M-N+1\le j<k\le M}(1+tz_jz_k)(1+tz_jz_k^{-1}) \\
&\quad \times o_\lambda^+(z_{M-N+1},\dots,z_M|\{\alpha\}|\{\gamma\}|t).
\end{aligned}$$

dual Cauchy formula for factorial Bump-Friedberg-Hoffstein Whittaker functions

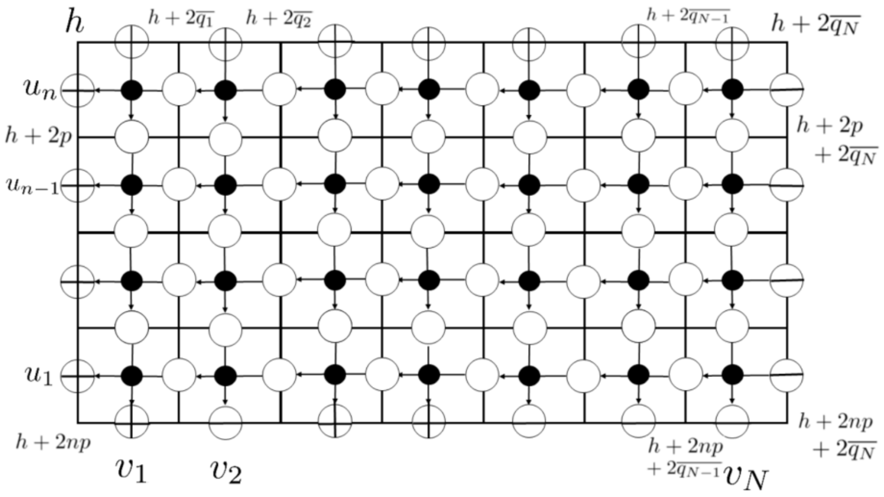
⟺

$$\begin{aligned}
&\sum_{\lambda\subseteq M^N} o_\lambda^+(\{x\}_N|\{\alpha\}|\{\gamma\}|t)\, o_{\hat{\lambda}}^-(\{y\}_M|\{-\alpha\}|\{-\gamma\}|t) \\
&= \prod_{j=1}^{M} y_j^{-N} \prod_{j=1}^{M+N}(1+\sqrt{-t}\gamma_j) \prod_{j=1}^{N}\prod_{k=1}^{M}(1+x_jy_k)(1+x_j^{-1}y_k) \\
&\quad \times \prod_{1\le j<k\le N+M}\{1+\alpha_j(\gamma_k-\gamma_j)\} \prod_{1\le j<k\le N+M}(1-\gamma_j\gamma_k),
\end{aligned}$$

see M-Sakai-Watanabe Journal of Geometry and Physics, 149 (2020), 103571 for details

using the notion of elliptic polynomials (Felder-Schorr, Rubtsov-Pakuliak-Silantyev),
can also deal with the elliptic models by Izergin-Korepin method
see also Felder-Varchenko, Rosengren, Borodin by other approaches

- Journal of Mathematical Physics, 59 (2018), 053505
  elliptic Felderhof model Deguchi-Akutsu Foda-Wheeler-Zuparic
  elliptic analogue of Brubaker-Bump-Friedberg, Bump-McNamara-Nakasuji
  elliptic analogue of Tokuyama formula



$$= \prod_{1\le j<k\le n} [u_k - u_j + 2p] S_{N,n}(u_1,\dots,u_n|v_1,\dots,v_N|x_1,\dots,x_n|h)$$

$$\begin{aligned}
&S_{N,n}(u_1,\dots,u_n|v_1,\dots,v_N|x_1,\dots,x_n|h)\\
&=\prod_{k=1}^{n} \frac{[h+2np+2\overline{q_{x^N_{k+1}-1}}]^{1/2}[h+2(n-k)p+2\overline{q_{x^N_k}}]^{1/2}}{[h+2np+2\overline{q_{x^N_k}}]^{1/2}[h+2(n-k)p+2\overline{q_{x^N_{k+1}-1}}]^{1/2}}\\
&\quad\times \sum_{\sigma\in S_n} \prod_{1\le j<k\le n} \frac{1}{[u_{\sigma(j)}-u_{\sigma(k)}]} \prod_{j=1}^{n} \prod_{k=x_j+1}^{N} [u_{\sigma(j)} - v_k - q_k + p]\\
&\quad\times \prod_{j=1}^{n} \frac{[2p]^{1/2}[2q_{x_j}]^{1/2}[h+2(n+1-j)p+2\overline{q_{x_j-1}}]^{1/2}}{[h+2np+2\overline{q_{x_j-1}}][h+2(n-j)p+2\overline{q_{x_j}}]^{1/2}}\\
&\quad\times \prod_{j=1}^{n} [-u_{\sigma(j)} + v_{x_j} + h + (2n-1)p + q_{x_j} + 2\overline{q_{x_j-1}}] \prod_{j=1}^{n} \prod_{k=1}^{x_j-1} [u_{\sigma(j)} - v_k + p + q_k]
\end{aligned}$$

- Journal of Mathematical Physics, 61 (2020), 053507

elliptic higher rank models

elliptic Foda-Manabe partition functions

↓ special cases

elliptic nested Bethe wavefunctions

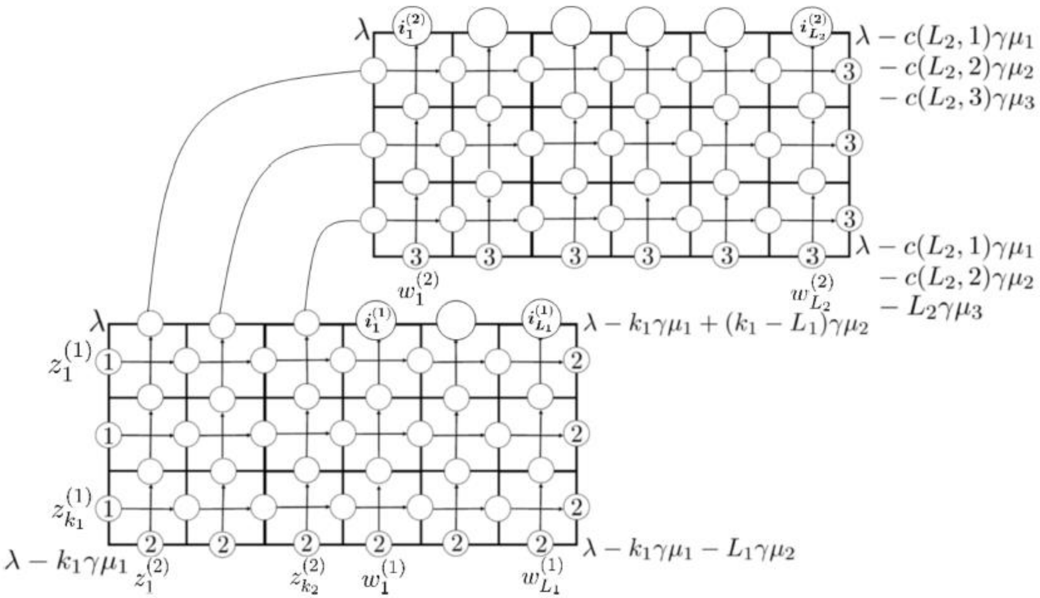
||

elliptic weight functions

recently introduced by Konno vertex operator

Rimanyi-Tarasov-Varchenko, Felder-Rimányi-Varchenko

bases for partial flag varieties in the theory of elliptic cohomology by Aganagic-Okounkov



$$
\begin{aligned}
=&[\gamma]^{k_1+k_2}\sum_{\sigma_1\in S_{k_1}}\sum_{\sigma_2\in S_{k_2}}\prod_{a=1}^{k_1}\left(\prod_{i=1}^{\widetilde{I_a^{(1)}}-1}[z^{(1)}_{\sigma_1(a)}-m_i^{k_2,L_1}(\{z^{(2)}_{\sigma_2}\}|\{\boldsymbol{w}^{(1)}\})]\right.\\
&\times\frac{[z^{(1)}_{\sigma_1(a)}-m^{k_2,L_1}_{\widetilde{I_a^{(1)}}}(\{z^{(2)}_{\sigma_2}\}|\{\boldsymbol{w}^{(1)}\})+\lambda_2-\lambda_1+\gamma(2a-1-\widetilde{I_a^{(1)}})]}{[\lambda_1-\lambda_2+(1-a)\gamma]}\\
&\left.\times\prod_{i=\widetilde{I_a^{(1)}}+1}^{k_2+L_1}[z^{(1)}_{\sigma_1(a)}-m_i^{k_2,L_1}(\{z^{(2)}_{\sigma_2}\}|\{\boldsymbol{w}^{(1)}\})-\gamma]\right)\\
&\times\prod_{1\le a<b\le k_1}\frac{[z^{(1)}_{\sigma_1(a)}-z^{(1)}_{\sigma_1(b)}+\gamma]}{[z^{(1)}_{\sigma_1(a)}-z^{(1)}_{\sigma_1(b)}]}\prod_{a=1}^{k_2}\left(\prod_{i=1}^{I_a^{(2)}-1}[z^{(2)}_{\sigma_2(a)}-w^{(2)}_i]\right.\\
&\times\frac{[z^{(2)}_{\sigma_2(a)}-w^{(2)}_{I_a^{(2)}}+\lambda_3-\lambda_{i^{(2)}_{I_a^{(2)}}}+(a-I_a^{(2)}-1+c(I_a^{(2)},i^{(2)}_{I_a^{(2)}}))\gamma]}{[\lambda_{i^{(2)}_{I_a^{(2)}}}-\lambda_3+(1-c(I_a^{(2)},i^{(2)}_{I_a^{(2)}}))\gamma]}\\
&\left.\times\prod_{i=I_a^{(2)}+1}^{L_2}[z^{(2)}_{\sigma_2(a)}-w^{(2)}_i-\gamma]\right)\prod_{1\le a<b\le k_2}\frac{[z^{(2)}_{\sigma_2(a)}-z^{(2)}_{\sigma_2(b)}+\gamma]}{[z^{(2)}_{\sigma_2(a)}-z^{(2)}_{\sigma_2(b)}]},
\end{aligned}
$$