How to come up with Solvable Lattice Models?

Slava Naprienko 03/10/2021.

* Sorry for the clickbait!
* It's work in progress.

Outline:

1. Spherical Lattice Models
2. Spherical Whittaker functions
3. How to come up with spherical models?
4. How to come up with colored models and my results.

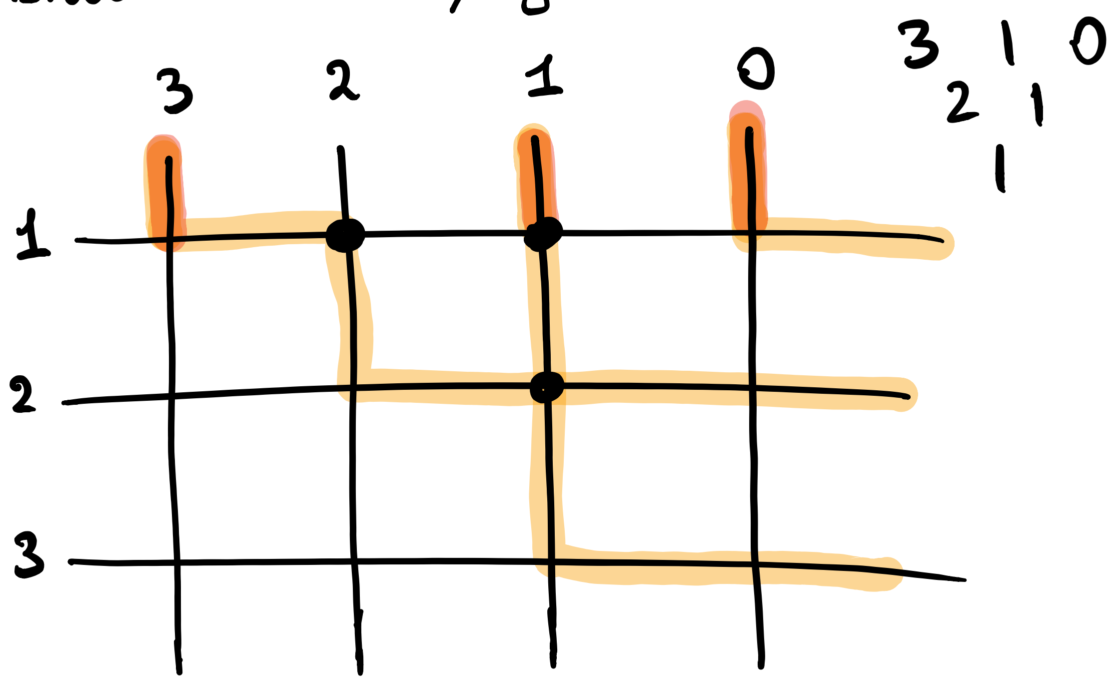
Main takeaway You can (without guessing)

1. start with functions from representation theory of p-adic groups,
2. decompose them into pieces geometrically,
3. describe pieces combinatorically,
4. realize combinatorical data as lattice models.

I've done it for spherical and Iwahori Whittaker f-ns of GL_n and got spherical and colored lattice models.

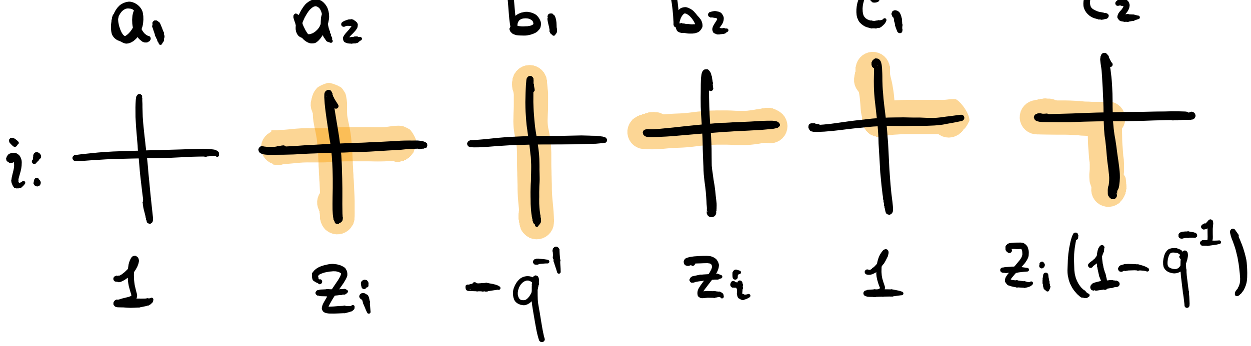
① Spherical Lattice Model

Consider model $\mathfrak{S}_\mu$ given as follows:



The top boundary is determined by strictly dominant partition μ. Above $\mu = (3,1,0)$.

Only the following 6 vertices are allowed:



Thm (Brubaker, Bump, Friedberg, '09)

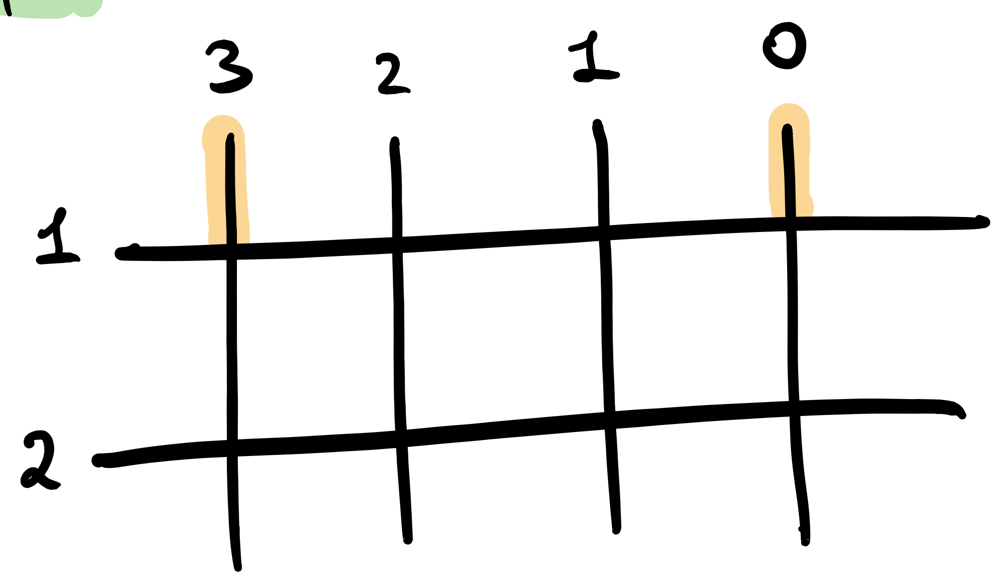
Let $\rho = (n-1, n-2, \ldots, 1, 0)$, and write $\mu = \lambda + \rho$. Then the partition function Z_μ of $\mathfrak{S}_\mu$ is given by

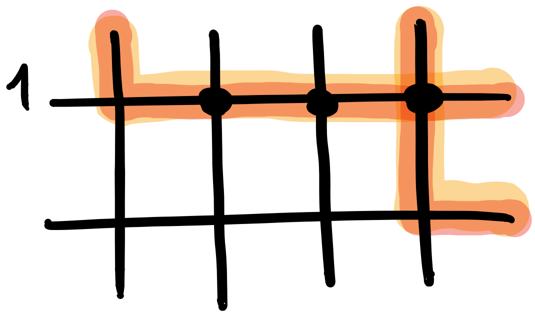
$$\prod_{i<j} (z_i - z_j)$$

$$\boxed{\begin{aligned} Z_\mu(q, z) &= \prod_{i<j} (z_i - q^{-1} z_j) \, s_\lambda(z), \\ &= z^\rho \prod_{i<j} (1 - q^{-1} z_i / z_j) \, s_\lambda(z), \end{aligned}}$$

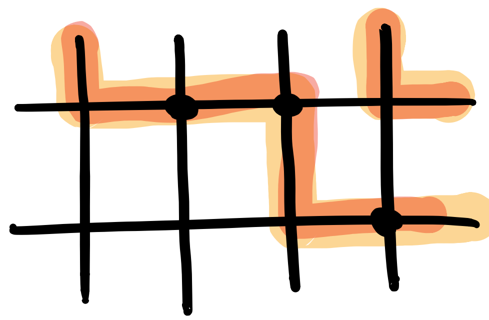
where s_λ is the Schur polynomial.

Example $\lambda = (2, 0)$, $\rho = (1, 0)$, $\mu = \lambda + \rho = (3, 0)$

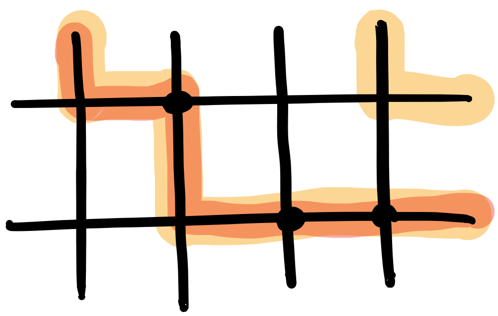




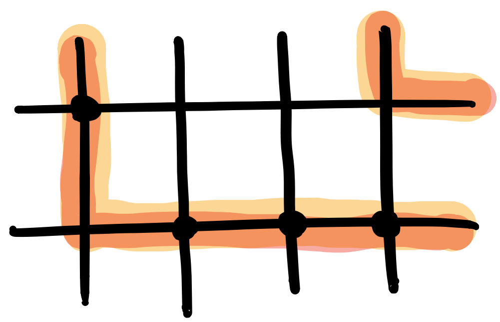
$z_1\cdot z_1\cdot z_1$



$z_1\cdot z_1(1-q^{-1})z_2$



$z_1(1-q^{-1})\cdot z_2\cdot z_2$



$-q^{-1}\cdot z_2\cdot z_2\cdot z_2$

$$Z_\mu(z)=z_1^3+z_1^2z_2(1-q^{-1})+z_1z_2^2(1-q^{-1})+(-q^{-1})z_2^3=$$

$$=(z_1-q^{-1}z_2)(z_1^2+z_1z_2+z_2^2)=$$

$$=\prod_{i<j}(z_i-q^{-1}z_j)\,s_\lambda(z)=z^\rho\prod_{i<j}\left(1-q^{-1}\frac{z_i}{z_j}\right)s_\lambda(z).$$

2 Spherical Whittaker function

Let $G = GL_n(F)$, where F local (non-arch) field (e.g. $\mathbb{Q}_p$). q - card of res field

$B = \begin{pmatrix} * & & * \\ & * & \\ & & \ddots \\ & & & * \end{pmatrix}$, $U = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & \ddots \\ & & & 1 \end{pmatrix}$

$\mathcal{O}_F$ - ring of integers.

$B^- = \begin{pmatrix} * & & & \\ & * & & \\ & & * \ddots & \\ * & & & * \end{pmatrix}$, $U^- = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{pmatrix}$.

$\mathfrak{p} = (\varpi)$, ϖ - uniformizer.

$T = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \ddots \\ & & & & * \end{pmatrix}$, $W = N_G(T)/T \simeq S_n$

(„permutation matrices")

$$K = GL_n(\mathcal{O}_F) = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \\ \vdots & & \ddots & \vdots \\ \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \end{pmatrix}.$$

$\Phi^+ = \{(i,j) \in [n]^2 \mid i<j\}$ — positive roots,

let $N = |\Phi^+| = \frac{n(n-1)}{2}$.

Also, for partition λ we denote $\varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \ddots & \\ & & & \varpi^{\lambda_n} \end{pmatrix}$.

Let $\psi : U^- \to \mathbb{C}^\times$ be given by

$$\psi\begin{pmatrix} 1 & & & & \\ x_1 & 1 & & & \\ & x_2 & 1 & & \\ & & & \ddots & \\ & & & x_{n-1} & 1 \end{pmatrix} = \psi_0(x_1)\cdot\ldots\cdot\psi_0(x_{n-1}),$$

where $\psi_0 : F \to \mathbb{C}^\times$ is a character which is trivial on $\mathcal{O}_F$ but not on $p^{-1}\mathcal{O}_F$.

Blackbox The spherical Whittaker function $W^z : G \to \mathbb{C}$ is given by the following integral.

$$W^z(g) = \int_{U^-} f^z(u \cdot g)\psi(u)\,du,$$

where $z \in \mathbb{C}^r$ and f^z is given by

$$f^z(g) = f(g) = f(u \cdot \varpi^\mu k) = z^\mu, \quad (z \text{ is fixed})$$

where we use the Iwasawa decomposition $G = UTK$.

Thm (Casselman–Shalika, '80) The values of the spherical Whittaker function are given by

$$W^z(g) = \begin{cases} 0, & \text{if } g \in U\varpi^\lambda K \text{ with } \lambda \text{ not dominant} \\ \prod_{i<j}\left(1 - q^{-1}\frac{z_i}{z_j}\right) s_\lambda(z), & \text{otherwise.} \end{cases}$$

In other words, the partition function Z_λ gives the values of spherical Whittaker function.

$$Z_\mu(z) = z^\rho \cdot W^z(g), \quad g \in U\varpi^\mu K, \ \lambda \in \Lambda^+.$$

The usual proof of the connection is ad hoc. You show that both sides satisfy the same functional equations imposed by the intertwining operators for $W^z(g)$, and Yang-Baxter equations for $Z_\mu(z)$. But how to come up with $\mathfrak{S}_\lambda$?

③ How to come up with spherical lattice model $\mathfrak{S}_\lambda$?

Let's start with $\boxed{W^z(g) = \int_{U^-} f(ug)\Psi(u)\,du.}$

Recall that $f(g) = f(u \cdot \varpi^\mu \cdot k) = z^\mu$.

It follows that $W^z(g)$ is determined on ϖ^λ.

So we get $\int_{U^-} f(u \cdot \varpi^\lambda)\Psi(u)\,du$. After change of variables $u \mapsto \varpi^\lambda \cdot u \cdot \varpi^{-\lambda}$, we get (up to constant) $\int_{U^-} f(u)\Psi_\lambda(u)\,du$, where

$$\Psi_\lambda(u) = \Psi(\varpi^\lambda \cdot u \cdot \varpi^{-\lambda}).$$

We study $\phi_\lambda(z) = \int_{U^-} f(u)\Psi_\lambda(u)\,du.$

Thm (McNamara, '09) We can write

$$U^- = \bigsqcup_{m \in \mathbb{N}^N} C_m, \quad \text{and} \quad f\big|_{C_m} = z^{wt(m)}.$$

(there is actually a lot more.) $\boxed{C_m \Leftrightarrow \text{MV cycles}}$

Using the theorem, we get

$$\varphi_\lambda(z) = \int_{U^-} f(u)\psi_\lambda(u)du = \sum_{m \in \mathbb{N}^N} z^{wt(m)} \cdot \int_{C_m} \psi_\lambda(u)du.$$

Denote $G_\lambda(m) = \int_{C_m} \psi_\lambda(u)du = \prod_{\alpha \in \Phi^+} G_\lambda(\alpha, m)$ Lusztig data

For almost all m, we have $G_\lambda(m) = 0$.

More precisely, there is finite set $Lu(\lambda+\rho)$ of N-tuples such that $G(m) = 0$ unless $m \in Lu(\lambda+\rho)$.

Punchline: States of spherical model correspond to $Lu(\lambda+\rho)$ and their weights to $z^{wt(m)} G_\lambda(m)$.

States have „geometric meaning"

Let's concentrate on bijection between states.

$Lu(\lambda+\rho) \longleftrightarrow GT(\lambda+\rho)$, Gelfand-Tsetlin patterns.

Ex

$GT(3,1,0)$:

$$\left\{\begin{array}{ccccc}3&&1&&0\\&1&&0&\\&&0&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&1&&0&\\&&1&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&1&&1&\\&&1&&\end{array}\right\}$$

$$\left\{\begin{array}{ccccc}3&&1&&0\\&2&&0&\\&&0&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&2&&0&\\&&1&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&2&&0&\\&&2&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&2&&1&\\&&1&&\end{array}\right\}$$

$$\left\{\begin{array}{ccccc}3&&1&&0\\&2&&1&\\&&2&&\end{array}\right\}
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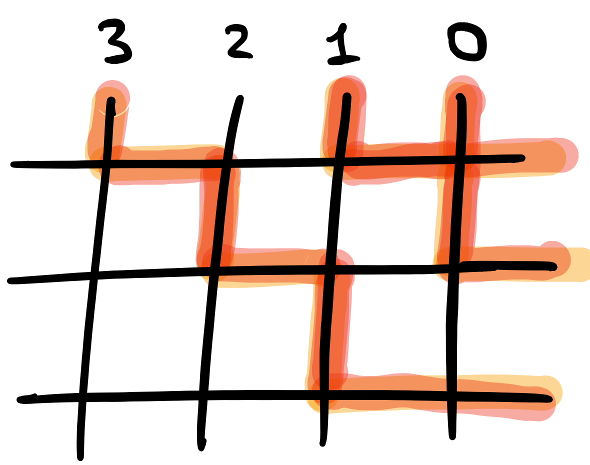
\left\{\begin{array}{ccccc}3&&1&&0\\&3&&0&\\&&2&&\end{array}\right\}$$

$$\left\{\begin{array}{ccccc}3&&1&&0\\&3&&0&\\&&3&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&3&&1&\\&&1&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&3&&1&\\&&2&&\end{array}\right\}
\left\{\begin{array}{ccccc}3&&1&&0\\&3&&1&\\&&3&&\end{array}\right\}.$$

Bijection:

$$\begin{array}{ccccc}3&&1&&0\\&2&&0&\\&&1&&\end{array} \longleftrightarrow$$



$$\underline{\text{States in } \mathcal{B}_{\lambda+\rho}} \longleftrightarrow \underline{\text{GT-patterns with } \lambda+\rho} \longleftrightarrow \underline{\text{Lusztig data } \lambda+\rho}$$

weight-preserving bijections.

Remark Calculations in terms of Lusztig data can be made uniformly for any split reductive group (e.g., $Sp(2n)$ or $O(n)$).

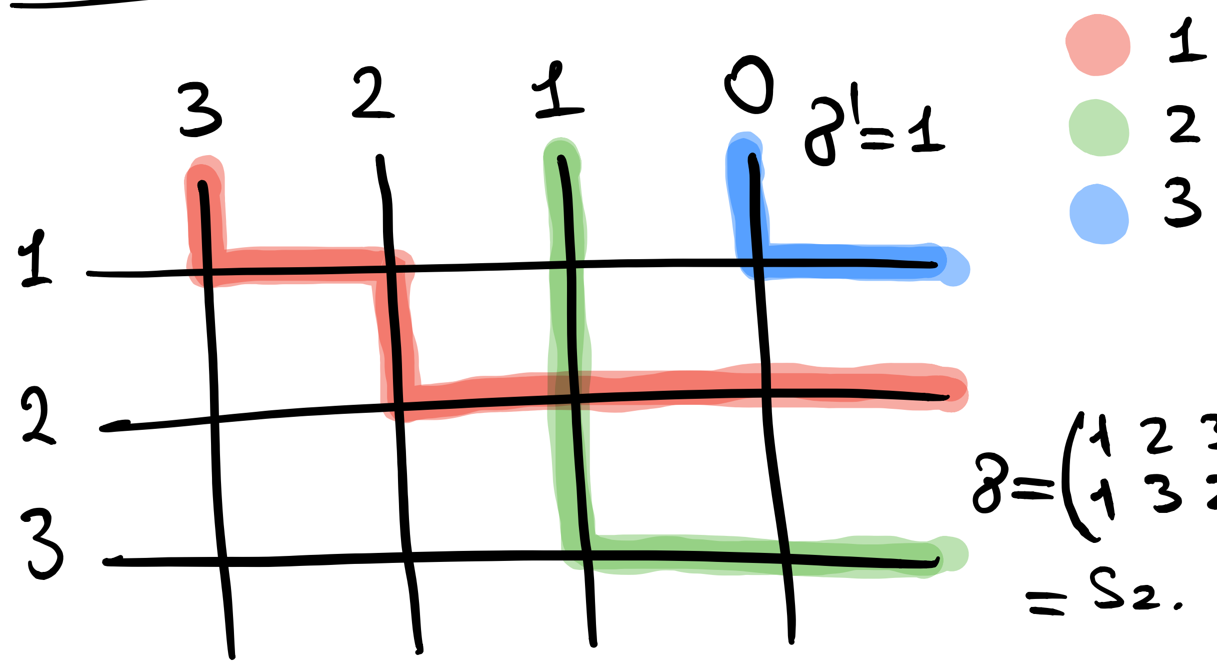
Summary

1. Start with spherical Whittaker function.
2. $\int_{U^-} = \sum_m \int_{C_m}$ by McNamara's decomposition.
3. $\int_{C_m} \neq 0$ for finite set $Lu(\lambda+\rho) \longleftrightarrow GT(\lambda+\rho)$.
4. $Lu(\lambda+\rho) \longleftrightarrow GT(\lambda+\rho) \longleftrightarrow$ states in $\mathcal{B}_{\lambda+\rho}$.

(4) How to come up with colored lattice models?

[show the table with patterns]

Colored lattice model $\mathfrak{S}_{\mu,\sigma,\sigma'}$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = s_2.$$

Top row is μ, input permutation is σ', and output permutation is σ. Models were introduced in 2019 by BBBG and BW.

Blackbox 2 Iwahori Whittaker functions

$W_\delta^z : G \to \mathbb{C}$ for $\delta \in W \simeq S_n$ are given by

$$W_\delta^z(g) = \int_{U^-} f_\delta^z(ug)\psi(u)\,du,$$ Iwahori decomp.

where $f_\delta^z(g) = f_\delta^z(u \cdot \varpi^\lambda \cdot \delta' \cdot k) = \begin{cases} 0, & \text{if } \delta' \neq \delta \\ z^\lambda & \text{if } \delta' = \delta. \end{cases}$

Then McNamara's decomposition

$$U^- = \bigsqcup_m C_m \quad \text{is not enough}$$

because f_δ^z is not constant on C_m

We need to explore $C_m \cap B\delta J(\delta')^{-1}$, the intersections of C_m with double Iwahori cosets. $\boxed{U^- = \bigsqcup_m C_m}$ and $\boxed{U^- = \bigsqcup_w BwJ}$

Thm (N, '20) $C_m \cap B\delta J(\delta')^{-1} = \bigsqcup_{\pi} S_{m,\pi}$.

Punchline Each $S_{m,\pi}$ corresponds to a state in colored lattice model with fixed input and output colors.

$\underbrace{\text{MV cycles}}$?? $\longleftrightarrow$ $\underbrace{C_m} \cap BwJw'$

Thm (N, '20) There are weight preserving bijections between

States of colored lattice model $\longleftrightarrow$ Colored GT patterns $\longleftrightarrow$ Colored Lusztig data

Thm (N, '20) We can express Iwahori Whittaker functions in terms of colored data above.

Questions:

A. Other groups besides $\underline{GL_n}$? $Sp(2n)$? $O(n)$? ???

B. Metaplectic covers? In progress.

C. Other decompositions of U^-?

There are two „good" ones.