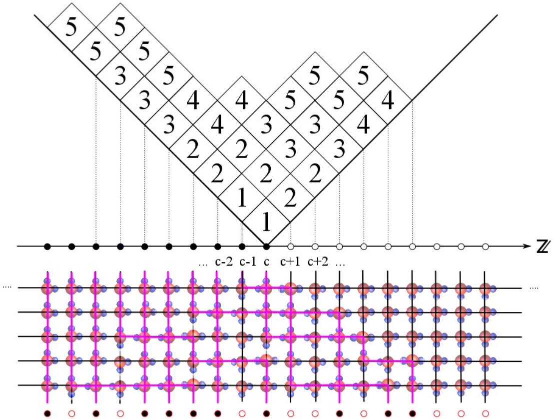
The asymmetric six-vertex model & cylindric Hecke characters

Based on COMM. MATH. PHYS. 381 (2021) 591-640



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Exactly solvable lattice models, Stanford, 8 Mar 2022

Outline

1. Macdonald's characteristic map & the Boson-Fermion Correspondence

2. Hecke characters & the asymmetric six-vertex model

3. Vertex operators

4. Periodic boundary conditions: cylindric Hecke characters & Gromov-Witten invariants

§1. Macdonald's characteristic map & the BF correspondence

R^n class functions of S_n

$$\begin{tikzcd} R = \bigoplus_{n\in\mathbb{N}} R^n \arrow[r, "\sim"] \arrow[dr, "ch"'] & Z = \bigoplus_{n\in\mathbb{N}} Z(\mathbb{C}S_n) \arrow[d, "F"] \\ & Sym = \bigoplus_{n\in\mathbb{N}} Sym^n \end{tikzcd}$$

centre of group algebra

Frobenius map

Macdonald's characteristic map

degree

ring of symmetric fctns

§1. Macdonald's characteristic map & the BF correspondence

R^n class functions of S_n

$$\begin{tikzcd}
R = \bigoplus_{n \in \mathbb{N}} R^n \arrow[r, "\sim"] \arrow[dr, "\mathrm{ch}"'] & Z = \bigoplus_{n \in \mathbb{N}} Z(\mathbb{C}S_n) \arrow[d, "F"] \\
& \mathrm{Sym} = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}^n
\end{tikzcd}$$

centre of group algebra

Macdonald's characteristic map

Frobenius map

degree

ring of symmetric fctns

Distinguished bases:

1. R^n: irreducible characters χ^λ (Specht module), $\lambda \vdash n$
2. $Z(\mathbb{C}S_n)$: sums over conjugacy classes $c_\lambda = \sum_{\text{cycle type } \lambda} w$
3. Sym^n: (a) power sums $\{p_\lambda : \lambda \vdash n\}$

   (b) Schur functions $\{s_\lambda : \lambda \vdash n\}$

§1. Macdonald's characteristic map & the BF correspondence

$$\begin{tikzcd}
R = \bigoplus_{n\in\mathbb{N}} R^n \arrow[r, "\sim"] \arrow[dr, "\mathrm{ch}(\chi^\lambda) = s_\lambda"'] & Z = \bigoplus_{n\in\mathbb{N}} Z(\mathbb{C}S_n) \arrow[d, "F(c_\mu) = \frac{p_\mu}{z_\mu}"] \\
& \mathrm{Sym} = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n
\end{tikzcd}$$

power sums

Schur function

<u>THM</u> (Macdonald) The above diagram of vector space isomorphisms commutes.

irred character

power sum

inverse

<u>COR</u> (Frobenius formula) $\displaystyle s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$

Schur function

cycle type

size of conjugacy class

§1. Macdonald's characteristic map & the BF correspondence

$$\begin{tikzcd}
R = \bigoplus_{n\in\mathbb{N}} R^n \arrow[rr, "\sim"] \arrow[drr, "ch"', "\text{Macdonald's characteristic map}"] & & Z = \bigoplus_{n\in\mathbb{N}} Z(\mathbb{C}S_n) \arrow[d, "F"', "\text{Frobenius map}"] \\
& & \mathrm{Sym} = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n
\end{tikzcd}$$

Hopf algebra structure :

$$\mathrm{Ind}_{S_m\times S_n}^{S_{m+n}} : R^m \otimes R^n \longrightarrow R^{m+n}$$

$$\mathrm{Res}^{S_N}_{\substack{S_m\times S_n \\ m+n=N}} : R^N \longrightarrow \bigoplus_{m+n=N} R^m \otimes R^n$$

$c^{\lambda}_{\mu\nu}$ = LR coefficients

$$s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu}\, s_\nu$$

$$\Delta s_\lambda = \sum_{\mu\subset\lambda} s_{\lambda/\mu} \otimes s_\mu$$

$$s_{\lambda/\mu} = \sum_\nu c^{\lambda}_{\mu\nu}\, s_\nu$$

§1. Macdonald's characteristic map & the BF correspondence

$$
\begin{tikzcd}
R \arrow[r, "\sim"] \arrow[d, "s"'] \arrow[dr, "ch"] & Z \arrow[d, "F"] \\
\Lambda^{\frac{\infty}{2},c} V \arrow[r, "BF"] & Sym
\end{tikzcd}
$$

$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j$

'charge'

fermionic Fock space $\Lambda^{\frac{\infty}{2},c} V$

e.g. [Kac, Raina]

Clifford algebra

ψ_j, ψ_j^*

Sym 'Bosonic Fock space'

Heisenberg algebra

$p_r, \ r\frac{\partial}{\partial p_r}$

<u>THM</u> The BF correspondence is an Heisenberg algebra module isomorphism

$$\mathcal{P}_r = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+r}^* \mapsto p_r, \qquad \mathcal{P}_r^* = \sum_{j \in \mathbb{Z}} \psi_{j+r} \psi_j^* \mapsto r\frac{\partial}{\partial p_r}$$

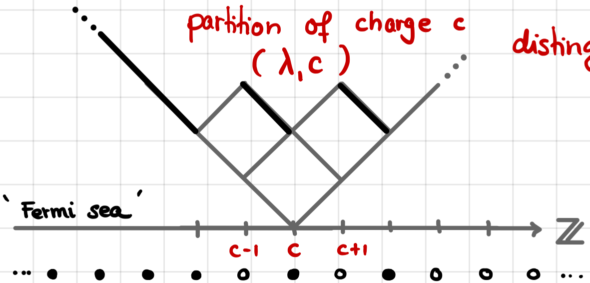
The fermionic Fock space

Identify elements in $\Lambda^{\frac{\infty}{2},c} V$ as the span of (half-) infinite wedge products

$$v = v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \dots \quad \text{with} \quad i_k = c - k \quad \text{for } k \ll -1$$

Define a Maya diagram $\sigma_{\lambda,c} : \mathbb{Z} \to \mathbb{Z}^2$ via $\sigma_{\lambda,c}(j) = \begin{cases} 1, & j = c+1+\lambda_i - i \\ 0, & \text{else} \end{cases}$

distinguished basis: Maya diagrams $\sigma : \mathbb{Z} \to \{0,1\}$



distinguished basis: power sums $p_r = \sum_{i \geq 1} x_i^r$

$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$

Clifford algebra

Heisenberg algebra

power sums / KP 'time parameters'

$$\sigma_{\lambda,c} \mapsto \langle \sigma_{\lambda,c} , e^{H[p]} \sigma_{\emptyset,c} \rangle = s_\lambda[X], \qquad H[p] = \sum_{r>0} \frac{p_r}{r} \Psi_r, \qquad \Psi_r = \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+r}^*$$

Boson-Fermion correspondence

SUMMARY

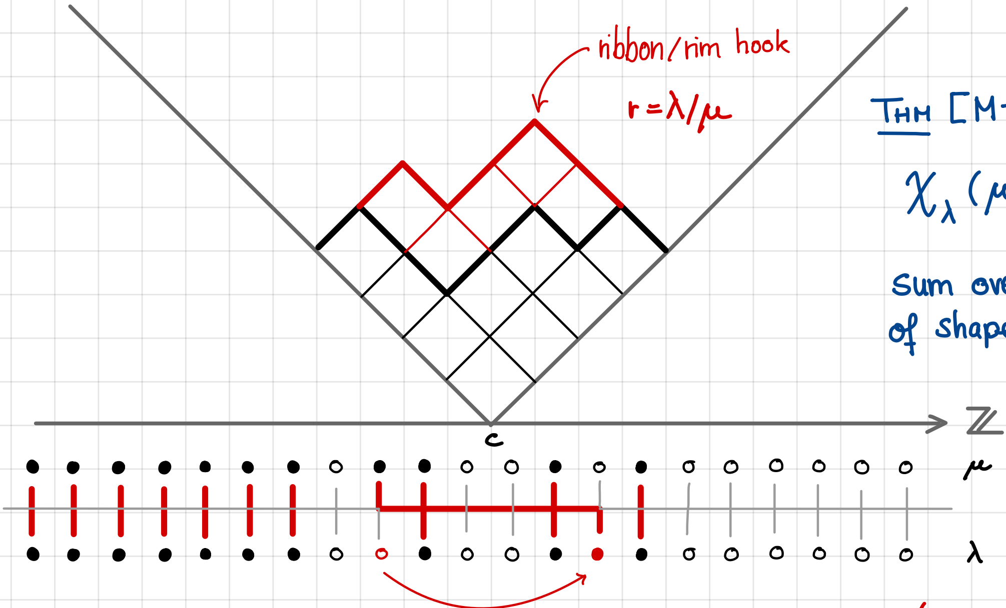
- The charater table of symmetric groups yields the change of basis $p_\mu \to s_\lambda$, $\lambda, \mu \vdash n$
- s_λ is a basis distinguished by positivity

$c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0} \rightsquigarrow$ intersection cohomology $H^*(Gr_{k,n})$

Q. How does one compute $\chi_\lambda(\mu)$?

A. Murnaghan-Nakayama rule
(combinatorial algorithm using ribbons/rim hooks)

THE MURNAGHAN-NAKAYAMA RULE

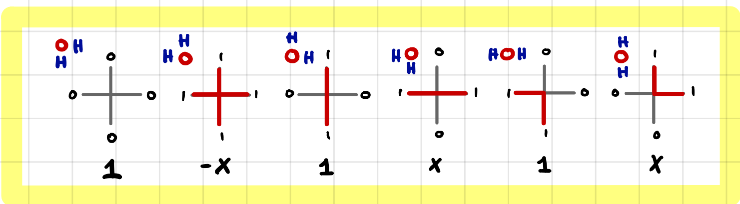


THM [M-N]

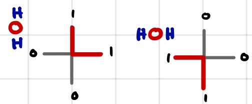
$$\chi_\lambda(\mu) = \sum_T \prod_{t \in T} (-1)^{ht(r)-1}$$

Sum over all rim hook tableaux of shape λ and weight μ.

$r = \lambda/\mu$



'special' configurations of the asymmetric 6-vertex model: only one pair of



allowed!

§2. The quantum characteristic map

Reminder: the Iwahori-Hecke algebra of type A

<u>DEF</u> $H_n = H_n(t)$ is the $\mathbb{C}(t)$-algebra with generators T_i, $i=1,\dots,n-1$

and relations

$$T_i^2+(1-t)T_i-t=0, \quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad T_iT_j=T_jT_i \quad |i-j|>1$$

Grothendieck ring of Hecke algebras

centre of $H_n(t)$

$$\begin{tikzcd}
\mathcal{R}_t = \bigoplus_{n\in\mathbb{N}} \mathcal{R}_t^n \arrow[rr, "\sim"] \arrow[drr, "ch_t"] & & Z = \bigoplus_{n\in\mathbb{N}} Z(H_n) \arrow[d, "F_t"] \\
& & \mathrm{Sym}_t = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}_t^n = \mathrm{Sym}^n \otimes_{\mathbb{C}} \mathbb{C}(t)
\end{tikzcd}$$

quantum characteristic map

quantum Frobenius map [Wan-Wang]

ring of symmetric fctns

Plethystic variable subsitution

THM [Wan-Wang] (inverse) 'quantum Frobenius formula'

$$s_\lambda[X] = \sum_\mu \chi_t^\lambda(T_{w_\mu})(t-1)^{\ell(\mu)} m_\mu\left[\frac{X}{t-1}\right]$$

Schur function in alphabet X

irreducible Hecke character $\lambda \vdash n$

$T_{w_\mu} = T_{i_1} \cdots T_{i_\ell}$ where $w_\mu = s_{i_1} \cdots s_{i_\ell}$ reduced and of cycle type μ

monomial symmetric function in alphabet $\frac{X}{t-1}$

$$p_r\left[\frac{X}{t-1}\right] = p_r[X]/(t^r-1)$$

Equivalently,

$$s_\lambda[(t-1)X] = \sum_\mu \chi_t^\lambda(T_{w_\mu})(t-1)^{\ell(\mu)} m_\mu[X]$$

'dual' Schur function w.r.t. $\langle P_\lambda, Q_\mu \rangle_t = \delta_{\lambda\mu}$

HL-functions

A Hecke version of the BF correspondence

Fermionic Fock space : $\Lambda^{\frac{\infty}{2},c} V_t = \Lambda^{\frac{\infty}{2},c} V \otimes_{\mathbb{C}} \mathbb{C}(t)$

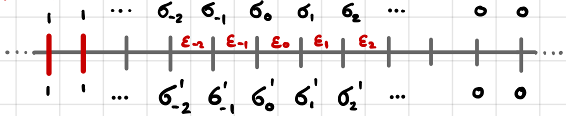
'free' six-vertex transfer matrix

PROP (CK) $\sigma_{\lambda,c} \longmapsto \langle \sigma_{\lambda,c} , \prod_{i\geq 1} A(x_i;t)\, \sigma_{\emptyset,c} \rangle = S_\lambda[(t-1)X]$

Maya diagram $\hat{=}$ spin configuration

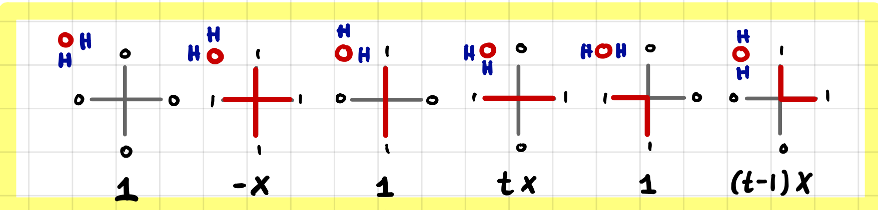
6v - transfer matrix

$\langle \sigma', A(x)\sigma \rangle = \sum_{\varepsilon_i}$



$A(x;t) = e^{H_t[p]}$

$H_t[p] = \sum_{r>0} \frac{(t^r-1)}{r} p_r \sum_{i\in\mathbb{Z}} \psi_i \psi^*_{i+r}$

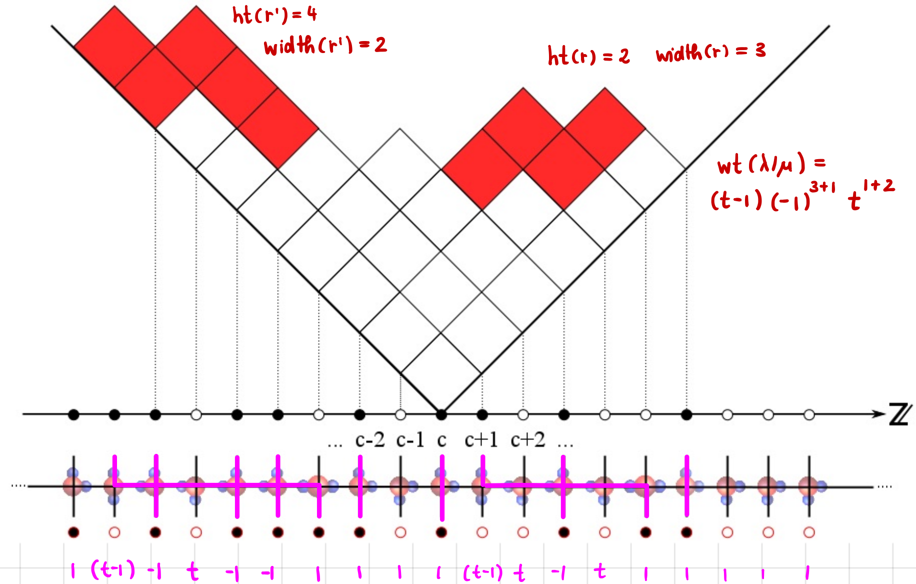


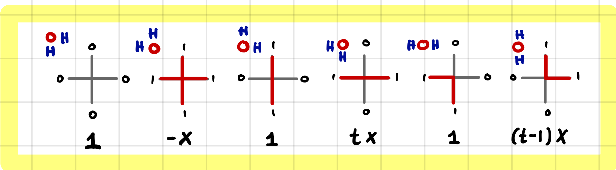
six vertex configurations & Boltzmann weights

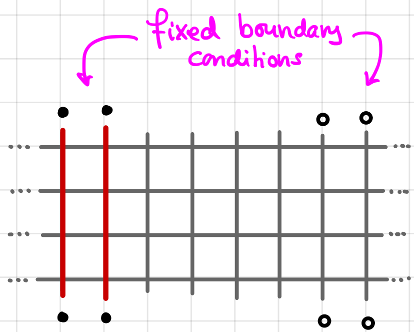
Rmk These are different from the weights for Tokuyama ice [Brubaker-Bump-Friedberg] and different boundary conditions

Broken rim hooks & asymmetric ice configurations

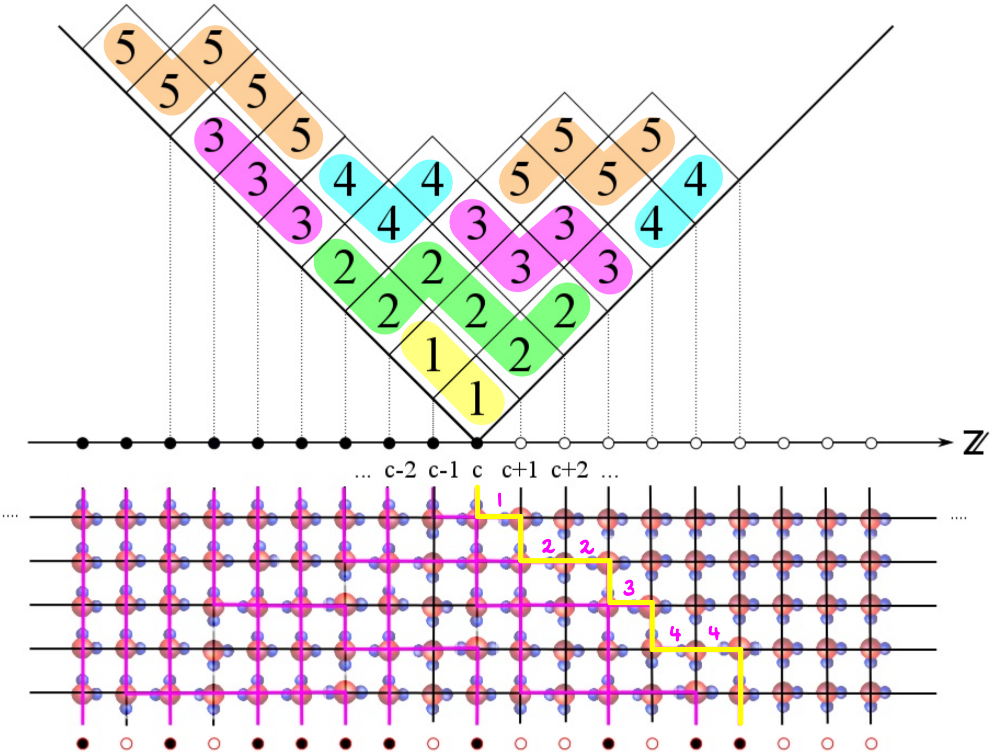
$$\mathrm{wt}(\lambda/\mu) = (t-1)^{\#(\lambda/\mu)-1} \prod_{\text{rim hooks } r} (-1)^{\mathrm{ht}(r)-1}\, t^{\mathrm{width}(r)-1}$$







Broken rim hook tableaux and Hecke characters



THM [Ram]

Hecke characters are given as weighted sums over broken rim hook tableaux.

$$\chi_t^\lambda(T_{w_\mu}) = \sum_{\mathcal{T}} wt(\mathcal{T})$$

$$wt(\mathcal{T}) = \prod_i wt(\lambda^{(i)}/\lambda^{(i+1)})$$

w_μ representative of conjugacy class fixed by μ

PROP The 6-vertex lattice configurations are in bijection with broken rim hook tableaux.

§ 3. Vertex operators & transfer matrices

__DEF__ (JING) Define the following (bosonic) vertex operators $\Phi^{\pm}(x;t) : \mathrm{Sym}_t \longrightarrow \mathrm{Sym}_t$

$$\Phi^{-}(x;t) = e^{\sum_{r>0} \frac{1-t^r}{r} p_r x^r} \, e^{\sum_{r>0} (t^r-1) \frac{\partial}{\partial p_r} x^{-r}}, \qquad \Phi^{+}(x;t) = e^{\sum_{r>0} \frac{t^r-1}{r} p_r x^r} \, e^{\sum_{r>0} (1-t^r) \frac{\partial}{\partial p_r} x^{-r}}$$

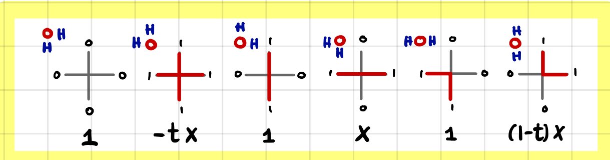
__THM__ (JING) $\Phi^{-}(x;t) = \sum_{r\in\mathbb{Z}} \Phi^{-}_r x^r$, $\quad \Phi^{-}_{\lambda_1} \cdots \Phi^{-}_{\lambda_\ell} 1 = Q_\lambda$ Hall-Littlewood function

Six-vertex construction

__THM__ (CK) Let $\imath_t : \Lambda^{\frac{\infty}{2},c} V_t \to \mathrm{Sym}_t$ be the t-extension of the BF-correspondence, then

$$\imath_t \circ A^{-1}(x) A^{*}(x) = \Phi^{-}(x;t) \circ \imath_t \qquad \text{and} \qquad \imath_t \circ A(x) (A^{-1}(x))^{*} = \Phi^{+}(x;t) \circ \imath_t$$

where $A^{-1}(x;t) = A(tx;t^{-1})$ is the six-vertex transfer matrix with weights



'fermionic' construction of Hall-Littlewood functions

q-boson construction: [Tsilevich '06]

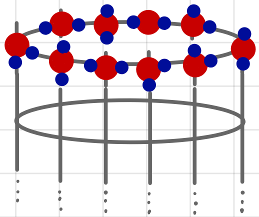
C.K. CMP 318 (2013) 173

LLT polynomials [Brubaker, Buciumas, Bump, Gustafsson]

§4. Periodic boundary conditions

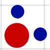
$$\Lambda^{\frac{\infty}{2},k} V\otimes \mathbb{C}(t) \longrightarrow \Lambda\, \mathbb{C}^n(t) = \bigoplus_{k=0}^{n} \Lambda^k\, \mathbb{C}^n(t)$$

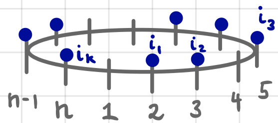
∞ wedge — finite wedge



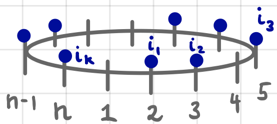
Denote by $T(x)$ the six-vertex transfer matrix with quasi-periodic boundary conditions and twist parameter q

$$T(x) = \sum_{r=0}^{n} x^r\, T_r\,, \quad T_r : \bigoplus_{d\in\mathbb{Z}} q^d \oplus \Lambda\, \mathbb{C}^n(t) \longrightarrow \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda\, \mathbb{C}^n(t)$$

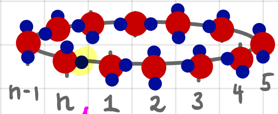




$\longmapsto\; V_{i_1} \wedge V_{i_2} \wedge \cdots \wedge V_{i_k}$, i_j positions of H-atoms ●



$\xmapsto{T_r} \sum q$

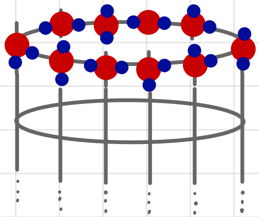


H-atom between site n and site 1 results in factor q!

§4. Periodic boundary conditions

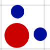
$$\Lambda^{\frac{\infty}{2},k} V \otimes \mathbb{C}(t) \longrightarrow \Lambda\, \mathbb{C}^n(t) = \bigoplus_{k=0}^{n} \Lambda^k \mathbb{C}^n(t)$$

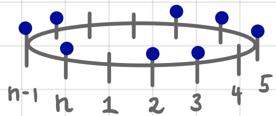
∞ wedge finite wedge



Denote by $T(x)$ the six-vertex transfer matrix with quasi-periodic boundary conditions and twist parameter q

$$T(x) = \sum_{r=0}^{n} x^r T_r\,, \quad T_r : \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t) \to \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t)$$





$\longmapsto v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, i_j positions of H-atoms •

Bethe ansatz: $T(x)$ is diagonalizable with eigenvalues

Bethe roots

$$\tau(x) = \left(1 + (-1)^k q\, x^n t^n\right) \prod_{i=1}^{k} \frac{1 - y_i x}{1 - t y_i x}\,, \qquad y_i^n = (-1)^{k-1} q$$

Define a 'renormalised' transfer matrix $H(x) : \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t) \to \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t)$

$$H(x) = \frac{T(x)}{1 + (-1)^k q\, x^n t^n} = \sum_{r\geq 0} x^r H_r$$

Cylindric Hecke characters

renormalised 6-vertex transfer matrix

DEF $q^{d}\chi_t^{\lambda/d/\mu}(\nu)(t-1)^{e} = \langle v_\lambda, H_{\nu_1}\cdots H_{\nu_e} v_\mu\rangle$, $\quad H(x) = \sum_{r\geq 0} x^r H_r$

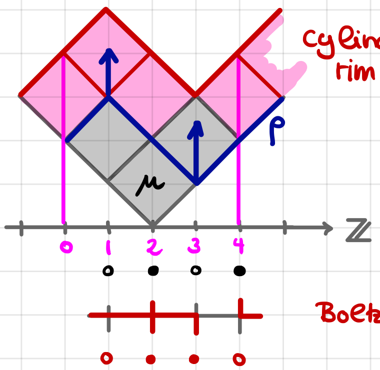
$d = \dfrac{|\mu|+|\nu|-|\lambda|}{n}$

LEMMA (Cylindric Murnaghan-Nakayama rule)

(i) $\chi_t^{\lambda/d/\mu}(\ldots,\nu_i+n,\ldots) = (-1)^{k-1}t^{n}\chi_t^{\lambda/d-1/\mu}(\ldots,\nu_i,\ldots)$

(ii) $\chi_t^{\lambda/d/\mu}(\nu, r) = \sum_\rho \chi_t^{\lambda/d/\rho}(\nu)\,\chi_t^{\rho/\mu}(r) + \underbrace{\sum_\rho \chi_t^{\lambda/d-1/\rho}(\nu)\,\chi_t^{\rho/1/\mu}(r)}_{\text{cylindric part}}$

The rules (i),(ii) allow one to compute $\chi_t^{\lambda/d/\mu}$.



Boltzmann weight $(t-1)(-1)t$

$n=4$, $k=2$ $\quad \chi_t^{(2,0)/1/(2,1)}(3) = (-1)^{2-1}t^{2-1}$

The coalgebra of cylindric Hecke characters

<u>Main Theorem</u> (C.K.) The cylindric Hecke characters $\chi_t^{\lambda/d/\emptyset}$ with $\lambda \subset \square_{n-k}^{k}$, $d \geq 0$ span an <u>∞-dimn'l</u> subcoalgebra of $\mathcal{R}_t$ (Grothendieck ring) with

$$\mathrm{Res}^{H_{m'+m''}}_{H_{m'}\otimes H_{m''}} \chi_t^{\lambda/d/\emptyset} = \sum_{d'+d''\leq d} \sum_{\substack{\mu \vdash m' \\ \nu \vdash m''}} C_{\mu\nu}^{\lambda, d-d'-d''} \, \chi_t^{\mu/d'/\emptyset} \otimes \chi_t^{\nu/d''/\emptyset}$$

where $C_{\mu\nu}^{\lambda,d} \in \mathbb{Z}_{\geq 0}$ are the 3-point genus 0 Gromov-Witten invariants of the Grassmannian $Gr_k(\mathbb{C}^n)$.

For $d=0$ one recovers $H^*(Gr_k(\mathbb{C}^n))$ as coalgebra with $C_{\mu\nu}^{\lambda,0} = c_{\mu\nu}^{\lambda}$ being the Littlewood-Richardson coefficients. (intersection cohomology)

Q. What is the image of cylindric Hecke characters under Macdonald's characteristic map?

Grothendieck ring of Hecke algebras

centre of $H_n(t)$

$$\begin{tikzcd}
\mathcal{R}_t = \bigoplus_{n \in \mathbb{N}} \mathcal{R}^n_t \arrow[rr, "\sim"] \arrow[drr, "ch_t"'] & & Z = \bigoplus_{n \in \mathbb{N}} Z(H_n) \arrow[d, "F_t"] \\
& & \mathrm{Sym}_t = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}^n_t = \mathrm{Sym}^n \otimes_{\mathbb{C}} \mathbb{C}(t)
\end{tikzcd}$$

quantum characteristic map

quantum Frobenius map [Wan-Wang]

ring of symmetric fctns

$\cup$

quantum cohomology of Grassmannians

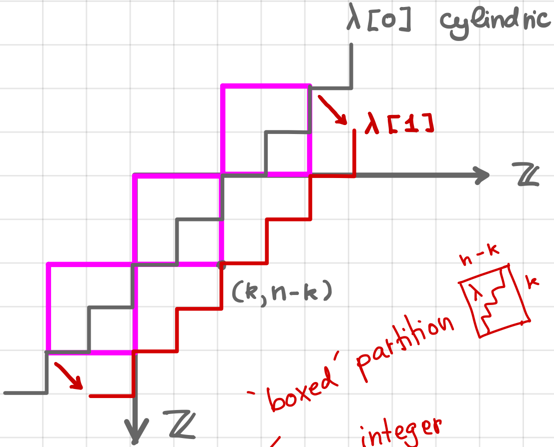
$qH^*(Gr_k(\mathbb{C}^n)) \leftsquigarrow$ **subcoalgebra of cylindric Schur functions**

(depend on $k \leq n$)

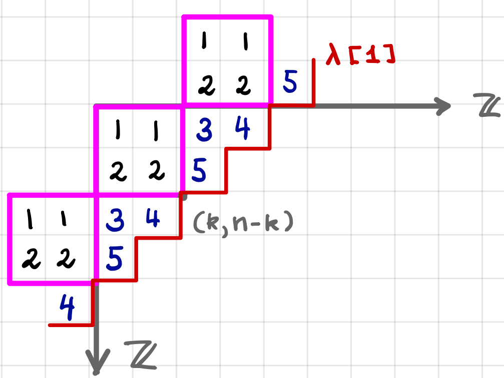
Combinatorial definition of cylindric Schur functions

[Gessel-Krattenthaler, Postnikov, Lam, McNamara ...] [D. Palazzo, C.K.]

↪ positive coalgebra



cylindric tableau



integer

<u>DEF</u> $\quad s_{\lambda/d/\emptyset} = \sum_{T} x^{T}$, T cylindric tableau of shape $\lambda[d]$

<u>THM</u> (C.K., D. Palazzo) Cylindric Schur functions form a positive subcoalgebra $\subset$ Sym

C.K. and D. Palazzo, Algebraic Combinatorics 3 (2020) 191-247

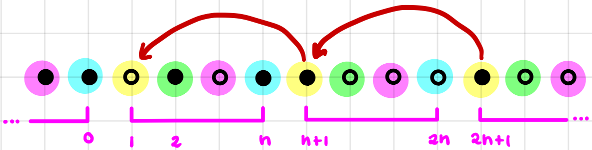
Rim hook algorithm & virtual Hecke characters

$$\begin{tikzcd}
\Lambda^{\frac{\infty}{2},k} V \otimes \mathbb{C}(t) \arrow[r, "\text{BF correspondence}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Sym}_t = \mathrm{Sym} \otimes_{\mathbb{C}} \mathbb{C}(t) \arrow[d, "\pi"'] \arrow[d, "\text{rim hook algorithm [Bertram, Ciocan-Fontanine, Fulton]}"] \\
\bigoplus_{d \geq 0} q^d \otimes \Lambda^k \mathbb{C}^n(t) \arrow[r, "\text{Satake correspondence}"'] & \mathbb{C}(t)[q, e_1, \dots, e_k, h_1, \dots, h_{n-k}]/\sim
\end{tikzcd}$$

[Ginzburg] [Golyshev, Manivel]

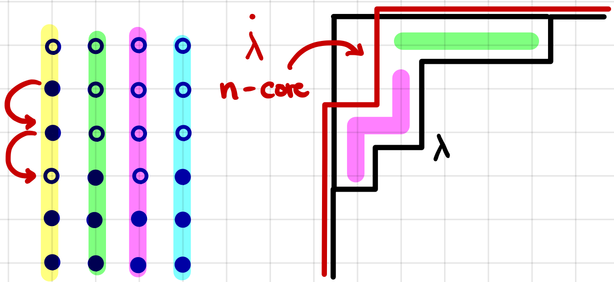
$$\left(\sum_{i=0}^{k} (-x)^i e_i\right)\left(\sum_{j=0}^{n-k} x^j h_j\right) = 1 + q(-1)^k x^n$$

Define a projection $\pi' : \Lambda^{\frac{\infty}{2},k} V \to \Lambda^k \mathbb{C}^n$ such that the above diagram commutes.



LEMMA $H(x) \circ \pi' = \pi' \circ A(x)$

periodic — open



COROLLARY $\chi_t^{\lambda/d/\emptyset} = \sum_{\mu = \lambda} \varepsilon(\mu/\lambda)\, \chi_t^{\mu}$

where $\varepsilon(\mu/\lambda) = (-1)^{kd + \sum_i \mathrm{width}(\rho_i)}$

removed n rim hook

A cylindric(al) Boson - Fermion correspondence

$$\begin{tikzcd}
\mathcal{R}_t = \bigoplus_{m\geq 0} \mathcal{R}_t^m \arrow[r, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\wr\ ch_t"] & \mathcal{Z}_t = \bigoplus_{m\geq 0} \mathcal{Z}(\mathcal{H}_m(t)) \arrow[d, "\wr\ F_t"] \\
\Lambda^{\frac{\infty}{2},c} V \otimes \mathbb{C}(t) \arrow[r, "\text{BF correspondence}"] \arrow[d, "c\,=\,k \bmod n"'] & \Lambda_t = \Lambda \otimes_{\mathbb{C}} \mathbb{C}(t) \arrow[d, "\text{rim hook algorithm}"] \\
\Lambda^k \mathbb{C}^n(t) \otimes \mathbb{C}[q] \arrow[r, "\text{Satake correspondence}"'] \arrow[ur, "J_t"] & qH^*(Gr_k(\mathbb{C}^n)) \otimes \mathbb{C}(t)
\end{tikzcd}$$

centres of Hecke algebras

Frobenius map

$qH^*(Gr_k(\mathbb{C}^n)) = \mathbb{C}[q, e_1, \ldots, e_k, h_1, \ldots, h_{n-k}]/\sim$

Bethe ansatz [C.K., Stroppel] $\Leftrightarrow \left(\sum_{i=0}^{k} (-x)^i e_i\right)\left(\sum_{j=0}^{n-k} x^j h_j\right) = 1 + q(-1)^k x^n$

cylindric boson-fermion correspondence

$$v_\lambda = v_{\lambda_1+k-1} \wedge \cdots \wedge v_{\lambda_k} \longmapsto \langle v_\lambda, H(x_1)H(x_2)\cdots v_\emptyset \rangle = \sum_{d\geq 0} q^d\, s_{\lambda'/d/\emptyset}[(t-1)X]$$

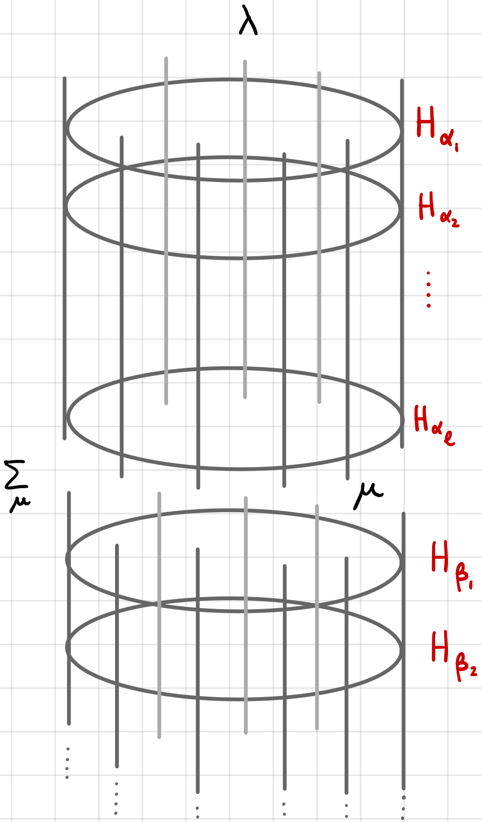
cylindric Schur functions

$$= \sum_{d\geq 0} q^d \sum_{\alpha} \chi_t^{\lambda/d/\emptyset}(\alpha)\,(t-1)^{\ell(\alpha)}\, m_\alpha[X]$$

cylindric Hecke characters

Proof (Sketch)

Use bra-ket notation: $\langle \lambda | \mathcal{O} | \mu \rangle \equiv \langle v_\lambda , \mathcal{O} v_\mu \rangle$



$$q^d \, \chi_t^{\lambda/d/\phi}(\alpha,\beta)\,(t-1)^{\ell(\alpha)+\ell(\beta)} = \langle \lambda | H_\alpha H_\beta | \phi \rangle$$

$$= \sum_\mu \langle \lambda | H_\alpha | \mu \rangle \langle \mu | H_\beta | \phi \rangle$$

$$= q^d (t-1)^{\ell(\alpha)+\ell(\beta)} \sum_{d'=0}^{d} \sum_\mu \underbrace{\chi_t^{\lambda/d-d'/\mu}(\alpha)}_{\text{skew cylindric Hecke character}} \chi_t^{\mu/d'/\phi}(\beta)$$

Use the Bethe ansatz (eigenbasis of H_α)

and Bertram-Vafa-Intriligator formula for $C_{\mu,\nu}^{\lambda,d}$

to show that $\chi_t^{\lambda/d/\mu} = \sum_{d'=0}^{d} \sum_\nu C_{\mu\nu}^{\lambda,d-d'} \chi_t^{\nu/d'/\phi}$

Proof (Sketch)

$$q^d \chi_t^{\lambda/d/\phi}(\alpha,\beta)(t-1)^{\ell(\alpha)+\ell(\beta)} = \langle \lambda | H_\alpha H_\beta | \phi \rangle = \sum_\mu \langle \lambda | H_\alpha | \mu \rangle \langle \mu | H_\beta | \phi \rangle$$

$$= q^d (t-1)^{\ell(\alpha)+\ell(\beta)} \sum_{d'=0}^{d} \sum_\mu \chi_t^{\lambda/d-d'/\mu}(\alpha)\, \chi_t^{\mu/d'/\phi}(\beta)$$

Employing eigenbasis of the H_r (Bethe ansatz)

$$q^d \chi_t^{\lambda/d/\mu}(\alpha)(t-1)^{\ell(\alpha)} = \sum_\xi \langle \lambda | H_\alpha \underbrace{|\xi\rangle\langle\xi|}_{\text{Bethe vectors}} \mu \rangle \,, \qquad \underbrace{\xi_i^n = (-1)^{k-1} q}_{\text{Bethe roots}}$$

$$= \sum_\xi h_\alpha[(t-1)\xi] \frac{s_\lambda(\xi)\, s_\mu(\xi^{-1})}{n^k \prod_{i<j} |\xi_i - \xi_j|^{-2}}$$

Bethe formula for cylindric Hecke characters

$$\sum_\nu \frac{s_\nu(\tilde\xi^{-1})\, s_\nu(\xi)}{n^k \prod_{i<j} |\tilde\xi_i - \tilde\xi_j|^{-2}} = \delta_{\xi\tilde\xi}$$

$$= \sum_\nu \underbrace{\left(\sum_{\tilde\xi} \frac{s_\lambda(\tilde\xi)\, s_\mu(\tilde\xi^{-1})}{n^k \prod_{i<j} |\tilde\xi_i - \tilde\xi_j|^{-2}} s_\nu(\tilde\xi^{-1}) \right)}_{q^{d-d'} C_{\mu\nu}^{\lambda, d-d'}} \underbrace{\sum_\xi h_\alpha[(t-1)\xi] \frac{s_\nu(\xi)}{n^k \prod_{i<j} |\xi_i - \xi_j|^{-2}}}_{q^{d'} \chi_t^{\nu/d'/\phi}(\alpha)}$$

[Bertram-Vafa-Intriligator]

THANK YOU FOR

YOUR ATTENTION!