Irreversible Markov Dynamics and Hydrodynamics for KPZ Phase in the Stochastic Six Vertex Model

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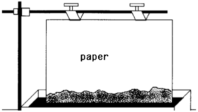
Outline

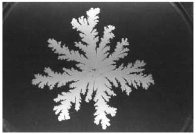
Surface Growth Models

Surface Growth Models

1. Models for surface growth are widely used in science and engineering to model various processes.
2. Examples include:



Fluid flow in a porous medium.



A bacterial colony

Figure: Natural examples of surface growth ([BS95]).

Surface Growth Models

Generally, we model the surface by a height function $h(x,t) = h_t(x)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, which is evolving in time.

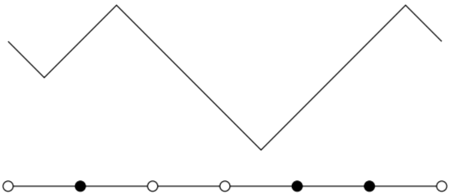


Figure: An ASEP configuration and its height function

KPZ Universality

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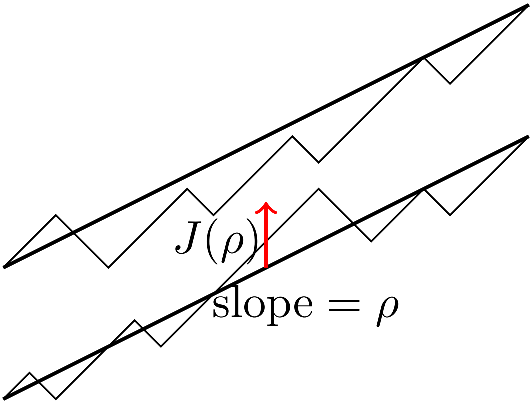
- **Smoothing**. The dynamics tends to force large fluctuations in the height function back towards the mean.
- **Slope dependent growth speed**. The average velocity of growth of the height function at a point only depends on its average slope around that point.
- **Space-time uncorrelated noise**. The randomness in the model comes from a random environment which is space-time uncorrelated, such as the white noise in the KPZ equation.

Current of a growth model

Suppose the averaged height function's slope near x is ρ. Then the *current* J, is defined by
$\overline{h}_{t+\Delta t}(x) - \overline{h}_t(x) = \Delta t J(\rho) +$ lower order.



Current and KPZ Equation

Hydrodynamic limit and fluctuations: We decompose the height function $h_t = \overline{h}_t + \tilde{h}_t$, where $\overline{h}$ is the mean height field and $\tilde{h}$ is the fluctuation field.

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2. For $\tilde{h}$, we have

$$\partial_t \tilde{h}_t = Q(\nabla \tilde{h}_t) + \nu \Delta \tilde{h}_t + \mathcal{W}$$

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3. It is conjectured that all of the surface growth models in the KPZ class are governed by the KPZ equation.

KPZ Equation in (d+1) dimensions

1. In (1+1) dimensions, the behavior of the solution to the KPZ equation is understood conjecturally ([KPZ86]), and expectations have been rigorously confirmed for several concrete KPZ growth models.
2. In (2+1) dimensions, it is conjectured (see [BT18]) that the large scale behavior depends on the signature of Q:
   - $\det Q \leq 0 \implies$ "Anisotropic KPZ class".
   - $\det Q > 0 \implies$ "Isotropic KPZ class" (this class has only been studied numerically).
3. In other words, the hessian of the current J determines the large scale behavior in (2+1) dimensions.

Previously studied models

1. In (1+1) dimensions: ASEP ([Lig05, TW09])
   - Bernoulli product measures are stationary, $\mathrm{Var}(h_t(x) - h_t(y))^{1/2} \propto |x-y|^{1/2}$ and $\mathrm{Var}(h_t(x) - h_0(x))^{1/2} \propto t^{1/3}$, current has simple explicit formula.
2. In (2+1) dimensions: Dynamics on dimer models ([Ton17, LT17, BF14])
   - Translation invariant Gibbs measures are stationary.
   - Asymptotically, in the stationary regime $\mathrm{Var}(h_t(x) - h_0(x))^{1/2} \propto \sqrt{\log t}$.
   - Current has been computed explicitly, and in particular $\det \mathrm{Hess}(J) \leq 0$.
   - In each (2+1) dimensional model above, the correlation structure at stationarity is *determinantal*.

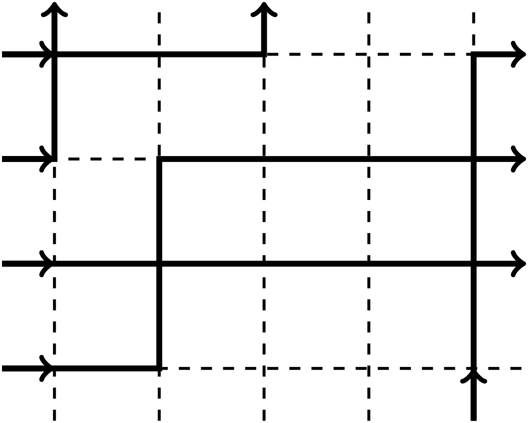
Bijectivisation dynamics

1. We will present a dynamics on height functions with stationary measures given by the Gibbs measures of the *Six Vertex Model*.
2. Unlike in the dimer case, the Gibbs measures of the six vertex model do not generally have a determinantal correlation structure.
3. In the case of *stochastic weights* we compute the current for the KPZ phase Gibbs measures.

Six Vertex Model

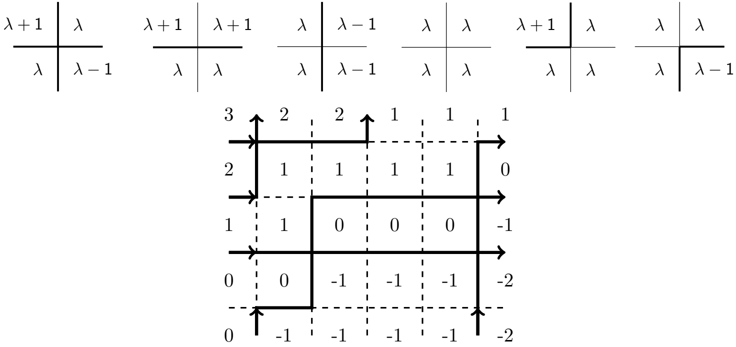
Non-intersecting path configurations

A *state* is a configuration of up-right paths in $\mathbb{Z}^2$ which can meet at a vertex but not cross or share an edge.



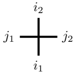
Height function

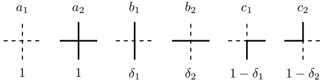
There is a correspondence between path configurations and height functions which satisfy certain local rules.



Stochastic Six Vertex Model

1. By "path conservation", there are six possible local configurations. Each has a *weight* $w(i_1, j_1; i_2, j_2)$.

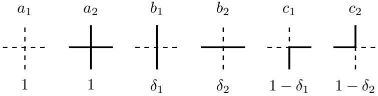




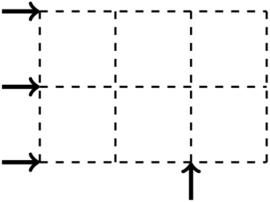
2. We consider *stochastic weights*: $a_1 = a_2 = 1, b_1 + c_1 = 1, b_2 + c_2 = 1$. We let

$$\delta_1 = b_1 \quad \delta_2 = b_2 \quad .$$

Markovian sampling of stochastic six vertex in quadrant



As a result, one can sample a configuration with some given entrance locations for paths and *free exit data* by sampling row by row in a Markovian way.



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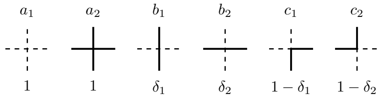

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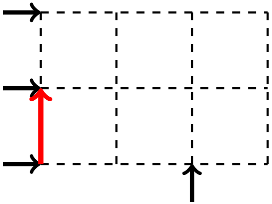

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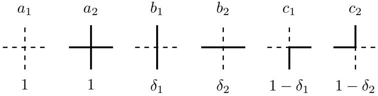

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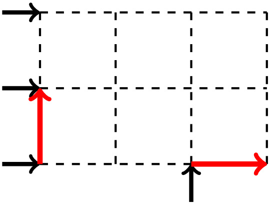

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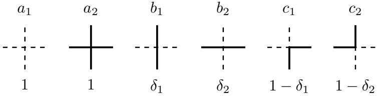

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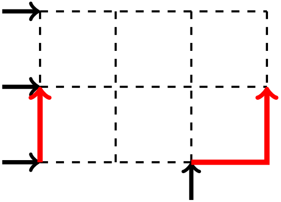

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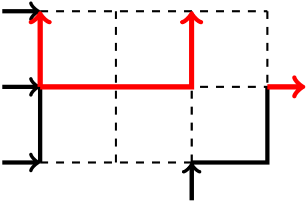

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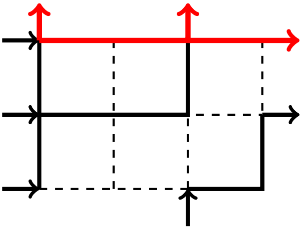
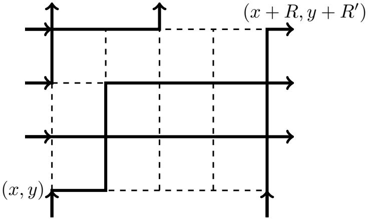


Six vertex model Gibbs measures

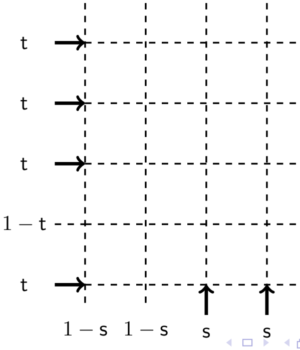
We say a probability measure μ on path configurations in $\mathbb{Z}^2$ is a *Gibbs measure* if the following holds: For any rectangle $\Lambda = \{x, x+1, \ldots, x+R\} \times \{y, y+1, \ldots, y+R'\}$, the conditional distribution of the state S inside Λ given some fixed boundary conditions on $\partial\Lambda$ is the *Boltzmann measure*

$$\mu(S \,|\, \text{boundary conditions}) = \frac{1}{Z} \prod_{v \in \Lambda} w(v, S) \quad .$$



Bernoulli entrance data in a quadrant

Now we define (s,t) *Bernoulli entrance data* for $0 < s,t < 1$. This is a probability measure on entrance locations for paths, under which incoming vertical edges have a path with probability s, and incoming horizontal edges have a path with probability t, all independently.



Full plane translation invariant stochastic six vertex Gibbs measure

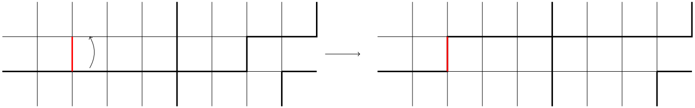
1. Let $\delta_1 < \delta_2$ define the stochastic weights. Given these, for each $0 < \rho < 1$ we define a translation invariant Gibbs measure.
2. Let $\varphi(\rho) = \frac{\rho}{u+\rho-u\rho}$, with $u = \frac{1-\delta_2}{1-\delta_1}$. Entrance data on any quadrant $\{x, x+1, \dots\} \times \{y, y+1, \dots\}$ is given by $(\rho, \varphi(\rho))$ Bernoulli entrance data.
3. Given the entrance data on any quadrant, the Markovian sampling procedure is used to sample the configuration in that quadrant, given the entrance data.
4. This defines a translation invariant Gibbs measure π_ρ, see [Agg18, Agg20] for details.

Continuous time dynamics

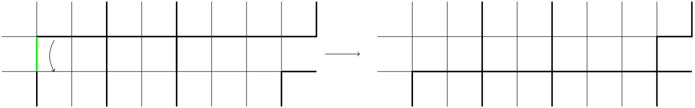
Now we define a Markov chain preserving π_ρ for each $0 < \rho < 1$.

Jumps

1. Occupied horizontal edges can jump up or down, and this jump propagates.
2. Propagation of up jump

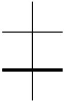
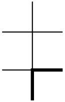
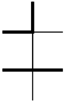


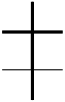
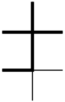
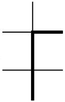
3. Propagation of down jump



Jump rates

The jump rates below depend on δ_1, δ_2 via $q = \frac{\delta_1}{\delta_2}$ and $u = \frac{1-\delta_1}{1-\delta_2}$. Any pair of vertically adjacent vertices in one of these local configurations can initiate a jump.

$R($$) = \mathfrak{c}$ $R($$) = \mathfrak{a}$ $R($$) = \mathfrak{b}$

$R($$) = \mathfrak{c}$ $R($$) = \mathfrak{a}$ $R($$) = \mathfrak{b}$

$$\mathfrak{c} := \frac{1-q}{(1-u)(1-qu)}, \qquad \mathfrak{b} := \frac{1-qu}{(1-u)(1-q)\,u},$$

$$\mathfrak{a} := \frac{(1-u)\,q}{(1-q)(1-qu)\,u}.$$

Full plane dynamics, stationarity of Gibbs measure, and current

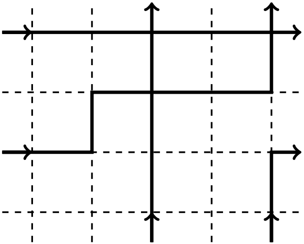
Theorem (M.N., L. Petrov, 2021)

- *The Markov chain is well defined from a set of states which is probability 1 under* π_ρ.
- *This Markov process preserves each translation invariant ergodic Gibbs measure* π_ρ.
- *Denote the current by* $J(\rho, u)$. *The current is given by*

$$J(\rho, u) = -\frac{\rho(1-\rho)}{(\rho + u - \rho u)^2} \quad .$$

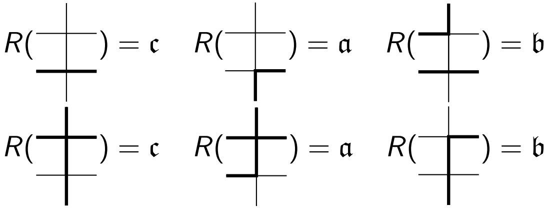
Torus Dynamics for General Weights

The state space $\mathcal{S}_{k_1,k_2}$ consists of up right path configurations on the lattice $\mathbb{T}_{M,N} = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ such that the height function has horizontal and vertical height change k_1, k_2.



Torus Dynamics for General Weights

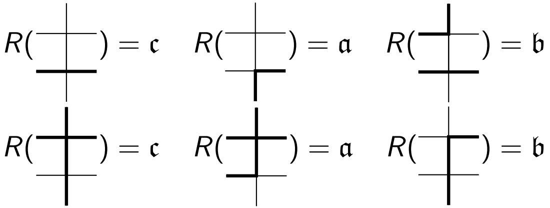
Let $a_1, a_2, b_1, b_2, c_1, c_2 > 0$. The jump rates $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ can be written in terms of the six vertex weights as follows:



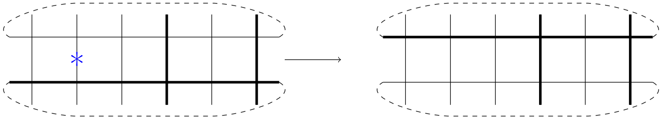
$$\mathfrak{c} = \frac{c_1 c_2}{\sqrt{b_1 b_2 a_1 a_2}}, \qquad \mathfrak{a} = \frac{\sqrt{b_1 b_2}}{\sqrt{a_1 a_2}}, \qquad \mathfrak{b} = \frac{\sqrt{a_1 a_2}}{\sqrt{b_1 b_2}}. \qquad (1)$$

Torus Dynamics

As before, jumps are initiated by vertically adjacent pairs.



Now propagation may "loop all the way around the torus". For example:



Torus Dynamics

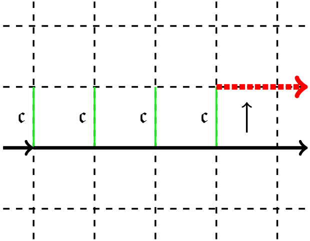
Let $\mu_{k_1,k_2}(S) = \frac{1}{Z} \prod_{v \in \mathbb{T}_{M,N}} w(v,S)$ denote the six vertex model Boltzmann measure on $\mathcal{S}_{k_1,k_2}$ given by the weights $a_1, a_2, b_1, b_2, c_1, c_2$.

Theorem (M.N., L. Petrov, 2021)

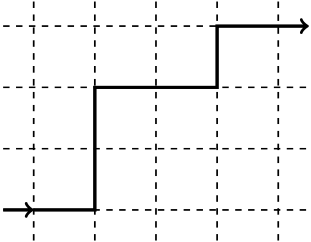
The torus dynamics preserves the measure μ_{k_1,k_2}.

See Borodin-Bufetov ([BB17]) for a different Markov chain preserving μ_{k_1,k_2}.

A Comment on Full Plane Dynamics



- Bad initial configuration

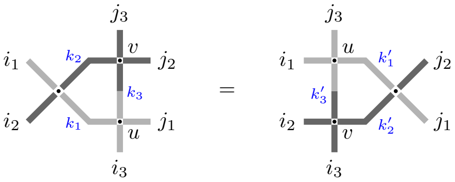


- Good initial configuration

Construction of dynamics: Bijectivisation of Yang–Baxter equation

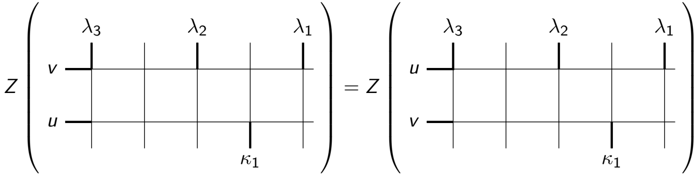
Yang–Baxter Equation

1. In general, stochastic weights can be parameterized by two parameters, the *spectral parameter* $0 < u < 1$, and another parameter q. Two weight functions with the same q value and different spectral parameters u and v can be related by the *Yang Baxter Equation*.
2. The picture on each side represents a partition function, which is a summation over possible values of internal edges, with boundary conditions fixed.



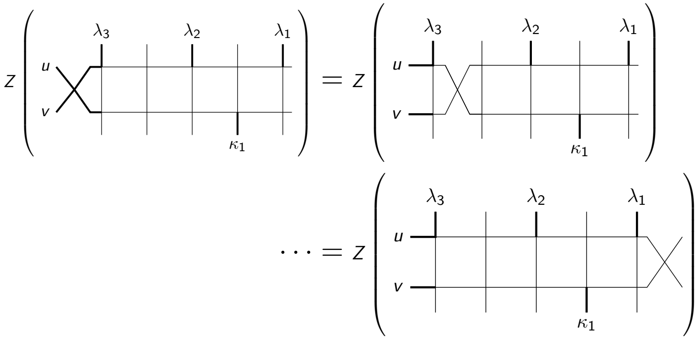
Swapping spectral parameters for two row partition function

From Yang–Baxter, we have the equality of the following partition functions.



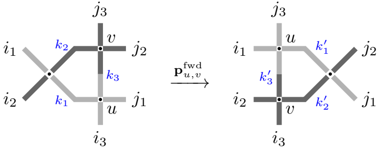
Swapping spectral parameters for two row partition function

Adding a fully occupied cross on the left does not change the partition function. Each time we drag it past a column, the partition function stays the same by the Yang–Baxter equation.



Bijectivisation of YBE

1. Our goal is to instead drag the cross through via random updates, and at each step have an equality of measures, instead of equality of partition functions.

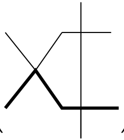


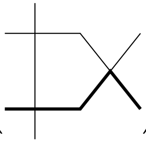
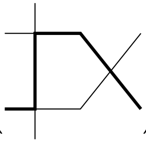
2. We would like transition probabilities $\mathbf{p}^{\text{fwd}}_{u,v}(S' \to S)$ that allow us to sample a configuration with the cross on the other side, in such a way that the transition is *weight preserving*:

$$\sum_{S'} \text{wt}_{u,v}(S')\mathbf{p}^{\text{fwd}}_{u,v}(S' \to S) = \text{wt}_{v,u}(S) \quad .$$

Bijectivisation of YBE

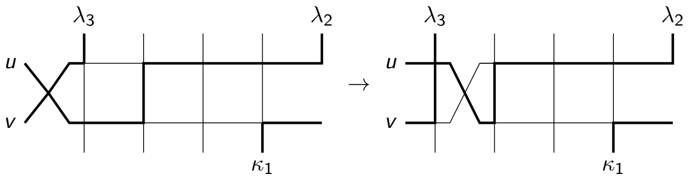
The kernel $\mathbf{p}^{\text{fwd}}_{u,v}$ allows us to sample one of the diagrams on the right, given the one on the left. An example where $\mathbf{p}^{\text{fwd}}_{u,v}$ is non-trivial:

$\mathrm{wt}_{u,v}$ () $=$

$\mathrm{wt}_{v,u}$ () $+\ \mathrm{wt}_{v,u}$ ()

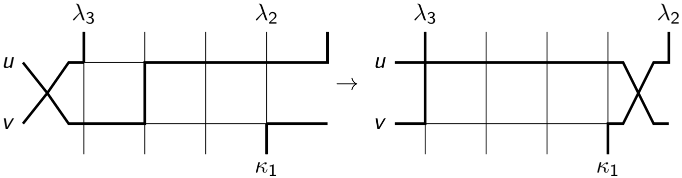
Row update

If we have two paths coming from the left, we can add a cross, and then we can update the configuration by dragging the cross through using the transition probabilities $\mathbf{p}^{\mathrm{fwd}}_{u,v}$.



Row update

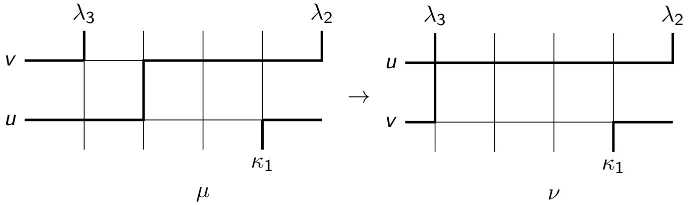
If we have a two row lattice, can update the configuration by dragging the cross through using the transition probabilities $\mathbf{p}^{\mathrm{fwd}}_{u,v}$.



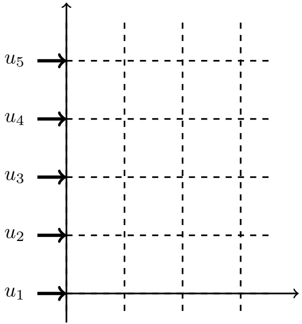
Swapping spectral parameters

Let $U^{\lambda,\kappa}_{u,v}(\mu \to \nu)$ denote the transition probabilities arising from moving the cross through randomly. If $P^{\lambda,\kappa}_{u,v}$ denotes the Boltzmann measure on a two row lattice with spectral parameter u on the bottom row and v on the top row, and boundary conditions specified by κ and λ, then

$$P^{\lambda,\kappa}_{u,v} U^{\lambda,\kappa}_{u,v} = P^{\lambda,\kappa}_{v,u} \quad .$$



Consider the step initial condition in the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. If the spectral parameters are $\mathbf{u} = (u_1, u_2, \dots)$, use the Markovian sampling procedure with u_i on row i to sample the configuration in the quadrant. We denote the resulting measure by $\mathbb{P}_{\mathbf{u}}$. If $u_i \equiv u$, we denote this measure by $\mathbb{P}_u$.



Sending u_1 up to infinity

$$
\begin{matrix} u_4 \\ u_3 \\ u_2 \\ u_1 \end{matrix}
\xrightarrow{U_{u_1,u_2}}
\begin{matrix} u_4 \\ u_3 \\ u_1 \\ u_2 \end{matrix}
\xrightarrow{U_{u_1,u_3}}
\begin{matrix} u_4 \\ u_1 \\ u_3 \\ u_2 \end{matrix}
\xrightarrow{U_{u_1,u_4}}
\cdots
$$

Sending u_1 up to infinity

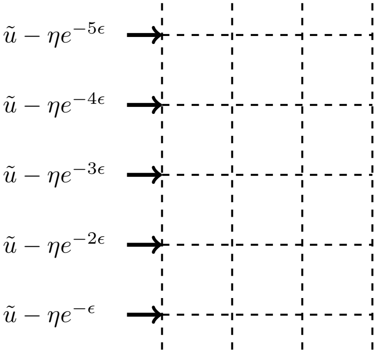
1. We can take a limit, taking u_1 up to "infinity" where it disappears.
2. Let $S(u_1, u_2, \dots) := (u_2, u_3, \dots)$. We obtain a Markov operator $L[\mathbf{u}]$ such that

$$\mathbb{P}_{\mathbf{u}} * L[\mathbf{u}] = \mathbb{P}_{S\mathbf{u}} \quad .$$

Continuous time limit

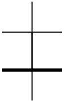
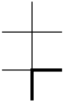
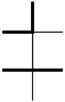
1. When transitioning according to $U_{u,v}$, the probability that any jump happens is proportional to $v - u$.
2. So if we set $u_i = \tilde{u} - \eta \exp(-\epsilon i)$, then on the k^{th} row, any transition probability will be proportional to $-\eta(\exp(-\epsilon(k+1)) - \exp(-\epsilon)) = \eta k\epsilon + O(\epsilon^2)$.

Choosing spectral parameters

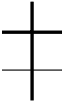
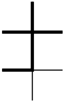
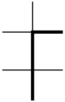


Continuous time limit

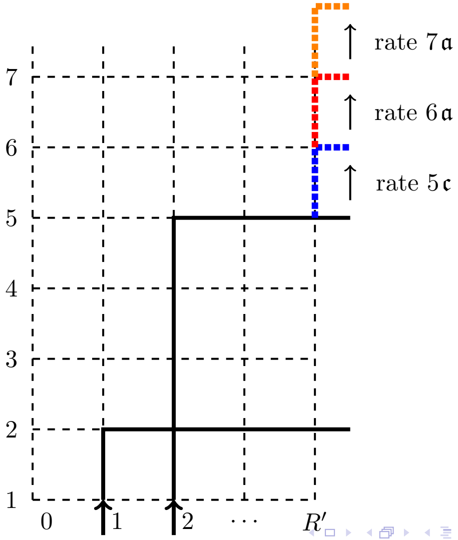
Make the variable change $t \to \tau/\epsilon$. The transition rates for vertical edge at height k as $\epsilon \to 0$:

$R($$) = k\mathfrak{c}(u)$ $R($$) = k\mathfrak{a}(u)$ $R($$) = k\mathfrak{b}(u)$

and

$R($$) = k\mathfrak{c}(u)$ $R($$) = k\mathfrak{a}(u)$ $R($$) = k\mathfrak{b}(u)$

Continuous time limit



Continuous time Limit

The limiting continuous time Markov semigroup $\mathcal{Q}_\tau$ satisfies

$$\mathbb{P}_u \mathcal{Q}_\tau = \mathbb{P}_{u+(1-e^{-\tau})\eta}. \tag{2}$$

1. Running this for time τ corresponds to doing $\lfloor \tau/\epsilon \rfloor$ steps of the discrete chain. We have

$$\mathbb{P}_{\mathbf{u}} \mathbf{Q}_{\lfloor \tau/\epsilon \rfloor} = \mathbb{P}_{\mathbf{u}[\tau]}, \qquad \mathbf{u}[\tau]_i := u + \left(1 - e^{-\epsilon(i+\lfloor \tau/\epsilon \rfloor)}\right) \eta.$$

2. As $\epsilon \to 0$, the parameters u_i become all equal to u, and $\mathbf{u}[\tau]_i$ become all equal to $u + (1 - e^{-\tau})\eta$.

Limit shape of stochastic six vertex model

The normalized height function converges to a deterministic limit: $\mathcal{H}(x, y) = \lim \frac{1}{N} H(Nx, Ny)$ exists in probability (see Borodin-Corwin-Gorin [BCG16].).

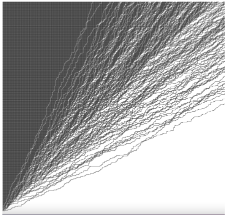
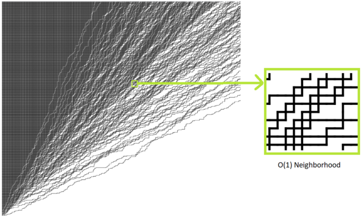


Figure: From L. Petrov (https://lpetrov.cc/2015/03/Spin-models/)

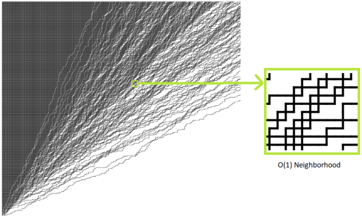
Local statistics and bulk limit

1. By the theorem, we know for each time τ, the normalized height function converges to a deterministic limit $\mathcal{H}(x, y, \tau)$. Zoom in on $(x/\epsilon, y/\epsilon)$, when lattice scale is ϵ. Local statistics here are given by the ergodic Gibbs measure with slopes $\nabla\mathcal{H}(x, y, \tau)$.



Local statistics and bulk limit

1. By the theorem, we know for each time τ, the normalized height function converges to a deterministic limit $\mathcal{H}(x, y, \tau)$. Zoom in on $(x/\epsilon, y/\epsilon)$, when lattice scale is ϵ. Local statistics here are given by the ergodic Gibbs measure with slopes $\nabla\mathcal{H}(x, y, \tau)$.
2. If we run the chain at speed ϵ, the jump rates around $(x/\epsilon, y/\epsilon)$ are converging to those of the full plane chain. Furthermore, after a finite time the Gibbs measure here is approximately preserved.



Hydrodynamic limit of dynamics

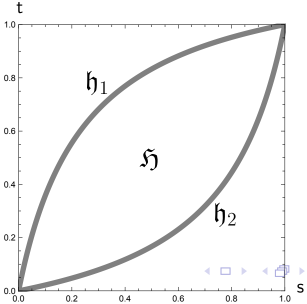
We expect that $\mathcal{H}$ satisfies a PDE of the form

$$\partial_\tau \mathcal{H} = e^{-\tau} y J(\nabla \mathcal{H}, u(\tau)) \ . \tag{3}$$

1. With step initial condition: $\mathcal{H}(x, y, \tau) = \frac{(\sqrt{y/u(\tau)} - \sqrt{x})^2}{1/u(\tau) - 1}$ ([BCG16]).
2. This $\mathcal{H}(x, y, \tau)$ indeed satisfies the equation (3).

Further directions

1. How does the height function evolve if we start from some fixed initial configuration, such that the local slopes (s, t) always satisfy $\mathsf{t} = \varphi(\mathsf{s})$?
2. What is the current, at points in the phase diagram away from the curve $\mathsf{t} = \varphi(\mathsf{s})$? Or for other choices of weights? (hard)
3. Is this model isotropic or anisotropic KPZ growth?



End!

Thank You!

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