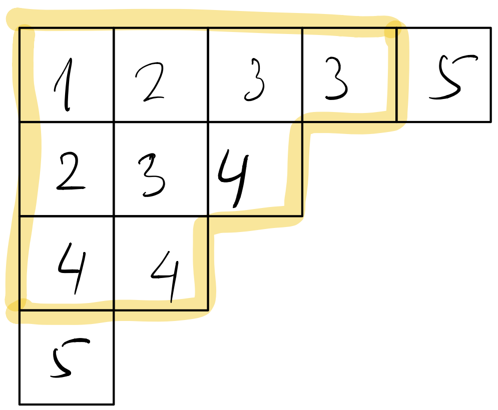
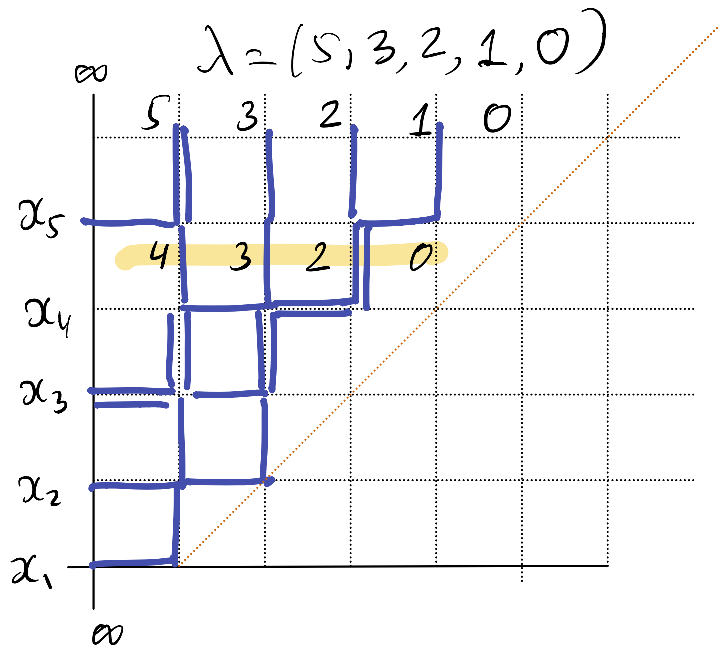
VERTEX MODELS & STOCHASTIC PARTICLE SYSTEMS

[Mucciconi – P. 2020]
https://arxiv.org/abs/2003.14260

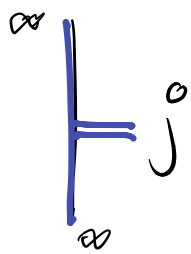
1. Vertex model for (spin) q-Whittaker

1.1. Semistandard tableaux

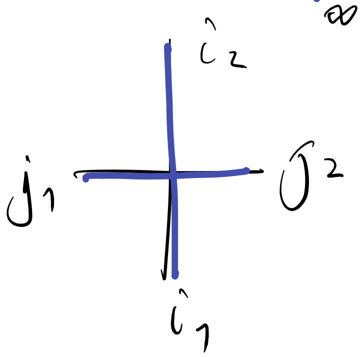




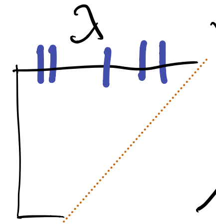
Schur weights :

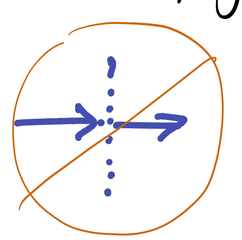


x^{j}



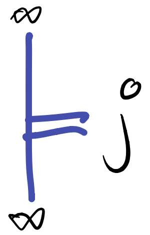
$x^{j_2} \mathbb{1}_{i_1+j_1=i_2+j_2} \mathbb{1}_{i_1 \geq j_2}$

$S_\lambda(x_1, \ldots x_N) = Z\Big($  $\Big).$

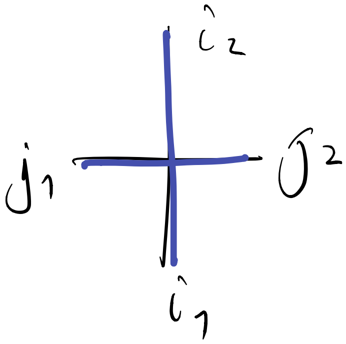


spin q-Whittaker weights

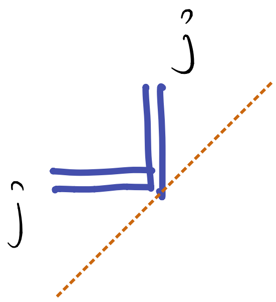
$$\boxed{(q,s)} \xrightarrow{s=0} \boxed{q\text{-Whittaker}} \xrightarrow{q=0} \boxed{\text{Schur}}$$



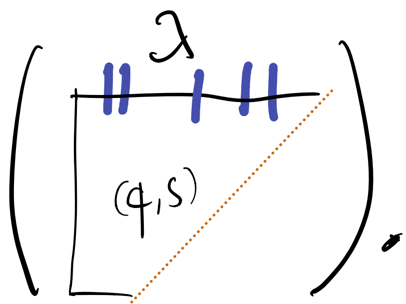
$$x^j \frac{(-s/x;q)_j}{(q;q)_j}$$



$$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}\, x^{j_2} \frac{(-s/x;q)_{j_2}(-sx;q)_{i_1-j_2}(q;q)_{i_2}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(s^2;q)_{i_2}}$$



$$\frac{(q;q)_j}{(-s/x;q)_j}$$

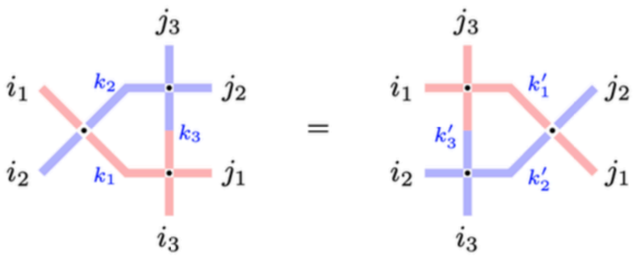
$$\mathbb{F}_\lambda(x_1,\ldots,x_N\,|\,q,s) = Z\Big($$  $$\Big).$$

$$\mathbb{F}_{\lambda/\mu}(x) := x^{|\lambda|-|\mu|}\prod_{i=1}^{k}\frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}$$

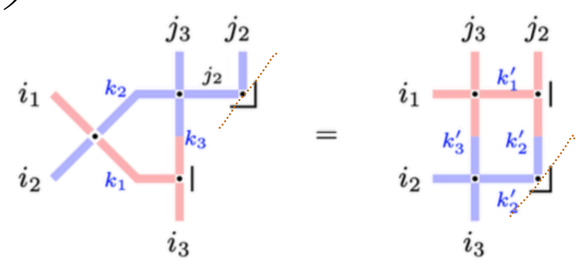
$$0\leq\lambda_{k+1}\leq\mu_k\leq\lambda_k\leq\ldots\mu_1\leq\lambda_1,\qquad \lambda_i,\mu_i\in\mathbb{Z}$$

1.2. Yang-Baxter Equation for Cauchy Id.

$(q=s=0)$

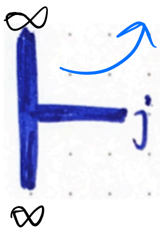


(a)



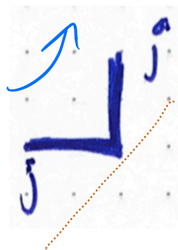
(b)

$$\mathbb{R}_{x,y} = \mathbf{1}_{i_1+j_2=j_1+i_2}(xy)^{\min(i_2,j_2)}$$

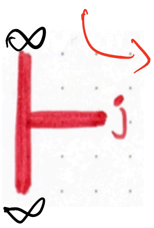


x^j

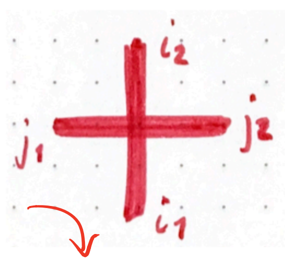
$\mathbf{1}_{i_1+j_1=i_2+j_2}\mathbf{1}_{i_1\geq j_2}\, x^{j_2}$



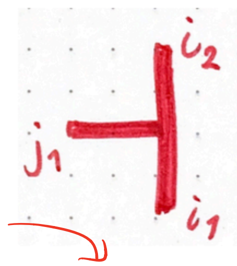
1



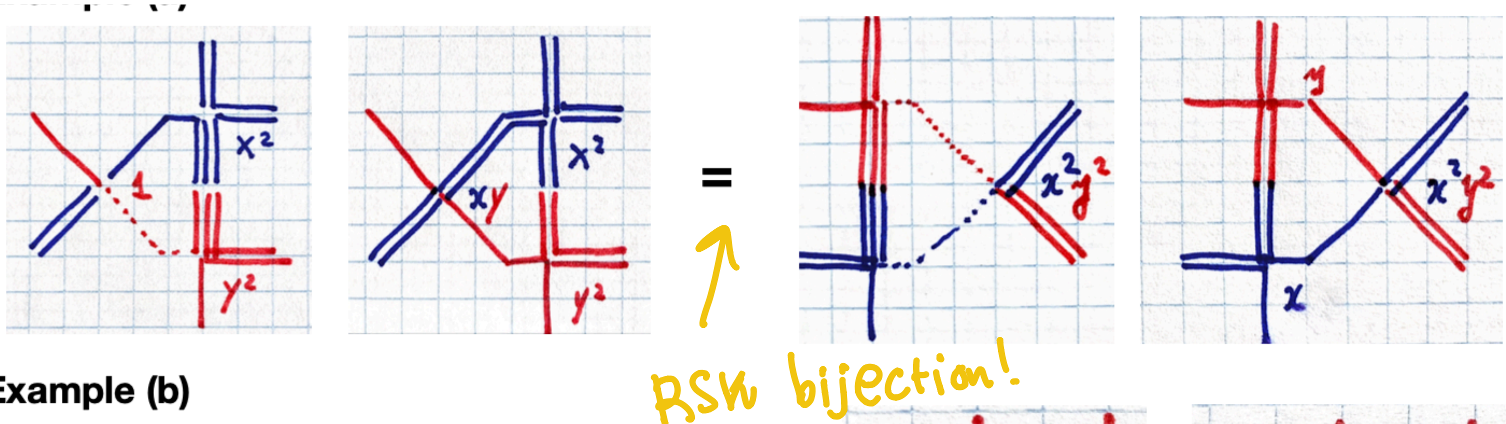
x^j



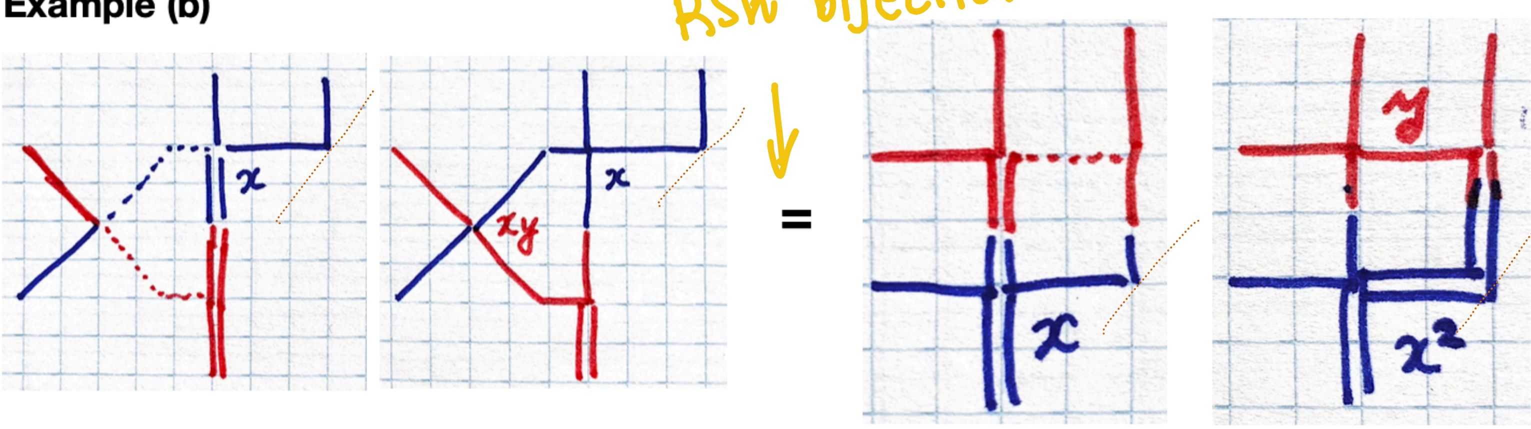
$\mathbf{1}_{i_2+j_1=i_1+j_2}\mathbf{1}_{i_2\geq j_2}\, x^{j_2}$



$\mathbf{1}_{i_1=j_1+i_2}$

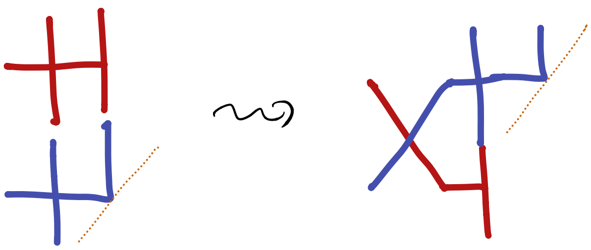


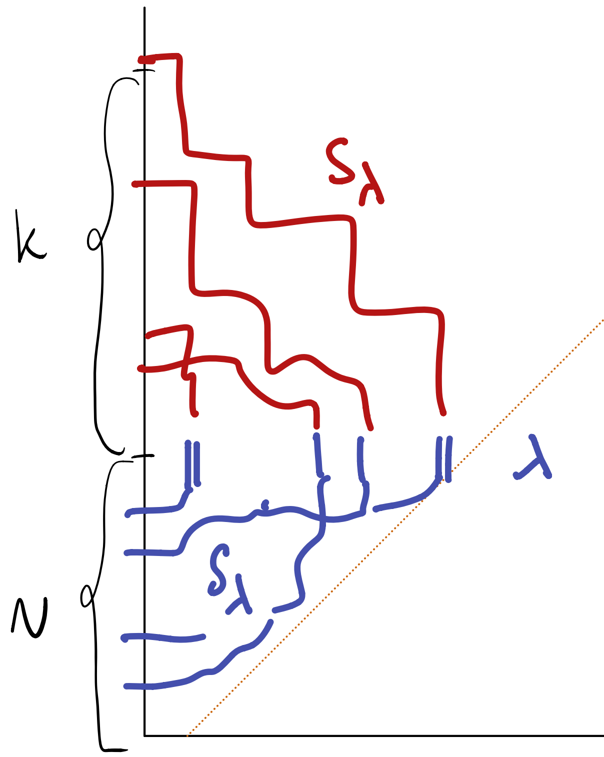
Example (b)



Thm. $$\sum_{\lambda_1\ge\dots\ge\lambda_N\ge 0} s_\lambda(x_1\dots x_N)\, s_\lambda(y_1\dots y_k) = \prod_{i,j}\frac{1}{1-x_iy_j}$$

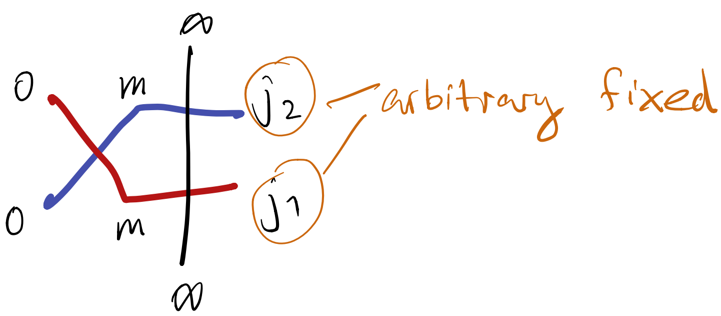
Proof. **Step 1**: Cross is born on the right:





Step 2. Move cross to left

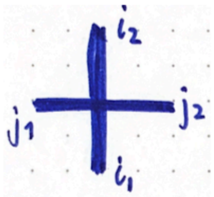
Step 3 Remove cross:



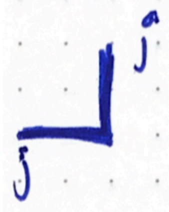
$$\sum_{m\ge 0}(xy)^m = \frac{1}{1-xy} \qquad \square$$

1.3. (s,q) weights & Cauchy (No RSK bijections)

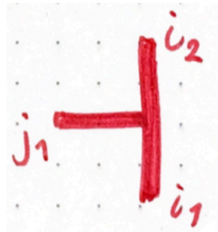
$$x^j \frac{(-s/x;q)_j}{(q;q)_j}$$



$$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}\, x^{j_2}\frac{(-s/x;q)_{j_2}(-sx;q)_{i_1-j_2}(q;q)_{i_2}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(s^2;q)_{i_2}}$$



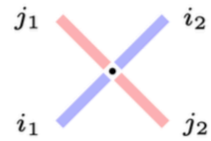
$$\frac{(q;q)_j}{(-s/x;q)_j}.$$



$$\mathbf{1}_{i_1=j_1+i_2}$$

$$x^j \frac{(-s/x;q)_j}{(q;q)_j}$$

$$\mathbf{1}_{i_2+j_1=i_1+j_2}\mathbf{1}_{i_2\geq j_2}\frac{y^{j_2}(q;q)_{i_2}(-s/y;q)_{j_2}(-sy;q)_{i_2-j_2}}{(q;q)_{i_2-j_2}(q;q)_{j_2}(s^2;q)_{i_2}}$$



$$\mathbb{R}_{x,y,s}(i_1,j_1;i_2,j_2) := \mathbf{1}_{i_2+j_1=i_1+j_2}\frac{q^{i_2 i_1+\frac{1}{2}j_2(j_2-1)}(sx)^{j_2}(q;q)_{j_1}}{(s^2;q)_{j_1+i_2}(q;q)_{j_2}(q;q)_{i_2}(-q/(sx);q)_{i_1-j_1}}$$

$$\times\ {}_4\overline{\phi}_3\left(\begin{matrix} q^{-i_2};q^{-i_1},-sy,-q/(sx) \\ -s/x, q^{1+j_2-i_2}, -yq^{1-i_1-j_2}/s \end{matrix}\,\Big|\, q,q\right).$$

$${}_4\overline{\phi}_3\left(\begin{matrix} q^{-n} & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3\end{matrix};q,z\right)$$
$$=\sum_{j=0}^{n} z^j\frac{(q^{-n};q)_j}{(q;q)_j}(a_1,a_2,a_3;q)_j\,(q^j b_1,q^j b_2,q^j b_3;q)_{n-j}.$$

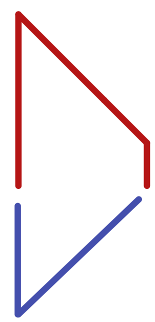
$\Rightarrow$ Cauchy:

$$\sum_{\lambda_1\geq\dots\geq\lambda_N\geq 0} \mathbb{F}_\lambda(x_1,\dots,x_N)\,\mathbb{F}^*_\lambda(y_1,\dots,y_K) =$$
$$= \prod_{j=1}^{K}\left(\frac{(-sy_j;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i,j}\frac{(-sx_i;q)_\infty}{(x_iy_j;q)_\infty}$$

Step 3: $$\sum_{m\geq 0}(xy)^m\frac{(-s/x,-s/y;q)_m}{(s^2,xy;q)_m} = \frac{(-sx,-sy;q)_\infty}{(s^2,xy;q)_\infty}$$

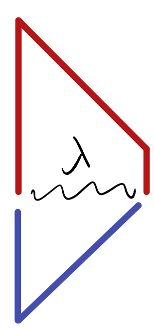
(q-Gauss)

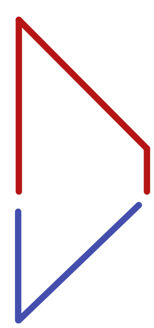
Summary

- (s,q) weights
- YBE $\Longrightarrow$ Cauchy
  $(s=q=0$: Schur/RSK$)$
- $Z\Big($  $\Big) =$ product

Next:

probability distribution of λ and λ_N in

$P\Big($ 

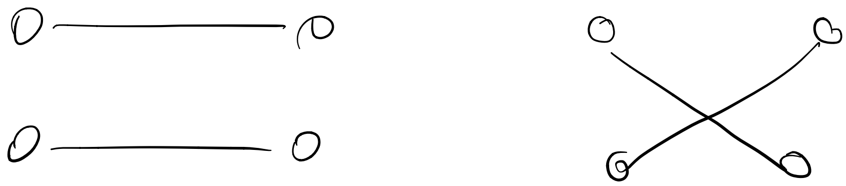
$\Big)$ / $Z\Big($  $\Big)$

2. From Cauchy / YBE to probability

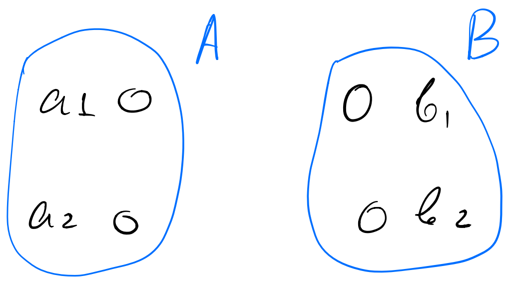
2.1. General formalism

("bijectivisation", "probabilistic bijection")

bijection



weighted spaces



$w(a_1)+w(a_2) =$
$= \hat{w}(b_1)+\hat{w}(b_2)$

(no weight-preserving bijection!)

we find $p(a_i \to b_j)$, $\hat{p}(b_j \to a_i)$ s.t.

$$w(a_i)\, p(a_i \to b_j) = \hat{w}(b_j)\, \hat{p}(b_j \to a_i)$$

Ex.

	1	3
2		
2		

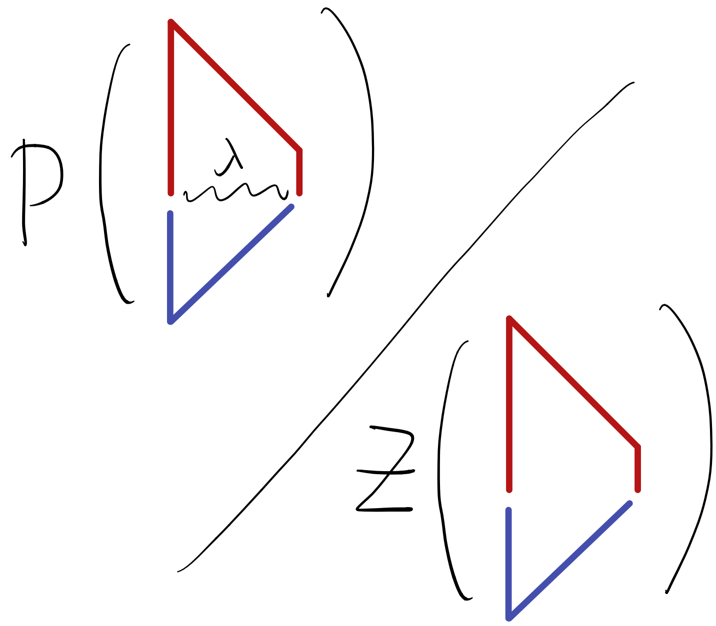
2.2 Application to YBE

YBE is an identity $\sum_a w(a) = \sum_b \hat{w}(b)$

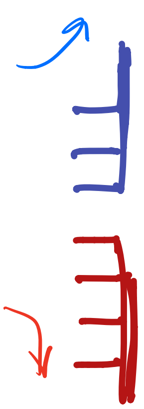
$(w, \hat{w} \geq 0)$

$\Rightarrow$ can choose a bijectivisation

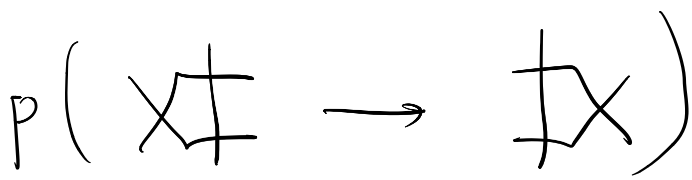
Goal: generate a measure by „elementary steps"



Start from

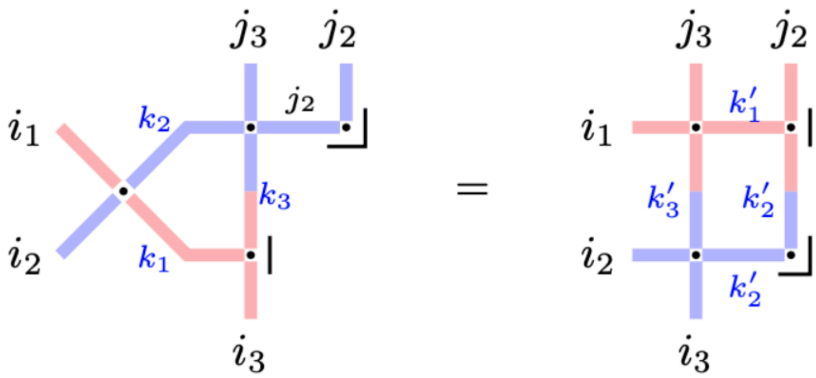


& apply bij from LHS to RHS



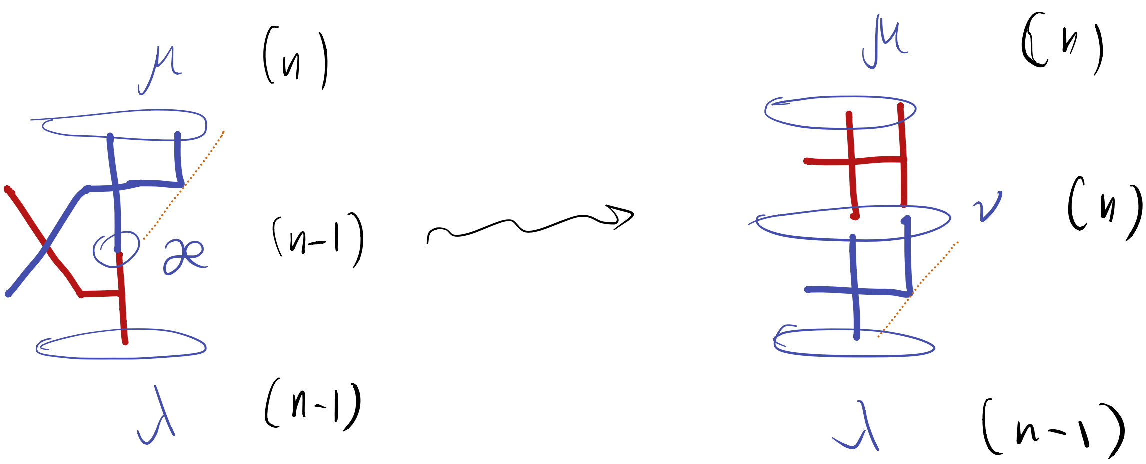
At right edge the behavior is independent of the rest

prob(LHS → RHS)



$= L(j_3 - i_2 + i_1 - k_1,\ k_1;\ j_3 - i_2 + i_1 - k_1',\ k_1')$

(ex. $L(\alpha_1,\beta_1;\alpha_2,\beta_2) = \mathbb{1}_{\alpha_1+\beta_1=\alpha_2+\beta_2}\, \mathbb{1}_{\beta_2 \le \alpha_1}\, (xy)^{\beta_2}\, (xy;q)_{\alpha_1-\beta_2} \begin{bmatrix}\alpha_1\\ \beta_2\end{bmatrix}_q$)
(s=0)



$$i_1 = \nu_{n-1} - \mu_{n-1}, \qquad i_2 = \nu_{n-1} - \lambda_{n-1}, \qquad j_3 = \mu_{n-1} - \mu_n,$$
$$k_1 = \lambda_{n-1} - \varkappa_{n-1}, \qquad k_1' = \nu_n - \mu_n.$$

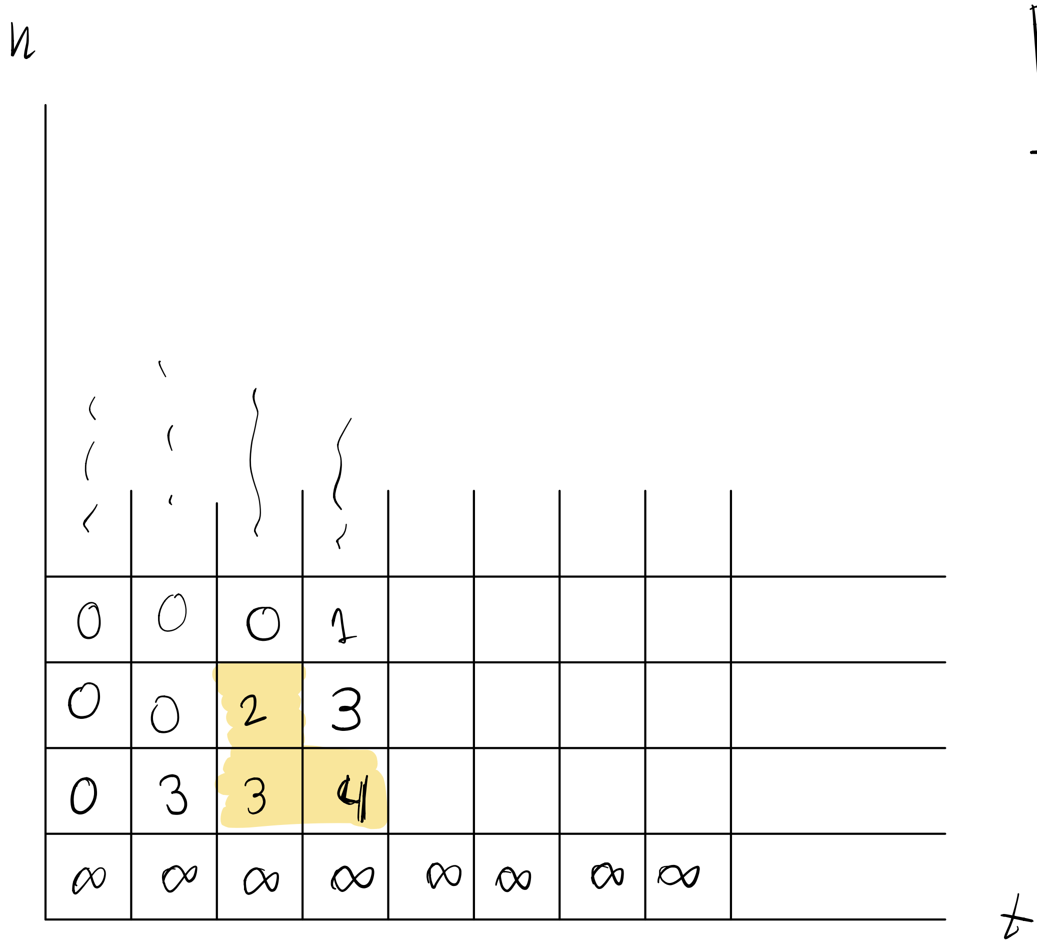
$$j_3 - i_2 + i_1 = \lambda_{n-1} - \mu_n$$

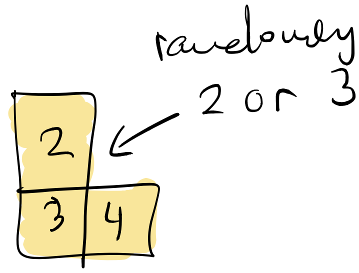
⇒ get a Marginally Markov evolution of last parts of λ's

$\Rightarrow$

Random field : $\lambda_n^{(n,t)} \in \mathbb{Z}_{\geq 0}$ $\rightsquigarrow$ Part. syst.

$$\boxed{x_n(t) = \lambda_n^{(n,t)} - n}$$

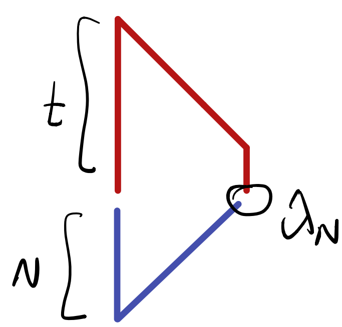




Summary Bijectivisation of YBE leads to random recursions (with values in $\mathbb{Z}_{\geq 0}$)

particle systems

for certain edge arrow counts :



Key: Evolution of partial information is independent of the rest (marginal Markov property)

3. Scaling $q \to 1$

3.1. Limit regime

$$\mathbb{F}_\lambda(x_1, \dots, x_N \mid q, s)$$

$0 < q < 1$, $q \to 1$

$x_i = q^{X_i}$, $L_i = q^{-\lambda_i}$

$s = -q^{S}$

$S > 0$, $|X_i| < S$,

$1 \leq L_N \leq \dots \leq L_2 \leq L_1$

Thm [MP '20]

$$\frac{\mathbb{F}_\lambda(x_1 \dots x_N)}{(-\log q)^{\frac{N(N-1)}{2}}} \longrightarrow f_{X_1 \dots X_N}(\underline{L}_N)$$

„spin Whittaker"

To gl_N Whittaker functions

$$L_i = S^{N+1-2i} e^{u_i}, \quad X_k = -i\lambda_k$$

$$\left(\frac{4\pi}{S \cdot 16^S}\right)^{\frac{N(N-1)}{4}} f_{\underline{X}}(\underline{L}) \xrightarrow{S \to \infty} \psi_{\underline{\lambda}}(\underline{u})$$

Examples

$f_x(L) = L^{-x}$ $(N=1)$

$(1 \leq L_2 \leq L_1)$ $\quad f_{x,y}(L_2, L_1)$ $(N=2)$

$$= \left(\frac{L_1}{L_2}\right)^s L_2^{-x-y} \, {}_2F_1\left(\begin{matrix} s+x, \; s+y \\ 2s \end{matrix} \middle| \, 1 - \frac{L_1}{L_2} \right)$$

$$\begin{aligned}
\mathfrak{f}_{x,y,z}(L_3, L_2, L_1) = {} & \frac{\Gamma(2s)}{\Gamma(s+x)\Gamma(s+y)} \frac{\Gamma(2s)}{\Gamma(s-z)\Gamma(s+z)} \frac{\Gamma(2s)}{\Gamma(s+z)} L_3^{-2s-x-y-z} L_2^s L_1^s \\
& \times \frac{1}{(2\pi \mathbf{i})^3} \int_{(\mathbf{i}\mathbb{R})^3} dt \, du_1 \, du_2 \, (L_1 - L_2)^{u_1} (L_2 - L_3)^{t-u_1+u_2} L_3^{-t-u_2} \\
& \times \frac{\Gamma(t+s+x)\Gamma(t+s+y)\Gamma(u_1-t)\Gamma(u_1+s+z)\Gamma(-u_1)}{\Gamma(t+2s)\Gamma(2s+u_1)} \\
& \times \frac{\Gamma(s+t-u_1-z)\Gamma(2s+t+x+y+u_2)\Gamma(s+z+u_2)\Gamma(-u_2)}{\Gamma(2s+t+x+y)\Gamma(2s+t-u_1+u_2)}.
\end{aligned}$$

etc?

3.2. Properties

1) Givental-type $\int$
2) Cauchy (- Bump - Stade - ...) identity
3) (conjectural) orthogonality
4) eigenoperators in x_p
5) Toda-like Hamiltonians in L_j

1)

Let $x_1, \dots, x_N, s \in \mathbb{R}$ be such that

("Combinatorial formula"; "spin Givental integral")

$$|x_i| < s \quad , \quad s > 0$$

"labels"

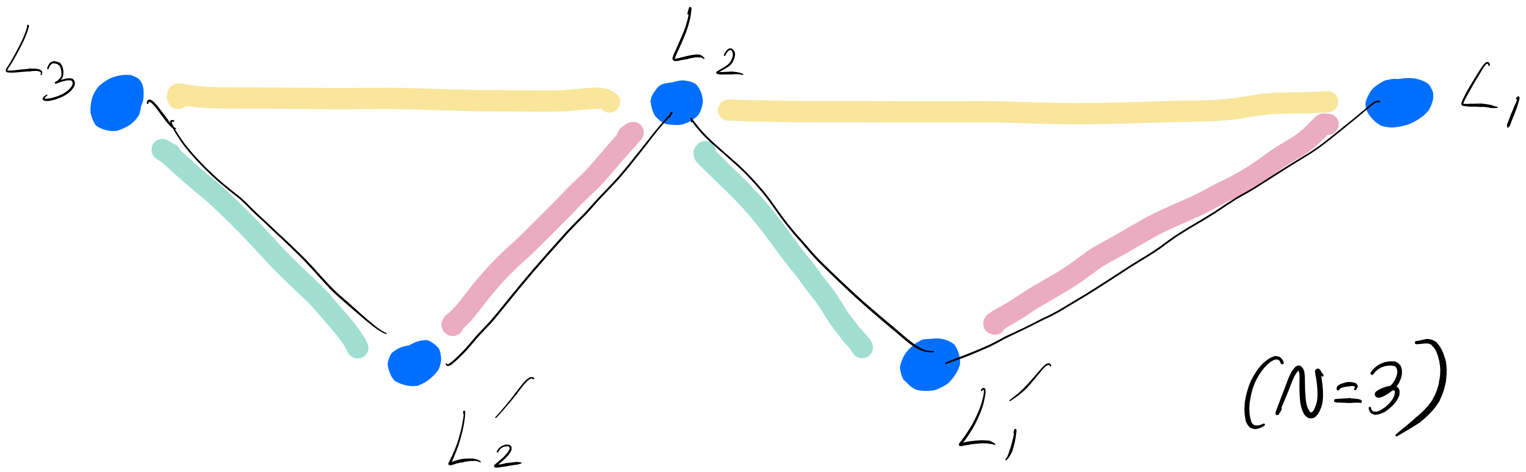
$$\underbrace{f_{x_1,\dots,x_N}}_{\text{"variables"}}(L_N, \dots, L_1) \doteq \int_{\vec{L}' < \vec{L}} f_{x_1,\dots,x_{N-1}}(L'_{N-1}, \dots, L'_1)\, f_{x_N}(\vec{L}', \vec{L})\, \frac{d\vec{L}'}{L'_1 \cdots L'_{N-1}}$$

$L_i, L'_i \in \mathbb{R}$

$$\boxed{s_\lambda(x_1 \dots x_N) = \sum_\mu s_{\lambda/\mu}(x_N)\, s_\mu(x_1 \dots x_{N-1})}$$

$x_N^{|\lambda|-|\mu|}$

$$1 \le L_N \le L'_{N-1} \le L_{N-1} \le \dots \le L'_1 \le L_1$$



$$f_x(\vec{L}', \vec{L}) = \left(\frac{\Gamma(2s)}{\Gamma(s+x)\,\Gamma(s-x)} \right)^{N-1} \left(\frac{L'_1 \cdots L'_{N-1}}{L_1 L_2 \cdots L_N} \right)^{x}$$

$$\times \left(1 - \frac{L'_1}{L_1}\right)^{s-x-1} \left(1 - \frac{L_2}{L'_1}\right)^{s+x-1} \left(1 - \frac{L_2}{L_1}\right)^{1-2s}$$

$$\times \left(1 - \frac{L'_2}{L_2}\right)^{s-x-1} \left(1 - \frac{L_3}{L'_2}\right)^{s+x-1} \left(1 - \frac{L_3}{L_2}\right)^{1-2s} \times \dots$$

2) Cauchy $\quad |x_i|, |y_j| < s, \qquad x_i + y_j > 0$

$$\int_{\mathbb{R}^N} f_{x_1,\dots,x_N}(\vec{L})\, g_{y_1,\dots,y_M}(\vec{L})\, \frac{d\vec{L}}{L_1 \cdots L_N} = \prod_{j=1}^{M} \left[\frac{\Gamma(x_1+y_j)}{\Gamma(s+x_1)} \prod_{i=2}^{N} \frac{\Gamma(x_i+y_j)\,\Gamma(2s)}{\Gamma(s+x_i)\,\Gamma(s+y_j)} \right].$$

3) Conjectural weak orthogonality with "spin Sklyanin measure"

Conj.

$$\frac{1}{N!\,(2\pi i)^N} \int_{(i\mathbb{R})^N} f_{\vec{x}}(\vec{L})\, f_{-\vec{x}}(\vec{L}') \prod_{i \neq j} \frac{\Gamma(s+x_i)\,\Gamma(s-x_j)}{\Gamma(2s)\,\Gamma(x_i-x_j)}\, d\vec{x} = \prod_{i=1}^{N-1} \left(1 - \frac{L_{i+1}}{L_i}\right)^{1-2s} \delta_{\vec{L},\vec{L}'}$$

4) Two eigenoperators

$$D_1 F = \sum_i \prod_{j \neq i} \frac{x_i + s}{x_i - x_j} F\Big/_{x_i \to x_i + 1}$$

$$\overline{D}_1 F = \sum_i \prod_{j \neq i} \frac{x_i - s}{x_i - x_j} F\Big/_{x_i \to x_i - 1}$$

Prop:

$$\boxed{\begin{aligned} D_1 f_{\vec{x}}(\vec{L}) &= L_N^{-1} f_{\vec{x}}(\vec{L}) \\ \overline{D}_1 f_{\vec{x}}(\vec{L}) &= L_1 f_{\vec{x}}(\vec{L}) \end{aligned}}$$

5) „Deformed Quantum Toda"

$$\mathcal{H} = -\frac{1}{2} \sum_i \left(\frac{\partial}{\partial u_i}\right)^2 + \sum_{i<j} s^{-2(i-j)} e^{u_j - u_i} \left(s - \frac{\partial}{\partial u_i}\right)\left(s + \frac{\partial}{\partial u_j}\right)$$

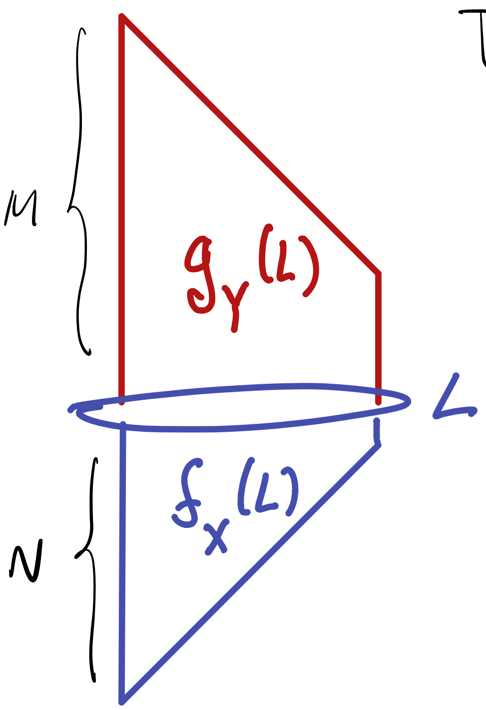
$$\left[s \to \infty : \quad -\frac{1}{2} \sum_i \left(\frac{\partial}{\partial u_i}\right)^2 + \sum_{\substack{i<j \\ j = i+1}} e^{u_j - u_i} \right]$$

Prop:

$$\boxed{\begin{aligned} &\mathcal{H} f_{\vec{x}}\left(L_i = s^{N+1-2i} e^{u_i}\right) = \\ &\qquad = -\frac{1}{2}\left(x_1^2 + \dots + x_N^2\right) f_{\vec{x}}\left(L_i = s^{N+1-2i} e^{u_i}\right) \end{aligned}}$$

3.3. Application to polymers

$q \to 1$ limit of YBE / random recursion:



Thm (MP '20)

$$L_N^{-1} \overset{d}{=} Z^{beta}(N,M)$$

Let $|x_i|, |y_j| < S$, $S > 0$, $x_i + y_j > 0$ for all i,j

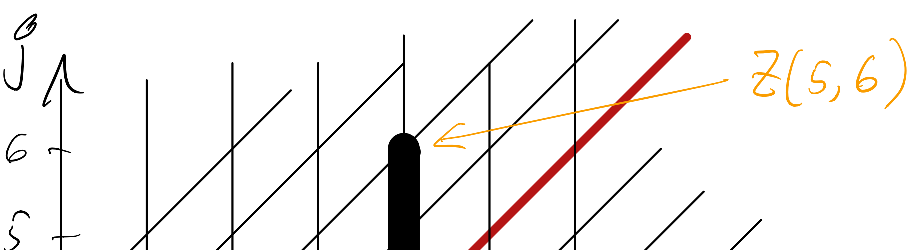
Beta (x,y)

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} t^{x-1}(1-t)^{y-1}$$

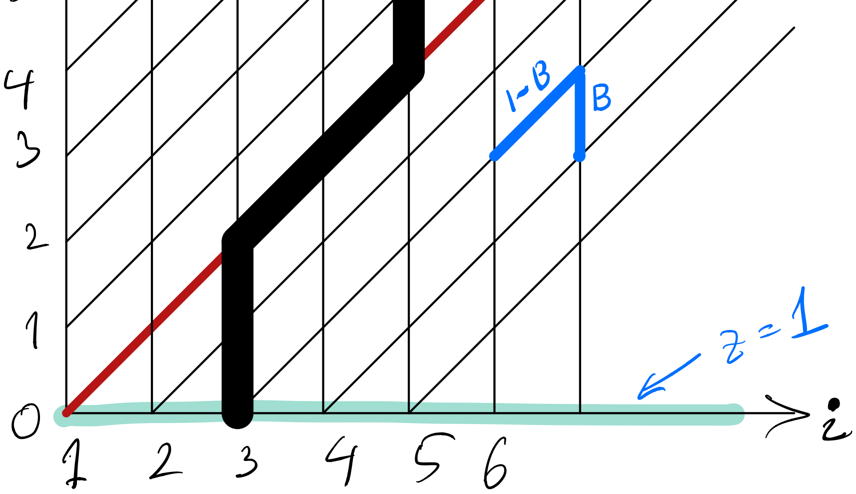
$t \in [0,1]$

$B_{ij} \sim \mathrm{Beta}(x_i + y_j, S - y_j)$ — independent Beta R.V.

$$\begin{cases} Z(i,j) = Z(i,j-1)B_{i,j} + Z(i-1,j-1)(1-B_{i,j}) & \text{for } 1 < i \le j; \\ Z(1,j) = Z(1,j-1)B_{1,j} & \text{for } j > 0; \\ Z(i,0) = 1 & \text{for } i > 0. \end{cases}$$



$Z(i,j)$ — strict-weak Beta polymer



partition function from the bottom line to the point (i,j)

Note: $Z(i,j) \equiv 1$ for $j \le i$.

$$Z(i,j) = \sum_{\text{paths } \pi} \prod_{e \in \pi} weight(e)$$

Barraquand – Corwin 2015:

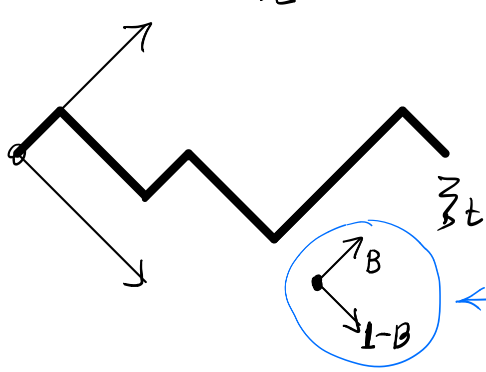
$x_i \equiv x, \; y_j \equiv y$

- Beta polymer as a limit of q-Hahn TASEP as $q \to 1$
- $\oint$ for $\mathbb{E}[Z(i,j)^k]$ via duality, and $\mathbb{E}[e^{-uZ(i,j)}]$
- KPZ class asymptotics $\leftarrow$ $\dfrac{Z(n,\alpha n) - h_\alpha n}{C_\alpha n^{1/3}} \xrightarrow{n \to \infty} F_2$ (*)
- Connection to random walks in random environment (RWRE):

$$Z(t,n) = \mathrm{Prob}_{(env.)}\left[\mathfrak{Z}_t \ge t - 2n + 2 \right]$$

$B \sim \mathrm{Beta}(x+y, s-y)$

(*) $\Leftrightarrow$ asymptotics of the RWRE in large deviations regime

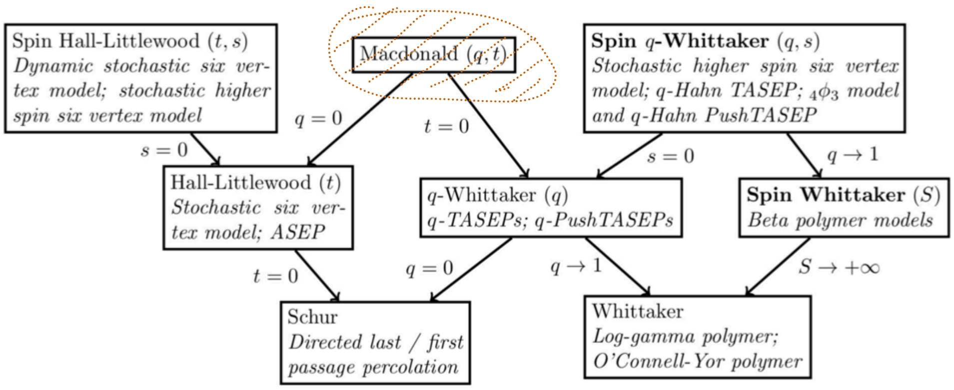


$\leftarrow$ prob. to go up / down depend on the random environment

Note: $S \to \infty$, log-gamma polymers / Whittaker processes

Summary

Vertex models and symmetric funct.



Bridges

RSK

randomized RSK
(Hall-Littlewood,
q-Whittaker)

bijectivised /
randomized YBE

Particle systems

Longest
increasing
subsequences

TASEP

ASEP

q-TASEP

random
polymers

stochastic
six vertex
model