Integrability of the Six-Vertex model

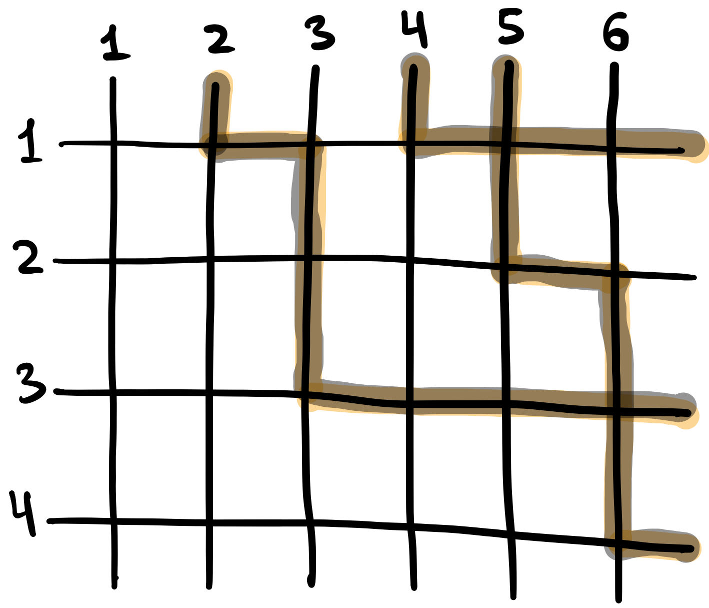
Slava Naprienko

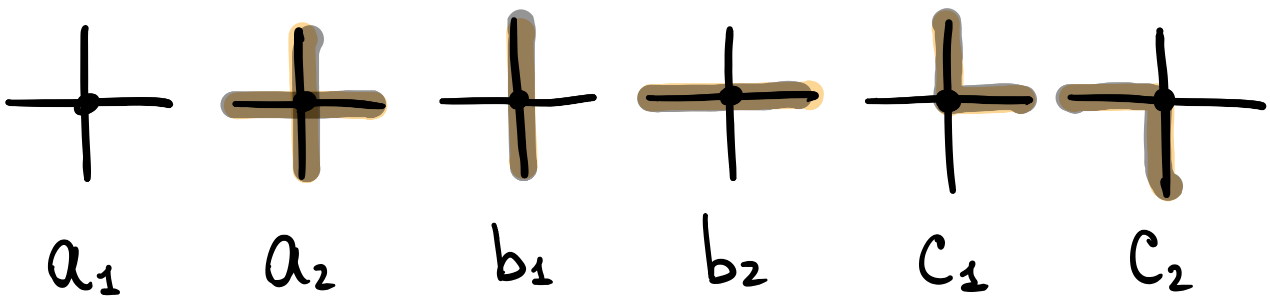
February 8, 2022

Outline:

1. Six-vertex model
2. The alternating sign matrices and the Izergin-Korepin formula
3. The Schur functions and the free-fermionic case
4. Integrability and the Yang-Baxter equation
5. Generalizations.
6. Generic weights
7. Overview

1. Six-vertex model



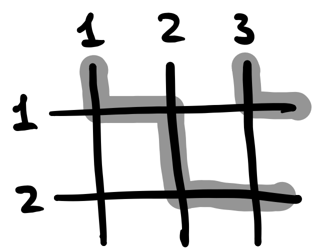


The partition function:

Interesting.

$$Z = \sum_{\text{states}} \text{weight}(\text{state}).$$

Example



$c_1(1,1)\,c_2(1,2)\,c_1(1,3)$

$a_1(2,1)\,c_1(2,2)\,b_2(2,3)$

2. The alternating sign matrices and the Izergin-Korepin formula

Alternating sign matrix has 0's, 1's, -1's such that sum in each row and column equals 1, and the nonzero entires alternate in sign.

Q: How many are there?

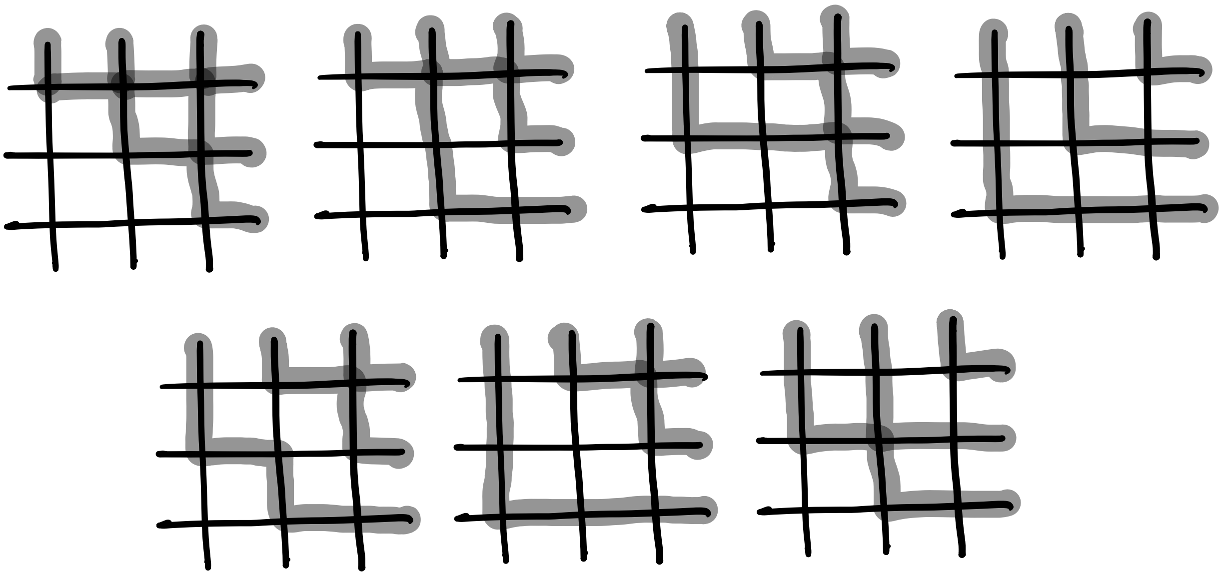
Example $(n=3)$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

The domain wall boundary conditions is full boundary on the top and on the right, and empty on the left and at the bottom. (DWBC)

Example ($n=3$)



Fact: States with the DWBC are in bijection with the alternating sign matrices.

The partition function:

$$\boxed{Z_n = \sum_{\text{states}} \text{weight}(\text{state}),}$$

so if weight $=1$, then Z_n is the number of the alternating sign matrices!

Greg Kuperberg ('96)

$$\boxed{Z_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}}$$

Theorem (Izergin, Korepin, ~92')

With a good choice of weights which depend on parameters $q, x_1, \ldots, x_n, y_1, \ldots, y_n$,

$$Z_n = \frac{\prod_{i,j}(x_i - q y_j)(x_i - q^{-1} y_j)}{\prod_{i<j}(x_i - x_j)(y_i - y_j)} \det\left(\frac{1}{(x_i - q y_j)(x_i - q^{-1} y_j)}\right).$$

Now take limit $x_i, y_i \to 1$, and $q = e^{2\pi i/3}$ to get weight $= 1$.

Takeaway: Six-vertex models are useful for enumeration of the alternating sign matrices.

3. The Schur functions and the free-fermionic case

The Schur polynomial can be given combinatorically as a sum over semistandard tableaux or Gelfand-Tsetlin patterns.

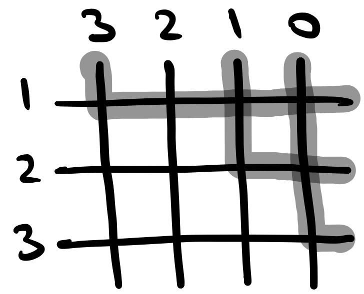
$$S_\lambda = \sum_{\text{patterns}} \text{weight}(\text{pattern}).$$

Example $\lambda = (2,0)$

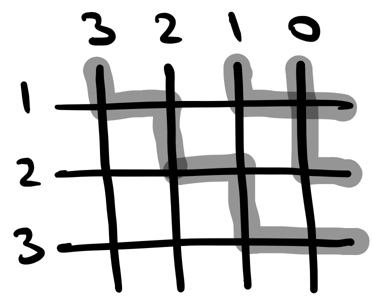
$$\begin{array}{ccc} \begin{matrix} 2 & & 0 \\ & 0 & \end{matrix} & \qquad \begin{matrix} 2 & & 0 \\ & 1 & \end{matrix} & \qquad \begin{matrix} 2 & & 0 \\ & 2 & \end{matrix} \\ \\ x_1^2 & \qquad x_1 x_2 & \qquad x_2^2 \end{array}$$

$$S_\lambda = x_1^2 + x_1 x_2 + x_2^2.$$

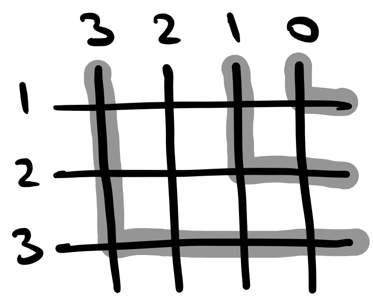
$$\begin{array}{ccccc} 3 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}$$



$$\begin{array}{ccccc} 3 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array}$$



$$\begin{array}{ccccc} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 3 & & \end{array}$$



$\vdots$ $\vdots$

Fact States of the six-vertex model with the top boundary defined by partition λ are in bijection with ~~strict~~ Gelfand-Tsetlin patterns.

$$S_\lambda = \sum_{\text{patterns}} \text{weight(pattern)}.$$

$$Z_\lambda = \sum_{\text{states}} \text{weight(state)},$$

So if weight(state) = weight(pattern), then the partition function gives us the Schur polynomial.

$$\left.\begin{array}{l} \text{– Schur polynomials} \\ \text{– Factorial Schur functions} \\ \text{– Spherical Whittaker functions} \\ \text{– } F_\lambda \text{ functions from ABPW '21...} \end{array}\right\} \text{free-fermionic weights}$$

Takeaway Six-vertex model is useful for the Schur-like special functions.

In all free-fermionic cases,

$$Z_\lambda = Z_0 \cdot s_\lambda, \text{ where}$$

Z_0 is an explicit product which we call the deformed Weyl denominator.

Notice that when $\lambda=0$ the boundary is the domain wall boundary conditions.

4. Integrability and the Yang-Baxter equation

Six-vertex model

Field-free case

$$a_1 = a_2 = a,$$
$$b_1 = b_2 = b,$$
$$c_1 = c_2 = c,$$

$$\Delta = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_1b_1}$$

$$= \frac{a^2 + b^2 - c^2}{2ab} = \text{const}$$

Izergin-Korepin det

Alternating sign matrices

$$Z_n \approx \det(\ldots)$$

Free-fermionic case

$$a_1a_2 + b_1b_2 - c_1c_2 = 0.$$

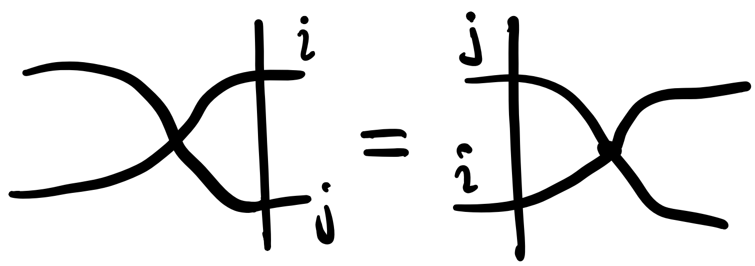
Special functions

Domino tilings

Deformed Weyl denominator

$$Z_n \approx \prod_{i<j}(\ldots)$$

In these cases one has the Yang-Baxter equation which gives functional equations for the partition function.



Now let's work backwards!

- Assume that the YB holds.
- Find integrability classes
- Find the generic weights
- Find the partition functions

5. Generalizations

(A) General fermionic case (better name?)

$a_1 = a_2 = a,$
$b_1 = b, \quad b_2 = r \cdot b_1,$
any c_1, c_2

Generalization of the field-free case

$$\Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2 a_1 b_1} =$$

$$= \frac{a^2 + r b^2 - c_1 c_2}{2ab} \text{ is constant.}$$

If $b_1 = b_2$, $c_1 = c_2$,
we get the field-free case.

Th (N., 21') The generalized Izergin-Korepin determinant holds:

$$Z_n = \frac{\prod_{i,j} a(i,j)\, b(i,j) \cdot \det\left(\dfrac{c_1(i,j)}{a(i,j)\, b(i,j)}\right)}{\prod_{i<j} B^H(i,j)\, B^V(i,j)}.$$

- Result outside of the field-free case
- Generic weights (more parameters, old results)
- Denominator is in terms of the weights

(B) Free-fermionic case:

$$a_1a_2+b_1b_2-c_1c_2=0.$$

Th (N. '21) The partition function with the DWBC is given by $(\lambda=0)$ the deformed Weyl denominator:

$$Z_0=\prod_{i<j}A^H(i,j)A^V(i,j).$$

With top boundary defined by λ,

$$Z_\lambda=Z_0\cdot S_\lambda,$$ where

S_λ is a Schur-like function.

- Generic weights (more parameters)
- The product is in terms of the weights.
- Comes from the Izergin-Korepin-like determinant which generalizes the Cauchy determinant:

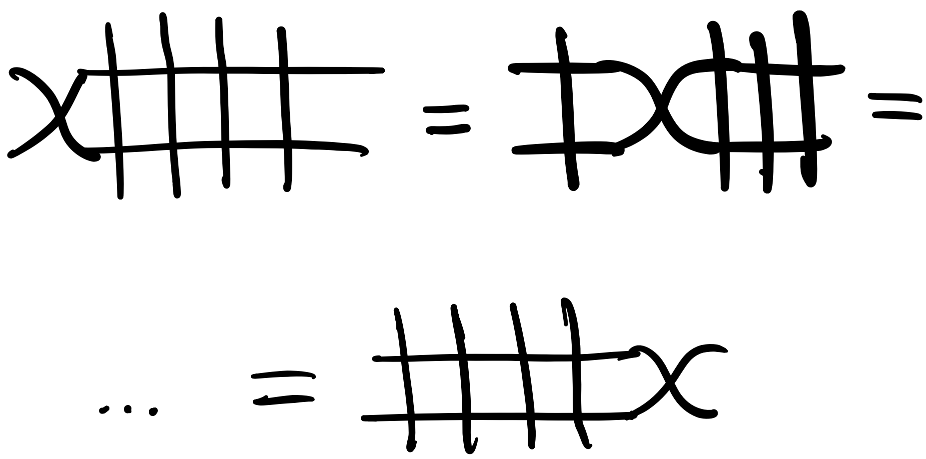
$$\det\left(\frac{c_1(i,j)}{a_1(i,j)}\right) = \frac{\prod_{i<j} B^H(i,j)B^V(i,j)}{\prod_{i,j} a_1(i,j)}$$

Compare:

$$\det\left(\frac{1}{x_i - y_j}\right) = \frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_j)}$$

6. Generic weights

The main application of the YB equation is the transfer argument:



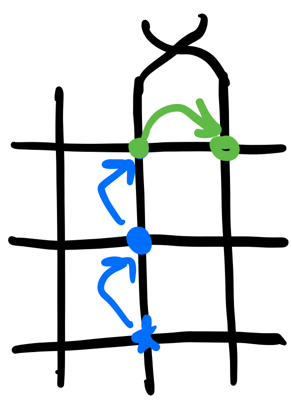
You get functional equations.

Usually, you pick weights such that the transfer argument is possible.

Transfer assumption

- The horizontal cross-weights don't depend on the column.
- The vertical cross-weights don't depend on the row.

But then one can express weights on any row in terms of the first row; and on any column in terms of the first column.



Th (N. '21) The transfer weights admit explicit parametrization in terms of the corner weights and the cross-weights both in the general fermionic case, and in the free-fermionic case.

Example Generalized Cauchy determinant:

Pick free parameters $a_1, a_2, b_1, b_2,$

$A_1^H(1), \dots, A_1^H(n);\ A_1^V(1), \dots, A_1^V(n),$

$B_2^H(1), \dots, B_2^H(n);\ B_2^V(1), \dots, B_2^V(n).$

Then:

$$\det(C(i,j))_{i,j} \approx \frac{\prod_{i<j} B_2^H(i,j) B_2^V(i,j)}{\prod_{i,j} C(i,j)},$$

where
$$C(i,j) = A_1^H(i)\left[a_1 A_1^V(j) - b_2 B_2^V(j)\right] - B_2^H(i)\left[b_1 A_1^V(j) + a_2 B_2^V(j)\right],$$

and
$$B_2^H(i,j) = A_1^H(j)B_2^H(i) - A_1^H(i)B_2^H(j),$$
$$B_2^V(i,j) = A_1^V(j)B_2^V(i) - A_1^V(i)B_2^V(j).$$

7. Overview

- The YB exists
  a. General fermionic case
  b. Free-fermionic case

- Transfer assumption holds
  - parametrization of the weights

- Partition functions
  a. The generalized Izergin-Korepin
  b. The deformed Weyl denominator and the Schur functions.

Possible applications:

- Refined alternating sign matrices enumeration ($b_1 \neq b_2$)
- F_λ, G_λ processes in the style of (ABPW '21)
- Domino tilings and asymptotics
- Larger integrability classes of the colored analogues of the six-vertex models
- New classes of the Schur functions (e.g. double factorial Schur functions)

- Thermodynamical limit with the DWBC in the presence of an electric field
- Stochastic six-vertex model for the generic weights.

Thank you!