OVERVIEW:

- Review many ways of thinking about lattice models.

(all in Baxter's "Exactly Solved Models in Statistical Mech.")

spins: + / −

choose orientation

arrows: $\rightarrow$ / $\leftarrow$

$\downarrow$ / $\uparrow$

particles: ∘ / •

"hole" "particle"

paths: no path / path on edge.

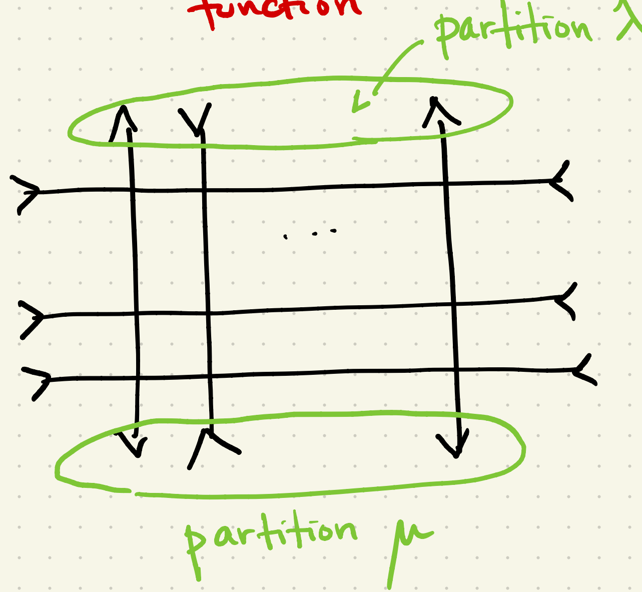
GAME: Given boundary conditions + Boltzmann weights for vertices $\rightsquigarrow$ Evaluate Partition function

$$\sum_{\text{admissible states}} \prod_{v \in \text{state}} \text{Boltz. wt}(v)$$

OR IN REVERSE: Find wts. + bdy conditions to recover it as partition function

Favorite special function (cx. geometry / Schubert calc. repn thy of p-adic gps, probability.)

Example:



boring on sides

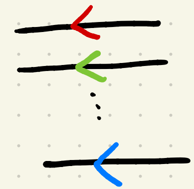
Solve for partition function $P_{\lambda,\mu}$

Additions: ① Allow wts. to depend on row, col. params

$x_1, \dots, x_r$, $y_1, \dots, y_t$, $z_{i,j} := x_i \oplus y_j$ $\quad \underline{z} = (z_{i,j})$

② Allow edge decorations to be colored.

$\leftarrow$ gets n colors



Similarly: $\rightarrow$ gets m (s)colors

partition λ + color (permutations)

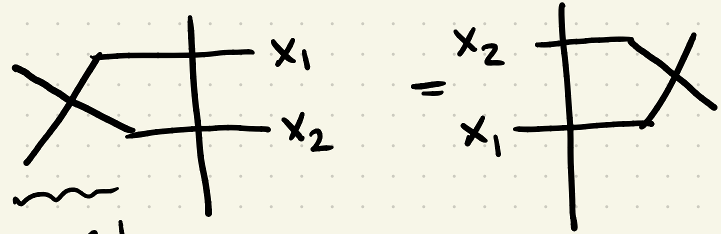
$\Rightarrow$ compositions. (weights $\simeq \mathbb{Z}^r$)

① + ②: Partition functions are multivariate functions indexed by partitions (oof!), permutations or compositions.

How do we prove that given set of wts., boundary conditions lead to desired partition function (i.e., special function)?

SOLVABILITY

(in pictures)



new set of wts.
(R-matrix)

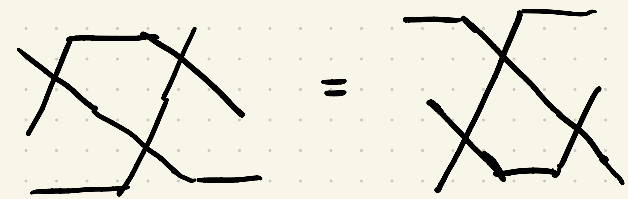
for any choice of 6 boundary edges.

When solvable, get relns between partition functions under $x_i \longleftrightarrow x_{i+1}$

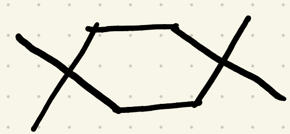
Also have following properties for R-matrix:

Hecke algebra

- braid relation $\rightsquigarrow$ ①



- quadratic relation $\rightsquigarrow$ ②



= constant.

Fomin-Kirillov notation: $H_{a,b}$ has general quad. rel'n

$$T_i^2 = aT_i + b$$

with braid rel'ns.

So these are of form $H_{0,b}$ in diagrammatic version today.

$$H_{0,1} = \mathbb{C}[S_n]$$

Upshot: Study special functions with Hecke algebra action using solvable lattice models.

uncolored models		colored models
symmetric functions	$\xrightarrow{\text{refine}}$	non-symmetric counterparts
K-fixed vectors in rep'n	$\rightsquigarrow$	Iwahori-fixed vectors in rep'n
		$K = \bigsqcup_{w \in W} IwI$

Aside about p-adic rep. thy.

Natural Hecke algebra $C^\infty(I\backslash K/I)$

has $T_i^2 = (q-1)T_i + q$

$[\text{ i.e. } (T_i - q)(T_i+1) = 0]$

A_i is natural in p-adic rep'n thy.

("standard intertwiners of prin. series rep'ns")

SLOGAN: R-matrices describe actions of intertwining operators.

So not of the above form.

But we can do affine linear change of coordinates.

$A_i = c_1 \cdot T_i + c_2$

for some c_1, c_2.

s.t. $A_i^2 = b$

some b.

Approaches using solvability / YBE :

Izergin/Korepin Kuperberg style...

① Solvability ⇒ Hecke algebra action. Use this

\+ other axioms to uniquely determine partition function

poly. of bounded degree, etc.

② pass to colored refinement of uncolored models.

See Henrik's talk in 2 wks.

e.g. colors record permutations $w \in W$. YBE gives recursion w.r.t. length in W.

$$P_\lambda(\underline{z}) = \sum_w \partial_w(\text{base case})$$

$$\partial_w = \partial_{i_1} \cdots \partial_{i_k} \quad \text{if} \quad w = s_{i_1} \cdots s_{i_k}$$

"Demazure-like" divided difference ops.

in simple 5-vertex case

$P_\lambda = s_\lambda(\underline{z})$, $\partial_w(z^\lambda)$: Demazure atoms in Demazure char. formula.

③ Algebraic Bethe Ansatz (mentioned in Amol's talk; paper of Borodin / Petrov "Higher Spin 6-vertex model...")

For example, my student Katy Weber has shown in above 5-vertex + 6-vertex gen'ztns

$$P_\lambda(\underline{z}) = \frac{\sum_w w\cdot(\underline{z}^{\lambda+\rho})}{\text{denom}}$$

(Weyl character formula)

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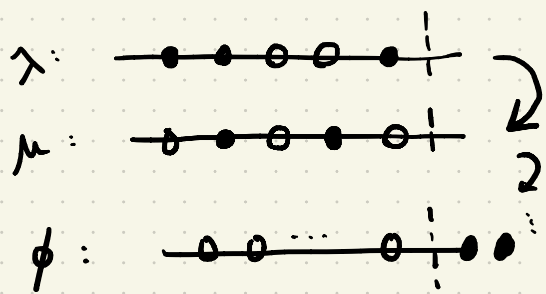
④ Hamiltonian methods:

Describe lattice model as discrete time evolution of particles



$$\langle \phi | \underset{\sim}{e^{H[t]}} | \lambda \rangle$$

prove commutation relns.

(talk to Andy Hardt, seminar attendee, on this)

So What?

What can we do once we have a connection between solvable models and special functions?

(1) Obtain all sorts of identities / relations which can be seen to follow from diagrammatic pfs.

(Cauchy identities, branching rules, ready-made generalizations)

from lattice approach

5-vertex to 6-vertex

uncolored to colored / Scolored

adjust formal gp.

lattice models for other Cartan types

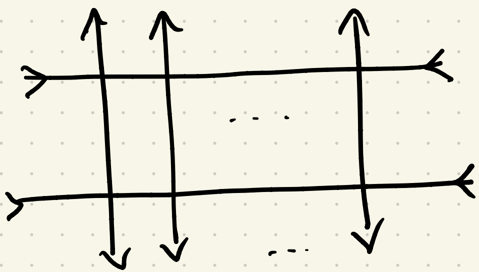
(2) YBEs should arise from modules for quantum groups.

(2) con't. So my dream: "Brubaker's Mittleren alters traum"

create dictionary between quantum gp modules and partition functions, with enough understanding to make predictions in either direction.

Some examples of diagrammatic identities.

Example 1 (Bump - McNamara - Nakasuji)

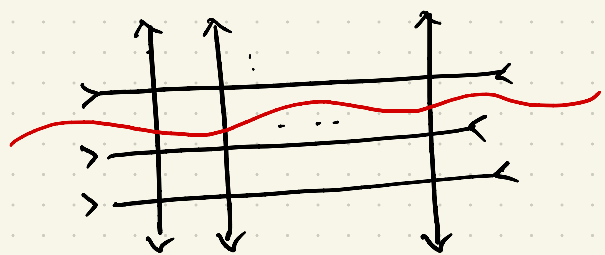


regular boundaries along every edge

can be evaluated directly via Yang-Baxter equations.

OR : we cut along middle:

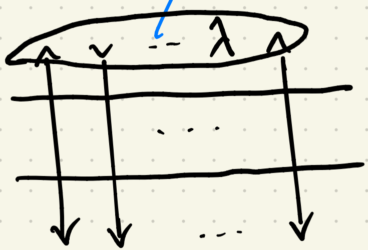
cut between fixed pair of rows



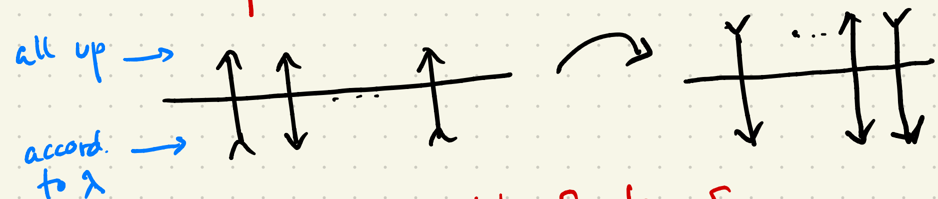
record up arrows as partition λ

then split lattice into two pieces

$$\sum_{\text{possible } \lambda} P_\lambda(\text{bottom row vars}) \cdot P'_\lambda(\text{top row vars.})$$



perform 180° rotation. -



resulting in partition function for conjugate partition.

Result: Dual Cauchy identity for (factorial) Schur functions in Bump-McNamara-Nakasuji.

Usual dual Cauchy identity for Schur polynomials:

$$\sum_{\lambda} s_{\lambda}(\underline{x})\, s_{\lambda'}(\underline{y}) = \prod_{i,j} (1 + x_i y_j)$$

conjugate partition

B-Mc-N:

$$\sum_{\lambda} s_{\lambda}(\underline{x} \mid \underline{\alpha})\, s_{\lambda'}(\underline{y} \mid -\underline{\alpha}) = \prod_{i,j} (1 + x_i y_j)$$

factorial Schur functions