The Stochastic Six-Vertex Model

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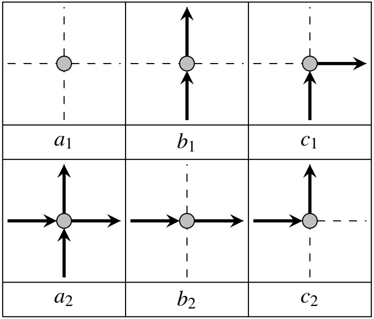
Outline

1. Stochastic six-vertex model
   - Definition
   - Height function
2. Asymptotic questions and examples of results
   - Limit shape
   - Fluctuations
   - Local statistics
3. Analysis through integrability
   - q-Moments of the height function
   - General idea of the method (Cauchy identities and orthogonality)

Local Configurations

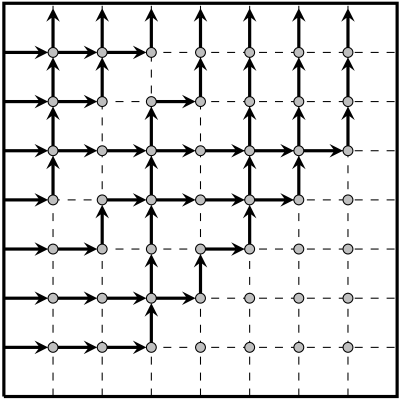
- Fix some domain $\Lambda \subseteq \mathbb{Z}^2$
- At each vertex $v \in \Lambda$, assign one of six **arrow configurations**



- These are the configurations satisfying **spin conservation**: Numbers of incoming and outgoing arrows at v are equal

Six-Vertex Ensemble

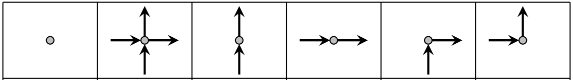
Six-vertex ensemble: Consistent assignment of arrow configuration to each vertex of Λ



- Arrows form up-right directed paths in Λ
- **Boundary conditions** prescribe where paths enter and exit Λ

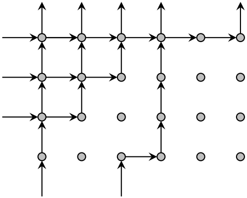
Six-Vertex Model

- Six-vertex ensemble $\mathcal{E}$ on some finite domain $\Lambda \subseteq \mathbb{Z}^2$
- Fix $a_1, a_2, b_1, b_2, c_1, c_2 > 0$
- Assign each arrow configuration a **vertex weight**



a_1	a_2	b_1	b_2	c_1	c_2

Ensemble **weight**: $w(\mathcal{E}) = a_1^{\#a_1} a_2^{\#a_2} b_1^{\#b_1} b_2^{\#b_2} c_1^{\#c_1} c_2^{\#c_2}$



$$w(\mathcal{E}) = a_1^8 a_2^7 b_1^3 b_2 c_1 c_2^4$$

Probability measure: $\mathbb{P}[\mathcal{E}] = Z^{-1} w(\mathcal{E})$, where $Z = \sum_{\mathcal{E}} w(\mathcal{E})$ is the **partition function**.

Physical Context

Set $\mathbb{P}[\mathcal{E}] = Z^{-1} w(\mathcal{E})$, where $Z = \sum_{\mathcal{E}} w(\mathcal{E})$ is the partition function.

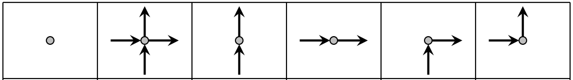
- Introduced by Pauling (1935)
  - Took all vertex weights equal to 1 and domain $\Lambda = [1, N] \times [1, N]$
  - Modeled residual entropy of ice by **free energy** $F = \lim_{N \to \infty} Z^{1/N^2}$
  - Proposed heuristic $F = \frac{3}{2}$
    - Each of about $2N^2$ edges in Λ can be occupied or unoccupied
    - At each of N^2 vertices, 6 of 16 local configurations are allowed
    - This gives $F \sim \left((2^{2N^2}) \left(\frac{6}{16} \right)^{N^2} \right)^{1/N^2} = \frac{3}{2}$
- Giauque–Stout: Experimental data for residual entropy $1.52 \ldots$
- Lieb (1967): $F = \left(\frac{4}{3} \right)^{3/2} \approx 1.5396 \ldots$
- Sutherland–Yang–Yang (1967): Free energy for arbitrary vertex weights

We are interested in the **asymptotic probabilistic behavior** of the six-vertex model on large domains

- Here, we specialize to the **stochastic six-vertex model**

Stochastic Six-Vertex Model

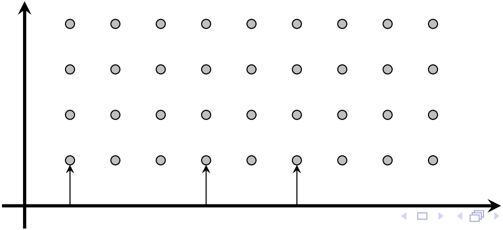
Stochastic six-vertex model (Gwa–Spohn, 1992): Set $a_1 = 1 = a_2$, $b_1 + c_1 = 1$, and $b_2 + c_2 = 1$,



1	1	b_1	b_2	$1 - b_1$	$1 - b_2$

- For fixed incoming arrows, sum of weights of all outgoing arrows is 1
- Under **free exit data**, enables a local, row by row, Markovian sampling
- Markov process on $\{0, 1\}^{\mathbb{Z}}$, with y-axis indexing time

$\mathbb{P}[\mathcal{E}] =$



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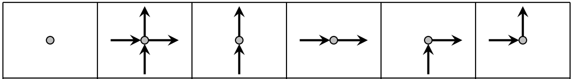

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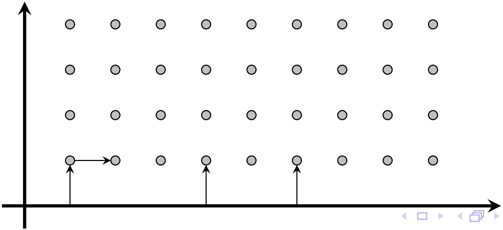

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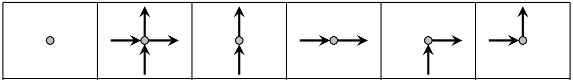

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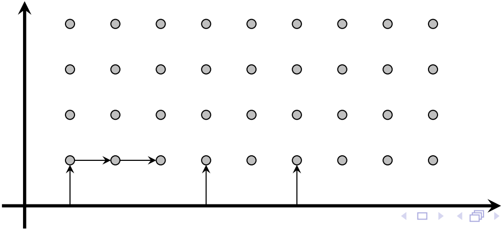

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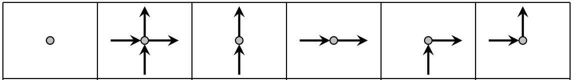

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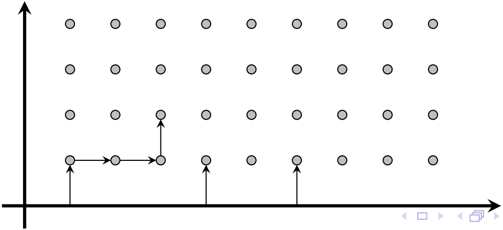

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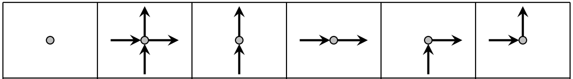

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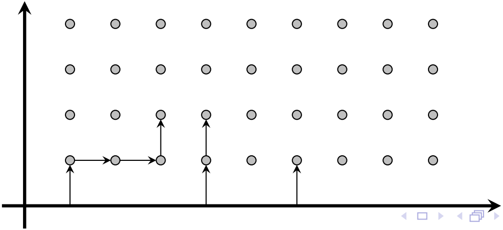

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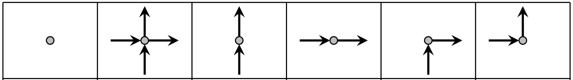

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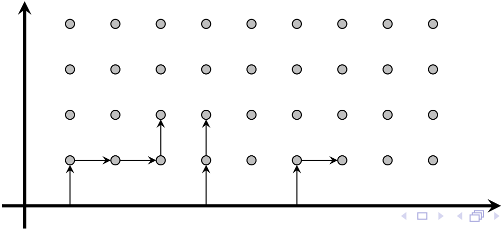

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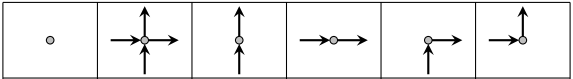

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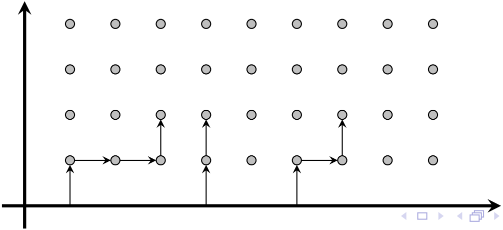

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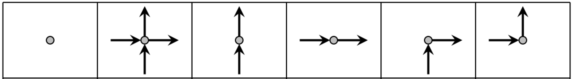

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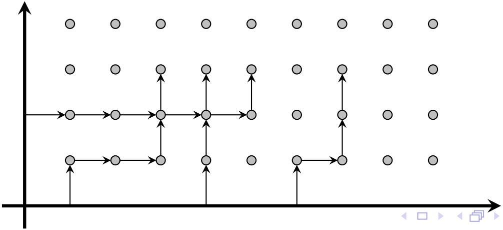

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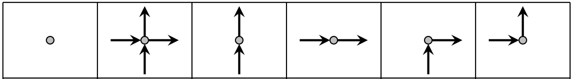

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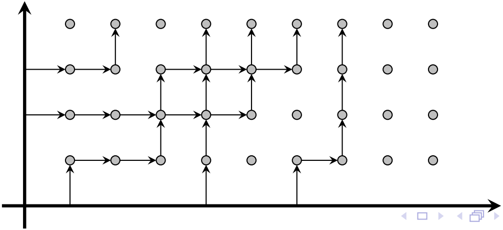

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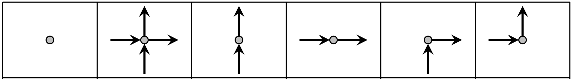

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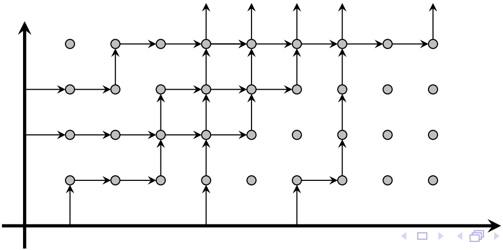


Asymptotics

Question

What is the asymptotic behavior of the stohastic six-vertex model?

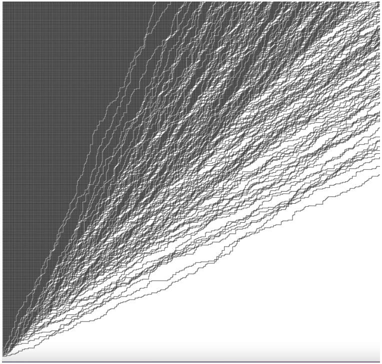
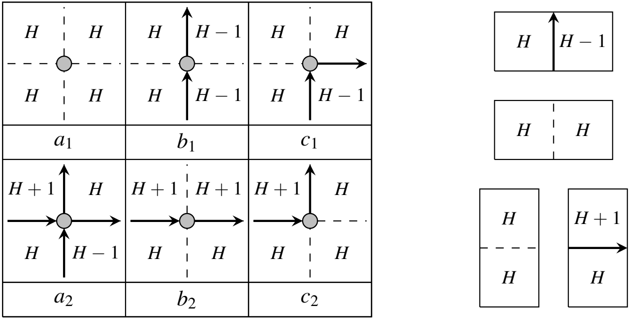


Figure by L. Petrov (`https://lpetrov.cc/2015/03/Spin-models`).

To precisely state this question and its associated results, it will be useful to associate a **height function** with the six-vertex model.

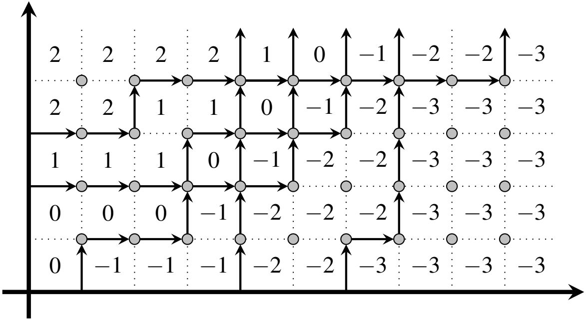
Height Functions



- Assign an integer to each face of the domain, satisfying the above local constraints around every vertex
- This produces a **height function** H on (the dual of) Λ
- Can view $H(u)$ as counting how many paths exist to the right of u
- Bijection between six-vertex ensembles and height function (up to shift)
    - We typically normalize $H(0,0) = 0$

Six-Vertex Ensembles and Height Functions

Bijection between six-vertex ensembles and height function (up to shift)



Asymptotic Questions

Question

What is the asymptotic behavior of the stohastic six-vertex model?

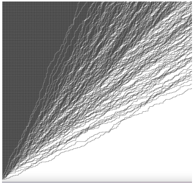
1. **Limit shape**: Law of large numbers for $H(xN, yN)$
2. **Fluctuations**: Behavior of $H(xN, yN) - \mathbb{E}\big[H(xN, yN)\big]$
3. **Local statistics**: Limits of local correlation functions

Many more questions not addressed here

- Boundary effects and inhomogeneities: Gwa–Spohn (1992); Reshetikhin–Sridhar (2016); A.–Borodin (2016); Borodin–Petrov (2017); A. (2016, 2019)
- Weakly asymmetric limits: Corwin–Ghosal–Shen–Tsai (2018)
- Translation-invariant Gibbs measures: Sheffield (2003); A. (2020)
- Colored stochastic vertex models: Borodin–Wheeler (2018); Borodin–Bufetov (2019); Borodin–Gorin–Wheeler (2019); Galashin (2020); Bufetov–Korokhtin (2020)
- Other (non-stochastic) six-vertex models: Vast literature since Lieb (1965)
- …

Limit Shape

Step boundary data: Paths enter $\mathbb{Z}^2_{>0}$ at y-axis



- H_N: Stochastic six-vertex height function
- Suppose $0 < b_1 < b_2 < 1$ and set

$$\varphi(z) = \frac{\kappa z}{(\kappa - 1)z + 1}, \quad \text{where} \quad \kappa = \frac{1 - b_1}{1 - b_2}$$

Theorem (Borodin–Corwin–Gorin, 2014)

Fix $x, y > 0$ with $\kappa^{-1} < \frac{x}{y} < \kappa$. We have $N^{-1}H(xN, yN) \to \mathfrak{H}(x, y)$ as N tends to ∞ in probability, where $\mathfrak{H}(x,y) = \frac{(\sqrt{y\kappa} - \sqrt{x})^2}{\kappa - 1}$.

A. (2019): Same result under general entrance data, where now $\mathfrak{H}$ satisfies

$$\partial_y \mathfrak{H} = \partial_x \big(\varphi(\mathfrak{H}) \big)$$

- κ: **Spectral parameter** of six-vertex model
- Limit shape independent of the **quantization parameter** $q = \frac{b_1}{b_2}$

Fluctuations

H_N: Stochastic six-vertex height function under step initial data

- Recall $0 < b_1 < b_2 < 1$ and $\kappa = \frac{1-b_1}{1-b_2}$
- Fix $x, y \in \mathbb{R}_{>0}$ with $\kappa^{-1} < \frac{x}{y} < \kappa$

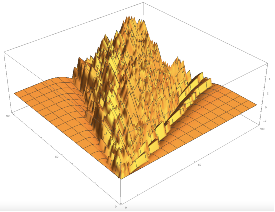


Figure by L. Petrov (`https://lpetrov.cc/2015/03/Spin-models`).

Theorem (Borodin–Corwin–Gorin, 2014)

There exists an explicit constant $C = C(x, y, \kappa) > 0$ such that, for any $s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\left[N\mathfrak{H}(x,y) - H(xN, yN) \leq CsN^{1/3}\right] = F_{TW}(s).$$

$N^{1/3}$ Fluctuations: Typical for models in **Kardar–Parisi–Zhang universality class**

- Different from $N^{1/2}$ fluctuations in random walks
- $F_{\mathrm{TW}}(s)$: Tracy–Widom GUE distribution

Local Statistics

- Consider stochastic six-vertex model on $\mathbb{Z} \times \mathbb{Z}_{>0}$ with boundary data
- Fix a vertex $v \in \mathbb{Z} \times \mathbb{Z}_{>0}$ and consider an $O(1)$-neighborhood of v

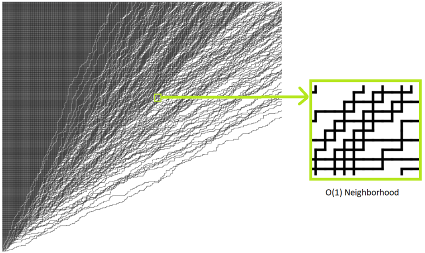


Figure by L. Petrov (`https://lpetrov.cc/2015/03/Spin-models`).

- This yields a random six-vertex ensemble on this $O(1)$-neighborhood.

Local statistics question: What is the law of this random ensemble?

- Contains information about all **local correlation functions** for the model
  - Joint law of nearly neighboring arrow configurations
    - For example, $\mathbb{P}\big[\text{Vertical arrow exits from both } (X, Y) \text{ and } (X+1, Y+1)\big]$

Local Statistics

- Consider stochastic six-vertex model on $\mathbb{Z}^2_{>0}$ with arbitrary entrance data
- Let N be a large integer
- Recall limit shape $\mathfrak{H}$ satisfying $\partial_y \mathfrak{H} = \partial_x \big(\varphi(\mathfrak{H}) \big)$
- Fix a continuity point $(x, y) \in \mathbb{R}^2_{>0}$ of $\nabla \mathfrak{H}$
- Let $v = (X, Y) = (X_N, Y_N) = \big(\lfloor xN \rfloor, \lfloor yN \rfloor \big) \in \mathbb{Z}^2_{>0}$

Theorem (A., 2019)

Let $\nabla \mathfrak{H}(x, y) = (-s, t)$. As N tends to ∞, the local statistics around $v = (X_N, Y_N)$ converge to a measure that ***depends in an explicit way*** *on (s, t).*

"**Depends in an explicit way**:" Local statistics around (X_N, Y_N) converge to unique **translation-invariant Gibbs measure** with slope (s, t)

- Explicit way of sampling from this measure
- Can be used to evaluate limiting local probabilities
  - $\chi(A, B) \in \{0, 1\}$: Indicator for a vertical arrow from (A, B) to $(A, B + 1)$
  - $\lim_{N \to \infty} \mathbb{P} \big[\chi(X_N, Y_N) = 1 \big] = s$
  - $\lim_{N \to \infty} \mathbb{P} \big[\chi(X_N, Y_N) = 1, \chi(X_N + 1, Y_N) = 1 \big] = s^2$
  - $\lim_{N \to \infty} \mathbb{P} \big[\chi(X_N, Y_N) = 1, \chi(X_N + 1, Y_N + 1) = 1 \big] = s^2 + s(1 - s)(1 - b_1)(1 - b_2)$

q-Moments of the Height Function

- From here, we consider stochastic six-vertex model with step boundary data
- Denote its height function at $(X, Y) \in \mathbb{Z}^2_{>0}$ by $H(X, Y)$
- Recall $0 < b_1 < b_2 < 1$ and set $q = \frac{b_1}{b_2} \in (0, 1)$

Define the q**-deformed moment** of the height function, $\mathbb{E}[q^{kH(x,y)}]$

- These are moments of $q^{H(x,y)} \in (0, 1)$
- Together, these q-moments determine the law of $q^{H(x,y)}$ (and so of $H(x, y)$)

One way to analyze model: Obtain **exact formulas for these** q**-moments**

- Eventually will take large scale limit, as x and y tend to ∞
  - Compexity of formulas for $\mathbb{E}[q^{kH(x,y)}]$ should not grow with $x + y$ (but they will grow with k)

Proposition (Borodin–Corwin–Gorin, 2014)

Letting $\psi_{x,y}(w) = \left(\frac{1 + w}{1 + q^{-1}w} \right)^y \left(\frac{1 - q^{-1}\kappa^{-1}w}{1 - \kappa^{-1}w} \right)^{x-1}$*, we have*

$$\mathbb{E}[q^{kH(x,y)}] = \frac{q^{\binom{k}{2}}}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{k} \psi_{x,y}(u_i) \frac{du_i}{u_i}.$$

Using q-Moments

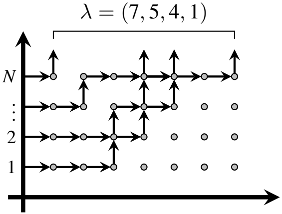
Letting $\psi_{x,y}(w) = \left(\frac{1+w}{1+q^{-1}w}\right)^y \left(\frac{1-q^{-1}\kappa^{-1}w}{1-\kappa^{-1}w}\right)^{x-1}$, we have

$$\mathbb{E}[q^{kH(x,y)}] = \frac{q^{\binom{k}{2}}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \prod_{1\le i<j\le k} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{k} \psi_{x,y}(u_i) \frac{du_i}{u_i}$$

- Take limit as q tends to 1
  - Can be used to analyze $q = 1$ case of the stochastic six-vertex model
    - Since $\lim_{q\to 1} \frac{q^A-1}{q-1} = A$, can be used to find standard moments $\mathbb{E}\left[H(x,y)^k\right]$
    - First two moments $\mathbb{E}\left[H(x,y)\right]$ and $\mathbb{E}\left[H(x,y)^2\right]$ suffice for limit shape
- Collect q-moments in a generating series
  - Standard exponential: $e^{\zeta H(x,y)} = \sum_{k=0}^{\infty} \frac{\zeta^k H(x,y)^k}{k!}$
  - q-Exponential: $\frac{1}{(\zeta q^{H(x,y)};q)_\infty} = \prod_{i=0}^{\infty} \frac{1}{1-q^{i+H(x,y)}\zeta} = \sum_{k=0}^{\infty} \frac{\zeta^k q^{kH(x,y)}}{(q;q)_k}$
- Match formulas with those of a simpler model
  - Exact expressions for q-moment type quantities in many exactly solvable systems admit similar contour integral formulas
  - In various cases, they can be matched exactly
    - Can be used to show that two systems have the same law

Random Partitions

- Run the stochastic six-vertex model under step boundary data for time N
- Random partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$: Sites where arrows exit row N



The height function $H(x, N)$ along the N-th row is **determined by** λ

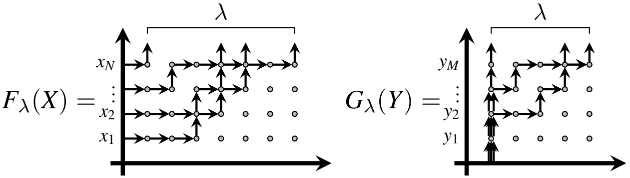
- $H(x, N) = \#\{j \in [1, N] : \lambda_j > x\} = \mathfrak{h}_\lambda(x)$

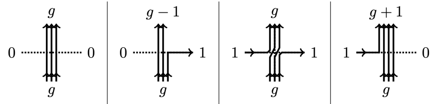
To analyze $\mathbb{E}[q^{H(x,N)}] = \mathbb{E}[q^{\mathfrak{h}_\lambda(x)}]$, this exposition follows Borodin–Petrov (2016)

- Uses **Cauchy identities** and **orthogonality** for symmetric functions
- Both originate from integrability
  - Cauchy identities: Yang–Baxter commutation relations for transfer operators
  - Orthogonality: Summation formulas for symmetric functions through algebraic Bethe ansatz

Let $X = (x_1, x_2, \ldots, x_N)$ and $Y = (y_1, y_2, \ldots, y_M)$ denote sequences of variables

1. Define **symmetric functions** $F_\lambda(X)$ and $G_\lambda(Y)$ as partition functions for certain lattice models, dependent on an extra parameter $s \in \mathbb{C}$ called **spin**





$w_{u,s}$	$\dfrac{1 - sq^g u}{1 - su}$	$\dfrac{(1 - s^2 q^{g-1})u}{1 - su}$	$\dfrac{u - sq^g}{1 - su}$	$\dfrac{1 - q^{g+1}}{1 - su}$

- Here, X and Y are spectral parameters for rows of these models
- These vertex models allow multiple arrows to occupy vertical edges
  - When $s = q^{-1/2}$, this is no longer be permitted for the F-model
- The symmetry of F and G will follow from Yang–Baxter equation for $w_{u,s}$

Outline of the Method

1. Define symmetric functions $F_\lambda(X)$ and $G_\lambda(Y)$ as partition functions
2. Show the **Cauchy identity** $\sum_\lambda F_\lambda(X) G_\lambda(Y) \sim \prod_{x \in X} \left(\frac{1}{1 - sx} \prod_{y \in Y} \frac{1 - qxy}{1 - xy} \right) = Z(X; Y)$
   - Also follows from suitably applying the Yang–Baxter equation
3. Define measure $\mathbb{P}[\lambda] = Z(X; Y)^{-1} F_\lambda(X) G_\lambda(Y)$, and realize six-vertex random partition as special case, $\mathbb{P}[\lambda] = F_\lambda(\boldsymbol{\kappa}) G_\lambda(\boldsymbol{\rho})$, for $s = q^{-1/2}$
   - Here, $\boldsymbol{\kappa} \sim (s\kappa, s\kappa, \ldots, s\kappa)$
     - Will make $F_\lambda(\boldsymbol{\lambda})$ nearly match stochastic six-vertex weight, up to $(-s)^{|\lambda|}$
   - Here, $\boldsymbol{\rho}$ is defined through a certain limiting procedure
     - Will make $G_\lambda(\boldsymbol{\rho}) \sim (-s)^{|\lambda|}$ to compensate F_λ
4. Insert additional variables into the Cauchy identity to obtain certain expectation values

Expectations Through the Cauchy Identity

- Recall $\mathbb{P}[\lambda] = F_\lambda(\boldsymbol{\kappa}) G_\lambda(\boldsymbol{\rho})$
- Let w be an additional variable, and let $(\boldsymbol{\rho}; w) = \boldsymbol{\rho} \cup \{w\}$
- Then, $\sum_\lambda F_\lambda(\boldsymbol{\kappa}) G_\lambda(\boldsymbol{\rho}; w) \sim \prod_{x \in \boldsymbol{\kappa}} \frac{1 - qxw}{1 - xw} = \left(\frac{1 - qs\kappa w}{1 - s\kappa w} \right)^N$

So,

$$\begin{aligned}
\left(\frac{1 - qs\kappa w}{1 - s\kappa w} \right)^N &\sim \sum_\lambda F_\lambda(\boldsymbol{\kappa}) G_\lambda(\boldsymbol{\rho}; w) \\
&= \sum_\lambda F_\lambda(\boldsymbol{\kappa}) G_\lambda(\boldsymbol{\rho}) \cdot \frac{G_\lambda(\boldsymbol{\rho}; w)}{G_\lambda(\boldsymbol{\rho})} \\
&= \sum_\lambda \mathbb{P}[\lambda] \frac{G_\lambda(\boldsymbol{\rho}; w)}{G_\lambda(\boldsymbol{\rho})} \\
&= \mathbb{E}_\lambda \left[\frac{G_\lambda(\boldsymbol{\rho}; w)}{G_\lambda(\boldsymbol{\rho})} \right] \sim \mathbb{E}_\lambda \left[(-s)^{-|\lambda|} G_\lambda(\boldsymbol{\rho}; w) \right]
\end{aligned}$$

Expectations Through the Cauchy Identity

Recall $\mathbb{E}_\lambda\left[(-s)^{-|\lambda|}G_\lambda(\boldsymbol{\rho}; w)\right] \sim \left(\frac{1 - qs\kappa w}{1 - s\kappa w}\right)^N$

- **Fact**: We have $(-s)^{-|\lambda|}G_\lambda(\boldsymbol{\rho}; w) = q^N + (1-q)\sum_{i=1}^{N} \frac{q^{i-1}}{1 - sw}\left(\frac{1 - s^{-1}w}{1 - sw}\right)^{\lambda_i - 1}$
- k**-Term**: $\frac{1}{2\pi\mathbf{i}}\oint \frac{dw}{1 - sw}\left(\frac{1 - s^{-1}w}{1 - sw}\right)^{-k} \cdot \frac{1}{1 - sw}\left(\frac{1 - s^{-1}w}{1 - sw}\right)^{\lambda_i - 1} = (s - s^{-1})\mathbf{1}_{\lambda_i = k}$
  - Under change of variables $u = \frac{1 - s^{-1}w}{1 - sw}$, becomes $\frac{1}{2\pi\mathbf{i}}\oint u^{\lambda_i - k - 1}du = \mathbf{1}_{\lambda_i = k}$
- Thus, $\frac{1}{2\pi\mathbf{i}}\oint \frac{dw}{1 - sw}\left(\frac{1 - s^{-1}w}{1 - sw}\right)^{-k} \cdot (-s)^{-|\lambda|}G_\lambda(\boldsymbol{\rho}; w) \sim (1-q)\sum_{i=1}^{N} q^{i-1}\mathbf{1}_{\lambda_i = k}$

So, $\mathbb{E}_\lambda\left[\sum_{i=1}^{N}(1-q)q^{i-1}\mathbf{1}_{\lambda_i = k}\right] \sim \frac{1}{2\pi\mathbf{i}}\oint \frac{dw}{1 - sw}\left(\frac{1 - sw}{1 - s^{-1}w}\right)^k \left(\frac{1 - qs\kappa w}{1 - s\kappa w}\right)^N$

- Summing over $k > x$ gives q-moment $\mathbb{E}_\lambda[q^{\mathfrak{h}_\lambda(x)}]$, since $q^{i-1}\mathbf{1}_{\lambda_i = k} = q^{\mathfrak{h}_\lambda(k)}\mathbf{1}_{\lambda_i = k}$
- To obtain higher moment $\mathbb{E}_\lambda[q^{m\mathfrak{h}_\lambda(x)}]$: Use m additional w-variables

Outline of the Method

1. Define symmetric functions $F_\lambda(X)$ and $G_\lambda(Y)$ as partition functions
2. Show Cauchy identity for $\sum_\lambda F_\lambda(X)G_\lambda(Y)$ by Yang–Baxter equation
3. Define measure $\mathbb{P}[\lambda] = Z(X;Y)^{-1}F_\lambda(X)G_\lambda(Y)$, and realize six-vertex random partition as special case, $\mathbb{P}[\lambda] = F_\lambda(\boldsymbol{\kappa})G_\lambda(\boldsymbol{\rho})$, for $s = q^{-1/2}$
4. Insert w-variables into Cauchy identity to obtain expectation values
   - Cauchy identity yields expression for $\mathbb{E}_\lambda\left[G_\lambda(\boldsymbol{\rho};w)/G_\lambda(\boldsymbol{\rho})\right]$
   - Use exact **formula** for $G_\lambda(\boldsymbol{\rho};w)$
   - Integrate with respect to w to obtain a q-moment type expectation

Exact formula for G_λ: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then

$$G_\lambda(y_1, y_2, \ldots, y_M) \sim \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{1 - qu_i y_j}{1 - u_i y_j}$$

$$\times \prod_{i=1}^{k} \left(\frac{1 - su_i}{u_i - s}\right)^{\lambda_i} \frac{du_i}{(u_i - s)(1 - su_i)}$$

Formula for G_λ

Integral formula

$$G_\lambda(y_1, y_2, \ldots, y_M) \sim \frac{1}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1\le i<j\le k} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{1 - qu_i y_j}{1 - u_i y_j} \times \prod_{i=1}^{k} \left(\frac{1 - su_i}{u_i - s} \right)^{\lambda_i} \frac{du_i}{(u_i - s)(1 - su_i)}$$

Follows from **orthogonality relation** for F_λ

- States $\oint \cdots \oint F_\lambda(u_1, \ldots, u_N) \prod_{1\le i<j\le N} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{N} \left(\frac{1 - su_i}{u_i - s} \right)^{\mu_i} \frac{du_i}{u_i - s} \sim \mathbf{1}_{\lambda=\mu}$
  - Proof uses **monomial expansion**

    $$F_\lambda(u_1, u_2, \ldots, u_N) = \sum_{\sigma \in \mathfrak{S}_N} \sigma \left(\prod_{1\le i<j\le N} \frac{u_i - qu_j}{u_i - u_j} \prod_{i=1}^{N} \frac{1-q}{1-su} \left(\frac{u_i - s}{1 - su_i} \right)^{\lambda_i} \right)$$

    - Proven by **algebraic Bethe ansatz**, again based on Yang–Baxter solvability
  - Gives $\oint \cdots \oint F_\lambda(u_1, \ldots, u_N) F_\mu(u_1^{-1}, \ldots, u_N^{-1}) \prod_{1\le i<j\le N} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^{N} du_i \sim \mathbf{1}_{\lambda=\mu}$
- Formula for G_λ: Orthogonality in $\sum_\lambda F_\lambda(U) G_\lambda(Y) = \prod_{i=1}^{N} \left(\frac{1}{1 - su_i} \prod_{j=1}^{M} \frac{1 - qu_i y_j}{1 - u_i y_j} \right)$

Summary

Stochastic six-vertex model

- Specialization of six-vertex model with stochastic / Markovian weights

Asymptotic questions for stochastic six-vertex model

- Limit shape: Law of large numbers for height function
- Fluctuations: Behavior of height fluctuations around limit shape
- Local statistics: Limits for local correlation functions

Analysis through integrability

- Explicit formulas for q-deformed moment of height function $\mathbb{E}[q^{kH(x,y)}]$
  - Complexity grows in k, but not in $|x| + |y|$, to allow large scale limit
- Obtained through Cauchy identities and orthogonality, of which are based on integrability
  - Both are based on integrability
    - Cauchy identity: Yang–Baxter commutation relations
    - Orthogonality: Algebraic Bethe ansatz